Supplementary Online Appendix

to

"Determining the Cointegration Rank in Heteroskedastic VAR Models of Unknown Order"

by

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August 17, 2016

Summary of Contents

This Supplement contains the Tables of results from the Monte Carlo experiments detailed in Sections 3 and 5 of our paper "Determining the Cointegration Rank in Heteroskedastic VAR Models of Unknown Order".

In particular, the results reported in Table 1 relate to the impact that the value of the lag length used has on the ability of information-based and PLR based procedures to determine the cointegration rank by reporting the frequencies with which the wild bootstrap PLR based procedure and a BIC-based procedure, which minimises the quantity BIC(k, r) defined in (3.5) over $r = 0, 1, ..., p$, select the true cointegration rank, in each case fixing the lag length used at each of $k = 1, ..., 6$. The results in Table 2 relate to the behaviour of the information criteria IC(k, p) defined in (3.10) in determining the autoregressive lag length k. The results in Tables 3.1, 3.2 and 3.3 relate to cointegration rank determination for the methods considered in section 3 of the paper for the cases of no cointegration ($r_0 = 0$) and GJR-GARCH innovations (Case A), Autoregressive Stochastic Volatility (Case B) and a single volatility break (Case C), respectively. Finally, the results in Tables 4.1, 4.2 and 4.3 relate to cointegration rank determination for these methods for the case of a single cointegration vector ($r_0 = 1$) in Cases A, B and C, respectively.

In all of the tables, the frequencies with which the various procedures select the true cointegration rank or true lag length are highlighted in bold text.

TABLE 1: Cointegration rank determination for different values of lag length k. VAR(2) model, i.i.d. Gaussian errors.

			BIC(k,r)						$Q^*_{r,k,T}$					
	T	γ	$k=1$	**2**	3	4	5	6	$k=1$	**2**	3	4	5	6
$r_0=0$	100	0.2	97.7	**98.9**	98.5	97.7	96.7	94.9	89.8	**95.7**	95.8	95.9	95.9	96.0
		0.5	62.5	**98.5**	97.9	96.9	94.7	91.8	42.8	**95.6**	95.8	95.8	95.8	95.7
		0.8	1.4	**96.0**	93.8	90.6	85.8	80.0	0.9	**95.1**	94.4	94.7	94.2	94.5
	400	0.2	99.9	**100**	100	100	100	100	89.6	**95.2**	95.0	95.2	95.2	95.4
		0.5	86.4	**100**	100	100	100	100	40.8	**95.2**	95.3	95.4	95.5	95.2
		0.8	5.9	**100**	100	100	99.9	99.9	0.9	**95.2**	95.4	95.4	95.2	95.2
$r_0=1$	100	0.2	8.5	**26.7**	14.6	11.0	9.2	9.6	28.5	**40.0**	23.1	15.4	10.7	8.0
		0.5	22.2	**76.1**	43.4	26.5	18.4	16.2	42.2	**73.2**	43.4	25.3	15.8	10.8
		0.8	43.3	**95.5**	89.4	66.2	43.2	33.6	44.2	**94.2**	78.5	48.1	26.7	16.0
	400	0.2	99.8	**99.9**	100	99.4	94.5	82.2	91.5	**95.3**	95.4	95.4	95.5	94.7
		0.5	89.2	**100**	100	100	99.8	97.9	58.0	**95.5**	95.6	95.5	95.5	95.4
		0.8	18.5	**99.9**	99.9	99.9	99.9	99.9	7.4	**95.5**	95.4	95.5	95.5	95.4
$r_0=2$	100	0.2	3.2	**21.0**	6.3	2.6	1.3	1.1	25.7	**40.6**	19.4	10.2	5.1	2.6
		0.5	2.2	**78.7**	36.1	13.7	5.7	3.3	18.6	**77.9**	43.8	21.8	9.6	4.7
		0.8	10.1	**94.9**	89.2	60.2	27.4	14.2	23.9	**94.3**	82.3	49.7	21.4	9.7
	400	0.2	99.3	**99.8**	99.8	99.7	99.2	94.3	93.1	**95.3**	95.6	95.5	95.5	95.2
		0.5	90.6	**99.8**	99.7	99.7	99.7	99.6	73.4	**95.5**	95.5	95.3	95.3	95.3
		0.8	40.0	**99.6**	99.7	99.7	99.7	99.6	26.0	**95.4**	95.4	95.1	95.3	95.2

TABLE 2: Sequential determination of lag length using IC(k,p): $\hat{k}_{\mathrm{IC}}$. VAR(2) model with rank $r_0 = 1$.

		HQC(k,p)			BIC(k,p)			HQC(k,p)			BIC(k,p)			HQC(k,p)			BIC(k,p)		
		GJR-GARCH errors [Case A]						Autoregressive Stochastic Volatility [Case B]						Single Volatility Break [Case C]					
γ	T	$k=1$	2	3,4,5,6	$k=1$	2	3,4,5,6	$k=1$	2	3,4,5,6	$k=1$	2	3,4,5,6	$k=1$	2	3,4,5,6	$k=1$	2	3,4,5,6
0.0	50	**99.2**	0.6	0.3	**100.0**	0.0	0.0	**89.9**	4.6	5.5	**99.0**	0.8	0.3	**35.6**	9.9	54.5	**95.4**	3.7	0.9
	100	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**93.7**	4.6	1.7	**99.5**	0.5	0.0	**80.9**	10.5	8.7	**99.6**	0.4	0.0
	200	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**90.4**	7.2	2.4	**98.9**	1.0	0.1	**92.7**	5.6	1.6	**100.0**	0.1	0.0
	400	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**83.2**	11.5	5.3	**97.5**	2.2	0.2	**96.3**	3.3	0.4	**100.0**	0.0	0.0
0.3	50	76.2	**22.7**	1.1	98.4	**1.7**	0.0	57.6	**31.6**	10.8	90.6	**8.8**	0.6	16.4	**18.5**	65.1	78.5	**18.7**	2.8
	100	39.1	**60.9**	0.0	89.6	**10.4**	0.0	32.3	**63.2**	4.5	74.2	**25.6**	0.2	17.6	**60.7**	21.7	68.5	**31.2**	0.3
	200	2.0	**98.0**	0.0	47.9	**52.1**	0.0	6.4	**87.3**	6.2	35.9	**63.6**	0.4	1.8	**89.4**	8.8	28.3	**71.6**	0.1
	400	0.0	**100.0**	0.0	0.7	**99.3**	0.0	0.4	**88.4**	11.3	5.7	**93.2**	1.1	0.0	**96.0**	4.0	0.6	**99.4**	0.0
0.5	50	11.2	**86.3**	2.5	53.2	**46.5**	0.3	11.1	**73.0**	15.9	44.5	**54.3**	1.2	2.8	**24.5**	72.7	36.3	**56.6**	7.1
	100	0.0	**100.0**	0.0	1.3	**98.7**	0.0	0.5	**93.8**	5.7	6.5	**93.1**	0.4	0.1	**75.8**	24.1	4.0	**95.2**	0.8
	200	0.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**93.4**	6.6	0.1	**99.3**	0.6	0.0	**91.0**	9.0	0.0	**99.9**	0.1
	400	0.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**88.0**	12.0	0.0	**98.8**	1.2	0.0	**95.9**	4.1	0.0	**100.0**	0.0
0.9	50	0.0	**95.9**	4.1	0.0	**100.0**	0.0	0.0	**76.5**	23.5	0.1	**97.8**	2.1	0.0	**17.3**	82.7	0.0	**86.2**	13.8
	100	0.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**92.9**	7.1	0.0	**99.4**	0.6	0.0	**73.9**	26.1	0.0	**99.1**	0.9
	200	0.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**91.5**	8.5	0.0	**99.1**	0.9	0.0	**90.6**	9.4	0.0	**99.9**	0.1
	400	0.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0	**84.6**	15.4	0.0	**98.1**	1.9	0.0	**95.8**	4.2	0.0	**100.0**	0.0

TABLE 3.1: Determination of cointegration rank. VAR(2) model with rank $r_0 = 0$, GJR-GARCH errors [Case A]

		$\mathrm{HQC}(k,r)$			$\mathrm{BIC}(k,r)$			$\mathrm{HQC}(\hat{k}_{\mathrm{AIC}},r)$			$\mathrm{HQC}(\hat{k}_{\mathrm{BIC}},r)$			$\mathrm{BIC}(\hat{k}_{\mathrm{AIC}},r)$			$\mathrm{BIC}(\hat{k}_{\mathrm{HQC}},r)$		
γ	T	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4
0.0	50	**61.8**	29.3	9.0	**95.4**	4.6	0.0	**50.6**	25.1	24.2	**61.8**	29.2	8.9	**80.9**	11.7	7.4	**95.4**	4.6	0.0
	100	**81.3**	15.9	2.7	**99.4**	0.6	0.0	**75.3**	20.8	4.0	**78.4**	17.6	4.0	**100.0**	0.0	0.0	**100.0**	0.0	0.0
	200	**93.1**	4.8	2.2	**98.9**	1.1	0.0	**85.9**	12.5	1.6	**86.9**	11.6	1.6	**100.0**	0.0	0.0	**100.0**	0.0	0.0
	400	**92.7**	6.8	0.4	**99.7**	0.3	0.0	**92.6**	7.0	0.4	**92.7**	6.9	0.4	**99.7**	0.3	0.0	**99.7**	0.3	0.0
0.3	50	**33.6**	39.8	26.6	**80.5**	15.2	4.3	**19.8**	25.2	55.0	**34.6**	41.5	23.9	**56.3**	22.9	20.8	**77.9**	16.9	5.2
	100	**65.1**	27.2	7.8	**95.7**	3.7	0.6	**66.4**	26.0	7.6	**53.0**	35.6	11.4	**97.2**	2.2	0.6	**96.6**	2.9	0.6
	200	**80.7**	15.9	3.4	**97.7**	2.3	0.0	**85.3**	12.5	2.2	**77.0**	18.6	4.5	**99.7**	0.3	0.0	**99.7**	0.3	0.0
	400	**92.0**	7.1	0.9	**99.7**	0.3	0.0	**91.9**	7.2	0.9	**92.1**	7.1	0.9	**99.7**	0.3	0.0	**99.7**	0.3	0.0
0.5	50	**28.4**	35.0	36.5	**64.2**	27.4	8.4	**14.0**	19.1	66.9	**11.4**	35.8	52.8	**45.5**	21.0	33.5	**70.3**	20.8	8.9
	100	**69.6**	25.8	4.7	**97.8**	1.9	0.3	**65.0**	28.0	7.1	**67.1**	26.2	6.8	**96.9**	2.8	0.3	**98.4**	1.3	0.3
	200	**84.7**	12.8	2.5	**99.7**	0.3	0.0	**84.0**	13.4	2.6	**84.7**	12.8	2.6	**99.4**	0.6	0.0	**99.7**	0.3	0.0
	400	**91.5**	7.4	1.1	**99.7**	0.3	0.0	**91.6**	7.2	1.2	**91.5**	7.4	1.1	**99.7**	0.3	0.0	**99.7**	0.3	0.0
0.9	50	**3.6**	8.5	88.0	**56.8**	24.9	18.3	**0.6**	3.2	96.2	**3.6**	9.2	87.2	**16.3**	11.0	72.6	**51.4**	23.0	25.6
	100	**28.5**	31.6	39.9	**92.8**	5.3	1.9	**22.1**	28.8	49.1	**26.7**	29.8	43.5	**87.1**	9.2	3.7	**93.2**	6.2	0.6
	200	**59.6**	29.9	10.5	**99.3**	0.7	0.0	**58.0**	31.5	10.6	**59.6**	29.9	10.5	**98.5**	1.6	0.0	**99.4**	0.6	0.0
	400	**81.9**	16.7	1.5	**99.9**	0.1	0.0	**81.6**	16.7	1.7	**81.8**	16.7	1.5	**99.9**	0.1	0.0	**99.9**	0.1	0.0

		$\mathrm{HQC}(1,r)$			$\mathrm{BIC}(1,r)$			$Q^*_{r,\hat{k}_{\mathrm{AIC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{HQC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{BIC}},T}$		
0.0	50	**59.9**	31.8	8.2	**96.3**	3.7	0.0	**92.1**	6.5	1.4	**96.3**	3.4	0.3	**96.2**	3.5	0.3
	100	**80.3**	17.4	2.3	**99.0**	1.0	0.0	**94.2**	5.4	0.4	**94.7**	4.7	0.7	**94.5**	4.9	0.6
	200	**86.0**	13.4	0.6	**100.0**	0.0	0.0	**95.2**	4.8	0.0	**95.2**	4.8	0.0	**95.2**	4.8	0.0
	400	**92.9**	6.5	0.6	**100.0**	0.0	0.0	**94.0**	5.2	0.8	**94.2**	5.1	0.7	**94.1**	5.1	0.8
0.3	50	**36.4**	42.3	21.4	**82.7**	15.8	1.5	**84.6**	14.0	1.4	**82.1**	15.6	2.3	**81.7**	15.8	2.5
	100	**55.4**	35.6	9.0	**92.1**	7.5	0.4	**93.0**	6.4	0.7	**88.1**	11.0	1.0	**83.2**	15.0	1.8
	200	**64.1**	32.8	3.1	**96.0**	4.1	0.0	**94.7**	4.6	0.7	**94.6**	4.7	0.8	**86.7**	12.0	1.3
	400	**73.6**	21.8	4.6	**98.6**	1.4	0.0	**93.6**	5.3	1.2	**93.7**	5.2	1.1	**93.6**	5.3	1.1
0.5	50	**11.4**	32.0	56.6	**44.8**	40.4	14.8	**86.6**	11.8	1.6	**79.2**	16.3	4.5	**54.5**	34.7	10.7
	100	**20.0**	41.8	38.2	**62.7**	32.3	5.0	**95.2**	4.4	0.4	**96.8**	2.9	0.4	**93.6**	5.4	0.9
	200	**29.8**	44.9	25.3	**70.3**	29.0	0.6	**93.5**	6.4	0.0	**93.4**	6.5	0.0	**93.5**	6.4	0.0
	400	**34.0**	43.8	22.2	**85.4**	13.6	1.0	**93.8**	5.1	1.1	**93.9**	5.0	1.1	**93.9**	4.9	1.2
0.9	50	**0.0**	0.5	99.5	**0.0**	1.7	98.3	**81.1**	15.6	3.3	**87.5**	10.1	2.4	**92.4**	6.0	1.6
	100	**0.0**	0.8	99.3	**0.0**	2.2	97.8	**90.5**	6.7	2.8	**93.7**	4.6	1.7	**93.7**	4.5	1.8
	200	**0.0**	0.6	99.4	**0.0**	3.5	96.5	**93.8**	4.9	1.3	**94.8**	4.0	1.3	**94.5**	4.2	1.3
	400	**0.0**	1.8	98.2	**0.0**	3.7	96.3	**94.6**	4.5	0.9	**95.0**	4.1	0.9	**94.9**	4.3	0.9

$\mathrm{IC}(k,r)$ denotes the joint IC-based procedure for determining k and r; $\mathrm{IC}(\hat{k}_{\mathrm{IC}},r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained in the first step; $\mathrm{IC}(1,r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k=1$; $Q^*_{r,\hat{k}_{\mathrm{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained by an IC-based procedure in the first step.

TABLE 3.2: Determination of cointegration rank. VAR(2) model with rank $r_0 = 0$, Autoregressive Stochastic Volatility [Case B]

		HQC(k,r)			BIC(k,r)			HQC$(\hat{k}_{\mathrm{AIC}},r)$			HQC$(\hat{k}_{\mathrm{BIC}},r)$			BIC$(\hat{k}_{\mathrm{AIC}},r)$			BIC$(\hat{k}_{\mathrm{HQC}},r)$		
γ	T	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4
0.0	50	**45.5**	34.5	20.0	**83.4**	14.6	2.0	**26.8**	25.3	47.9	**46.6**	35.6	17.9	**54.1**	24.7	21.3	**78.5**	16.1	5.3
	100	**60.1**	29.4	10.6	**91.2**	8.2	0.7	**52.4**	32.1	15.6	**60.4**	29.1	10.5	**85.9**	12.3	1.8	**90.2**	8.9	0.9
	200	**68.0**	25.8	6.2	**94.5**	5.3	0.2	**64.7**	27.9	7.4	**67.5**	26.1	6.4	**93.7**	6.1	0.3	**94.0**	5.8	0.2
	400	**75.6**	21.0	3.5	**97.1**	2.9	0.1	**75.3**	21.1	3.6	**74.6**	22.0	3.4	**97.4**	2.6	0.1	**97.0**	2.9	0.0
0.3	50	**29.9**	37.1	33.0	**69.9**	24.5	5.6	**13.2**	22.0	64.7	**31.1**	39.2	29.7	**38.3**	29.2	32.6	**62.7**	25.5	11.8
	100	**51.4**	34.6	13.9	**84.6**	13.8	1.7	**45.2**	35.6	19.2	**46.2**	37.2	16.6	**83.6**	14.2	2.2	**85.6**	12.7	1.7
	200	**65.9**	27.6	6.6	**93.4**	6.3	0.3	**63.0**	29.1	7.9	**61.7**	30.3	8.0	**94.0**	5.8	0.3	**94.2**	5.6	0.2
	400	**76.4**	20.2	3.5	**97.3**	2.6	0.0	**76.2**	20.3	3.5	**75.1**	21.3	3.6	**97.6**	2.4	0.0	**97.5**	2.5	0.0
0.5	50	**20.3**	33.5	46.3	**56.8**	31.1	12.1	**8.7**	15.7	75.6	**13.9**	34.8	51.3	**30.0**	25.6	44.4	**54.1**	27.0	19.0
	100	**49.9**	34.8	15.3	**87.2**	11.4	1.4	**42.0**	35.8	22.2	**47.0**	34.9	18.1	**82.3**	15.1	2.6	**86.8**	11.6	1.6
	200	**65.3**	27.9	6.8	**94.9**	5.0	0.2	**61.8**	29.8	8.5	**65.0**	28.1	6.8	**93.6**	6.1	0.3	**94.5**	5.3	0.2
	400	**76.4**	20.1	3.4	**97.4**	2.6	0.0	**76.0**	20.4	3.6	**75.8**	20.6	3.6	**97.6**	2.4	0.0	**97.5**	2.5	0.0
0.9	50	**2.9**	8.9	88.2	**42.8**	28.7	28.5	**0.7**	2.2	97.1	**3.5**	11.1	85.4	**9.1**	9.2	81.7	**32.5**	22.8	44.8
	100	**20.3**	31.0	48.7	**75.7**	19.6	4.7	**14.3**	25.8	59.9	**20.6**	31.5	47.9	**63.4**	25.1	11.5	**73.6**	20.4	6.0
	200	**47.6**	34.8	17.5	**90.7**	8.6	0.7	**42.7**	35.8	21.4	**47.7**	35.0	17.4	**88.6**	10.5	0.9	**90.4**	8.9	0.8
	400	**69.6**	25.2	5.2	**96.8**	3.2	0.0	**68.5**	25.6	5.9	**69.0**	25.7	5.3	**97.0**	2.9	0.1	**96.9**	3.1	0.0

		HQC$(1,r)$			BIC$(1,r)$			$Q^*_{r,\hat{k}_{\mathrm{AIC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{HQC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{BIC}},T}$		
0.0	50	**46.8**	35.7	17.5	**83.6**	14.6	1.9	**86.2**	11.8	2.0	**91.6**	7.1	1.4	**94.8**	4.5	0.7
	100	**60.4**	29.1	10.5	**91.1**	8.2	0.7	**89.7**	8.9	1.4	**93.7**	5.4	0.8	**94.8**	4.6	0.6
	200	**67.5**	26.0	6.5	**94.4**	5.4	0.2	**91.6**	7.5	0.9	**94.0**	5.4	0.7	**94.5**	4.9	0.6
	400	**74.6**	22.0	3.4	**97.0**	3.0	0.1	**93.4**	5.9	0.8	**94.2**	5.1	0.7	**94.6**	4.8	0.6
0.3	50	**32.9**	39.7	27.5	**69.6**	24.8	5.6	**83.0**	14.8	2.2	**81.9**	15.4	2.7	**84.3**	13.5	2.2
	100	**47.0**	36.5	16.6	**81.5**	16.3	2.1	**88.7**	9.7	1.5	**89.2**	9.3	1.5	**86.4**	11.9	1.7
	200	**56.6**	33.1	10.3	**88.3**	10.8	1.0	**91.8**	7.2	1.0	**93.2**	6.0	0.8	**91.0**	8.0	0.9
	400	**64.5**	29.4	6.1	**93.1**	6.7	0.2	**93.6**	5.7	0.7	**94.3**	5.2	0.6	**94.1**	5.4	0.5
0.5	50	**12.3**	35.8	51.9	**39.8**	39.9	20.3	**83.4**	14.0	2.6	**78.6**	16.8	4.6	**68.3**	24.2	7.5
	100	**21.4**	40.4	38.3	**55.3**	33.8	11.0	**89.3**	9.1	1.7	**92.7**	6.2	1.2	**89.9**	8.1	2.0
	200	**28.7**	43.1	28.2	**67.0**	27.2	5.8	**91.9**	7.1	1.0	**94.0**	5.2	0.8	**94.4**	4.9	0.7
	400	**35.3**	43.4	21.3	**76.4**	21.0	2.6	**93.4**	6.0	0.7	**94.2**	5.2	0.6	**94.2**	5.3	0.5
0.9	50	**0.0**	0.5	99.5	**0.0**	2.4	97.6	**77.9**	18.3	3.8	**77.6**	17.5	4.9	**87.4**	8.9	3.7
	100	**0.0**	0.9	99.1	**0.1**	3.2	96.8	**83.9**	13.7	2.4	**89.7**	8.7	1.6	**91.5**	7.2	1.3
	200	**0.0**	1.2	98.8	**0.1**	4.9	94.9	**90.0**	8.6	1.4	**92.3**	6.6	1.1	**92.9**	6.1	1.0
	400	**0.0**	1.7	98.3	**0.4**	7.6	92.1	**93.6**	5.8	0.6	**94.2**	5.3	0.5	**94.3**	5.2	0.5

IC(k,r) denotes the joint IC-based procedure for determining k and r; IC$(\hat{k}_{\mathrm{IC}},r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained in the first step; IC$(1,r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k=1$; $Q^*_{r,\hat{k}_{\mathrm{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained by an IC-based procedure in the first step.

TABLE 3.3: Determination of cointegration rank. VAR(2) model with rank $r_0 = 0$, Single Volatility Break [Case C]

		HQC(k,r)			BIC(k,r)			HQC$(\hat{k}_{\text{AIC}},r)$			HQC$(\hat{k}_{\text{BIC}},r)$			BIC$(\hat{k}_{\text{AIC}},r)$			BIC$(\hat{k}_{\text{HQC}},r)$		
γ	T	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4
0.0	50	**14.4**	26.9	58.7	**76.1**	20.4	3.6	**1.2**	12.0	86.8	**26.9**	39.2	33.9	**16.6**	40.9	42.5	**33.3**	33.6	33.1
	100	**41.5**	39.6	19.0	**90.3**	9.1	0.6	**17.6**	39.8	42.5	**43.2**	39.7	17.1	**68.9**	26.5	4.5	**84.6**	13.5	1.9
	200	**56.4**	33.0	10.6	**96.5**	3.4	0.1	**41.5**	39.3	19.2	**57.0**	32.6	10.4	**91.0**	8.6	0.4	**95.9**	4.0	0.2
	400	**66.8**	27.1	6.1	**98.8**	1.2	0.0	**58.3**	32.4	9.4	**67.0**	26.9	6.1	**97.8**	2.1	0.1	**98.7**	1.3	0.0
0.3	50	**6.8**	21.7	71.5	**52.3**	36.5	11.2	**0.6**	8.2	91.2	**13.6**	38.0	48.4	**11.4**	36.6	52.0	**19.0**	36.5	44.5
	100	**30.3**	42.0	27.7	**73.2**	23.5	3.3	**15.5**	39.2	45.3	**25.0**	45.5	29.5	**65.8**	28.7	5.5	**72.6**	23.1	4.4
	200	**51.9**	34.7	13.4	**92.0**	7.4	0.6	**39.5**	40.1	20.4	**44.8**	38.0	17.2	**91.3**	8.1	0.6	**94.3**	5.4	0.4
	400	**65.0**	28.1	6.9	**98.6**	1.4	0.0	**57.7**	32.8	9.5	**64.7**	28.4	6.9	**97.9**	2.0	0.1	**98.5**	1.5	0.0
0.5	50	**3.5**	15.0	81.5	**40.0**	40.0	20.0	**0.2**	4.6	95.1	**5.3**	27.6	67.1	**7.6**	30.4	62.1	**14.6**	30.7	54.7
	100	**29.7**	40.4	29.9	**83.0**	15.0	2.1	**13.7**	36.9	49.4	**29.7**	40.8	29.5	**63.2**	30.4	6.4	**76.1**	20.0	3.8
	200	**50.0**	35.8	14.3	**95.3**	4.5	0.2	**37.4**	40.6	22.0	**50.3**	35.7	14.0	**90.1**	9.2	0.7	**94.2**	5.5	0.3
	400	**63.6**	29.1	7.3	**98.5**	1.5	0.0	**56.6**	33.4	10.0	**63.6**	29.2	7.3	**97.7**	2.3	0.1	**98.4**	1.6	0.0
0.9	50	**0.2**	2.0	97.8	**29.3**	33.8	36.9	**0.0**	0.1	99.9	**1.5**	9.2	89.3	**0.6**	5.3	94.1	**3.7**	9.8	86.5
	100	**12.0**	30.8	57.3	**70.0**	24.5	5.5	**3.0**	16.9	80.1	**13.8**	33.4	52.8	**39.2**	39.2	21.6	**58.1**	30.2	11.7
	200	**32.7**	40.2	27.1	**90.4**	9.0	0.6	**22.2**	39.4	38.4	**33.2**	40.3	26.5	**81.8**	16.0	2.2	**89.0**	10.0	1.0
	400	**51.5**	35.5	13.0	**97.5**	2.5	0.1	**44.6**	39.0	16.4	**51.5**	35.5	13.0	**95.5**	4.3	0.2	**97.2**	2.7	0.1

		HQC$(1,r)$			BIC$(1,r)$			$Q^*_{r,\hat{k}_{\text{AIC}},T}$			$Q^*_{r,\hat{k}_{\text{HQC}},T}$			$Q^*_{r,\hat{k}_{\text{BIC}},T}$		
0.0	50	**28.5**	40.1	31.4	**76.5**	20.1	3.3	**82.2**	15.8	2.0	**72.1**	23.6	4.2	**91.2**	7.1	1.7
	100	**43.4**	39.6	17.0	**90.3**	9.1	0.6	**74.9**	21.4	3.7	**88.0**	9.9	2.1	**93.6**	5.5	0.9
	200	**57.0**	32.6	10.4	**96.5**	3.4	0.1	**84.8**	13.2	2.1	**92.9**	6.4	0.7	**93.8**	5.6	0.6
	400	**67.0**	26.9	6.1	**98.8**	1.2	0.0	**90.9**	7.8	1.2	**94.5**	4.9	0.6	**94.7**	4.7	0.6
0.3	50	**16.9**	41.2	42.0	**52.8**	36.2	11.0	**82.7**	15.3	2.0	**67.4**	27.4	5.3	**72.7**	22.7	4.6
	100	**25.4**	46.0	28.6	**67.0**	28.5	4.5	**75.4**	20.9	3.8	**79.0**	17.7	3.3	**76.9**	20.2	3.0
	200	**33.8**	44.4	21.8	**77.7**	20.2	2.1	**85.1**	12.8	2.1	**91.1**	7.7	1.2	**85.4**	12.7	1.9
	400	**39.8**	43.4	16.8	**85.2**	13.7	1.0	**91.1**	7.7	1.3	**94.0**	5.4	0.7	**94.1**	5.3	0.6
0.5	50	**5.9**	31.6	62.5	**24.3**	46.1	29.6	**82.7**	15.2	2.1	**68.7**	26.2	5.1	**57.4**	32.3	10.4
	100	**8.7**	39.1	52.2	**32.9**	47.2	19.9	**76.0**	20.6	3.4	**84.2**	13.2	2.6	**86.5**	10.5	3.0
	200	**11.9**	41.5	46.7	**42.6**	43.4	14.0	**85.1**	12.8	2.1	**91.8**	7.1	1.0	**93.1**	6.1	0.8
	400	**14.0**	44.7	41.3	**51.7**	38.7	9.6	**90.7**	8.1	1.3	**94.0**	5.4	0.7	**94.2**	5.2	0.6
0.9	50	**0.0**	0.8	99.2	**0.0**	3.5	96.5	**75.7**	20.5	3.7	**62.0**	31.1	6.9	**72.3**	19.8	7.9
	100	**0.0**	0.9	99.1	**0.0**	3.4	96.5	**72.5**	23.6	3.9	**79.7**	17.3	3.0	**89.3**	9.2	1.5
	200	**0.0**	1.0	99.0	**0.0**	3.8	96.2	**84.9**	12.8	2.4	**91.1**	7.8	1.1	**92.7**	6.4	0.9
	400	**0.0**	1.2	98.8	**0.1**	5.2	94.7	**90.3**	8.4	1.3	**93.9**	5.4	0.7	**94.2**	5.1	0.6

IC(k,r) denotes the joint IC-based procedure for determining k and r; IC$(\hat{k}_{\text{IC}},r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained in the first step; IC$(1,r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k=1$; $Q^*_{r,\hat{k}_{\text{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained by an IC-based procedure in the first step.

TABLE 4.1: Determination of cointegration rank. VAR(2) model with rank $r_0 = 1$, GJR-GARCH errors [Case A]

		HQC(k, r)			BIC(k, r)			HQC$(\hat{k}_{\mathrm{AIC}}, r)$			HQC$(\hat{k}_{\mathrm{BIC}}, r)$			BIC$(\hat{k}_{\mathrm{AIC}}, r)$			BIC$(\hat{k}_{\mathrm{HQC}}, r)$		
γ	T	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4
0.0	50	37.5	**44.4**	18.2	89.4	**10.3**	0.3	30.1	**39.1**	30.8	37.2	**44.6**	18.3	78.4	**13.9**	7.7	88.8	**10.7**	0.6
	100	10.7	**74.0**	15.3	78.1	**21.3**	0.6	14.4	**68.9**	16.7	15.2	**69.7**	15.1	73.7	**25.5**	0.8	76.0	**23.2**	0.8
	200	0.0	**86.3**	13.7	8.3	**91.7**	0.0	0.0	**86.3**	13.7	0.0	**86.3**	13.7	9.3	**90.7**	0.0	8.3	**91.7**	0.0
	400	0.0	**90.3**	9.7	0.0	**99.7**	0.3	0.0	**90.3**	9.7	0.0	**90.4**	9.6	0.0	**99.7**	0.3	0.0	**99.7**	0.3
0.3	50	37.7	**36.3**	25.9	85.7	**12.4**	1.9	15.1	**31.1**	53.8	42.1	**36.9**	21.0	51.9	**28.4**	19.7	76.2	**20.5**	3.3
	100	15.8	**66.7**	17.6	86.3	**13.3**	0.4	6.2	**73.8**	20.1	24.4	**52.5**	23.1	56.8	**42.3**	0.9	63.5	**35.7**	0.9
	200	0.0	**86.2**	13.8	27.8	**71.3**	1.0	0.0	**86.2**	13.9	0.0	**79.3**	20.7	5.7	**94.0**	0.3	5.4	**93.9**	0.7
	400	0.0	**90.9**	9.1	0.0	**99.7**	0.3	0.0	**91.0**	9.0	0.0	**91.0**	9.0	0.1	**99.6**	0.3	0.0	**99.7**	0.3
0.5	50	10.7	**38.3**	51.1	56.9	**36.1**	7.1	4.6	**27.1**	68.3	11.7	**39.2**	49.0	36.5	**37.7**	25.8	52.6	**37.0**	10.4
	100	0.6	**77.2**	22.2	25.5	**71.8**	2.8	0.6	**75.9**	23.5	0.7	**76.2**	23.1	23.2	**73.8**	3.1	23.1	**74.1**	2.8
	200	0.0	**85.2**	14.8	0.0	**98.9**	1.1	0.0	**84.7**	15.3	0.0	**84.4**	15.6	0.3	**99.1**	0.6	0.0	**99.7**	0.3
	400	0.0	**90.7**	9.3	0.0	**99.7**	0.3	0.0	**90.5**	9.5	0.0	**90.6**	9.4	0.1	**99.6**	0.3	0.0	**99.7**	0.3
0.9	50	0.3	**8.9**	90.8	1.7	**56.6**	41.6	0.3	**2.5**	97.3	0.3	**8.8**	91.0	0.8	**25.9**	73.2	1.4	**54.5**	44.1
	100	0.0	**41.4**	58.6	0.0	**89.8**	10.2	0.0	**39.3**	60.8	0.0	**41.4**	58.6	0.0	**88.1**	11.9	0.0	**89.9**	10.1
	200	0.0	**72.8**	27.2	0.0	**97.7**	2.3	0.0	**72.7**	27.3	0.0	**72.7**	27.3	0.0	**97.7**	2.3	0.0	**97.7**	2.3
	400	0.0	**84.9**	15.1	0.0	**99.7**	0.3	0.0	**84.5**	15.5	0.0	**85.0**	15.0	0.0	**99.7**	0.3	0.0	**99.7**	0.3
		HQC$(1, r)$			BIC$(1, r)$			$Q^*_{r,\hat{k}_{\mathrm{AIC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{HQC}},T}$			$Q^*_{r,\hat{k}_{\mathrm{BIC}},T}$					
0.0	50	40.3	**39.8**	19.9	90.1	**9.4**	0.5	87.7	**11.1**	1.3	90.3	**8.8**	1.0	90.8	**8.6**	0.6			
	100	12.4	**70.8**	16.8	76.7	**23.0**	0.3	55.9	**40.7**	3.4	56.6	**40.2**	3.2	56.8	**40.1**	3.1			
	200	0.0	**84.7**	15.3	9.9	**90.1**	0.0	2.2	**91.7**	6.1	2.1	**91.9**	6.0	2.1	**92.0**	5.9			
	400	0.0	**90.2**	9.8	0.0	**99.9**	0.1	0.0	**94.4**	5.6	0.0	**94.6**	5.4	0.0	**94.6**	5.4			
0.3	50	37.6	**43.2**	19.2	86.0	**12.3**	1.7	83.4	**15.5**	1.2	83.7	**14.7**	1.6	86.4	**12.1**	1.5			
	100	28.4	**49.3**	22.4	88.8	**10.4**	0.9	51.1	**44.6**	4.3	53.0	**42.9**	4.1	64.9	**29.8**	5.4			
	200	0.6	**73.9**	25.6	49.4	**49.4**	1.2	1.1	**93.7**	5.2	0.8	**94.0**	5.3	5.6	**83.6**	10.8			
	400	0.0	**76.0**	24.0	0.0	**97.9**	2.1	0.1	**95.0**	4.9	0.0	**94.8**	5.2	0.0	**94.7**	5.3			
0.5	50	20.6	**46.3**	33.2	59.1	**33.6**	7.3	82.2	**17.0**	0.9	76.0	**21.5**	2.5	66.3	**29.6**	4.2			
	100	23.2	**39.0**	37.8	72.5	**24.4**	3.1	26.9	**68.7**	4.4	26.9	**69.3**	3.8	26.9	**68.9**	4.2			
	200	0.6	**50.8**	48.6	64.9	**27.0**	8.1	0.3	**94.3**	5.4	0.0	**95.2**	4.8	0.0	**95.2**	4.8			
	400	0.0	**46.8**	53.2	0.0	**86.1**	13.9	0.0	**94.7**	5.3	0.0	**94.5**	5.5	0.0	**94.6**	5.5			
0.9	50	0.5	**6.9**	92.6	1.2	**16.9**	81.9	50.9	**44.4**	4.7	17.4	**77.0**	5.6	17.0	**78.2**	4.8			
	100	0.2	**9.7**	90.1	1.2	**22.6**	76.2	1.7	**90.4**	7.8	0.2	**94.2**	5.7	0.1	**94.2**	5.7			
	200	0.0	**1.2**	98.8	1.8	**25.7**	72.5	0.0	**94.8**	5.2	0.0	**95.3**	4.7	0.0	**95.0**	5.0			
	400	0.0	**0.6**	99.4	0.6	**1.8**	97.6	0.0	**95.3**	4.7	0.0	**95.5**	4.6	0.0	**95.4**	4.6			

IC(k, r) denotes the joint IC-based procedure for determining k and r; IC$(\hat{k}_{\mathrm{IC}}, r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained in the first step; IC$(1, r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k = 1$; $Q^*_{r,\hat{k}_{\mathrm{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\mathrm{IC}}(p)$ obtained by an IC-based procedure in the first step.

TABLE 4.2: Determination of cointegration rank. VAR(2) model with rank $r_0 = 1$, Autoregressive Stochastic Volatility [Case B]

		HQC(k,r)			BIC(k,r)			HQC$(\hat{k}_{\text{AIC}},r)$			HQC$(\hat{k}_{\text{BIC}},r)$			BIC$(\hat{k}_{\text{AIC}},r)$			BIC$(\hat{k}_{\text{HQC}},r)$		
γ	T	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4	$r=0$	1	2,3,4
0.0	50	29.7	**39.6**	30.8	75.5	**20.2**	4.3	18.0	**30.2**	51.7	30.3	**40.4**	29.4	51.3	**29.8**	19.0	71.5	**21.4**	7.1
	100	14.1	**54.9**	31.0	63.2	**32.2**	4.7	13.9	**52.6**	33.6	13.9	**55.1**	31.1	61.7	**33.1**	5.2	62.0	**33.0**	5.1
	200	1.4	**68.7**	29.9	18.3	**76.1**	5.6	3.4	**66.8**	29.8	1.0	**68.6**	30.3	30.3	**65.2**	4.4	19.0	**75.4**	5.6
	400	0.2	**75.2**	24.6	0.9	**95.6**	3.5	0.7	**76.0**	23.3	0.1	**74.6**	25.3	10.4	**86.9**	2.8	1.6	**95.0**	3.5
0.3	50	27.8	**36.1**	36.1	72.6	**22.4**	5.0	10.1	**24.6**	65.3	31.9	**36.9**	31.1	36.4	**34.9**	28.7	58.4	**29.1**	12.5
	100	13.9	**51.4**	34.7	70.4	**25.0**	4.7	8.6	**52.6**	38.8	20.6	**45.8**	33.7	50.1	**42.8**	7.1	52.4	**40.7**	6.9
	200	0.8	**69.5**	29.8	22.9	**71.3**	5.9	1.9	**67.4**	30.8	0.9	**67.2**	31.9	20.7	**74.4**	4.9	11.8	**82.5**	5.7
	400	0.1	**77.0**	22.9	0.6	**96.1**	3.3	0.2	**77.2**	22.5	0.1	**76.1**	23.8	4.8	**92.4**	2.8	1.0	**95.9**	3.2
0.5	50	8.5	**35.2**	56.4	47.7	**39.2**	13.1	3.5	**19.4**	77.1	10.3	**36.5**	53.2	24.2	**36.8**	39.0	38.5	**40.5**	21.0
	100	2.7	**56.9**	40.4	27.4	**63.8**	8.8	3.5	**53.1**	43.5	3.0	**56.0**	41.0	29.5	**61.1**	9.5	24.8	**65.6**	9.6
	200	0.2	**69.5**	30.4	2.3	**91.5**	6.2	0.8	**67.4**	31.8	0.1	**69.3**	30.6	10.7	**83.6**	5.7	2.8	**91.0**	6.2
	400	0.0	**77.2**	22.8	0.2	**96.7**	3.2	0.1	**77.2**	22.7	0.0	**76.6**	23.4	2.3	**94.8**	2.9	0.3	**96.5**	3.1
0.9	50	0.1	**9.8**	90.2	3.2	**48.6**	48.2	0.0	**3.1**	96.9	0.1	**11.9**	88.0	1.5	**19.7**	78.8	2.5	**40.5**	57.0
	100	0.0	**35.0**	65.0	0.3	**78.7**	21.0	0.1	**29.9**	70.0	0.0	**35.3**	64.6	3.9	**71.0**	25.2	0.4	**77.7**	21.9
	200	0.0	**58.8**	41.2	0.0	**91.0**	9.0	0.1	**56.0**	43.9	0.0	**58.7**	41.3	2.0	**88.4**	9.5	0.1	**90.9**	9.0
	400	0.0	**74.6**	25.4	0.0	**96.3**	3.7	0.1	**74.5**	25.5	0.0	**74.3**	25.8	0.3	**96.2**	3.5	0.0	**96.3**	3.7

		HQC$(1,r)$			BIC$(1,r)$			$Q^*_{r,\hat{k}_{\text{AIC}},T}$			$Q^*_{r,\hat{k}_{\text{HQC}},T}$			$Q^*_{r,\hat{k}_{\text{BIC}},T}$		
0.0	50	30.4	**40.5**	29.1	75.6	**20.3**	4.2	81.8	**15.2**	3.0	85.7	**11.3**	3.0	89.0	**8.8**	2.3
	100	13.9	**55.2**	31.0	63.1	**32.3**	4.7	62.4	**31.9**	5.7	65.0	**29.7**	5.3	66.5	**28.5**	5.0
	200	1.0	**68.7**	30.3	18.1	**76.3**	5.6	25.0	**67.7**	7.3	18.8	**74.3**	6.8	19.0	**74.6**	6.5
	400	0.0	**74.6**	25.4	0.7	**95.7**	3.6	6.1	**87.0**	6.9	1.9	**92.0**	6.1	1.5	**92.7**	5.8
0.3	50	34.9	**37.8**	27.4	73.4	**22.1**	4.5	79.7	**17.2**	3.2	79.6	**16.9**	3.5	84.5	**12.9**	2.5
	100	26.9	**43.0**	30.1	76.9	**19.1**	4.0	55.7	**37.9**	6.4	57.1	**36.3**	6.5	66.6	**27.6**	5.8
	200	2.7	**63.3**	34.0	45.5	**47.7**	6.8	17.9	**74.7**	7.4	12.6	**80.5**	6.9	18.4	**73.3**	8.3
	400	0.1	**69.6**	30.3	2.3	**91.1**	6.6	3.1	**90.1**	6.8	1.0	**93.1**	5.9	0.7	**93.3**	5.9
0.5	50	20.4	**39.4**	40.3	53.8	**34.6**	11.6	77.4	**19.3**	3.3	72.0	**23.6**	4.4	69.2	**25.7**	5.1
	100	21.7	**37.0**	41.3	64.7	**27.8**	7.5	40.5	**52.6**	6.9	37.3	**56.2**	6.5	37.4	**56.0**	6.6
	200	3.3	**43.9**	52.8	53.8	**32.7**	13.5	10.6	**82.2**	7.1	4.7	**89.2**	6.2	4.3	**89.9**	5.8
	400	0.1	**48.3**	51.6	3.9	**78.1**	18.0	1.7	**91.7**	6.6	0.5	**93.6**	5.9	0.3	**93.9**	5.8
0.9	50	0.1	**8.2**	91.7	0.9	**17.7**	81.3	60.9	**32.9**	6.2	28.3	**61.7**	10.0	26.2	**64.8**	8.9
	100	0.3	**9.7**	90.0	1.6	**23.6**	74.7	12.3	**77.8**	9.9	2.8	**88.9**	8.2	2.3	**90.3**	7.4
	200	0.1	**1.1**	98.8	2.7	**25.5**	71.8	3.5	**88.9**	7.6	0.3	**93.4**	6.2	0.1	**93.9**	6.0
	400	0.0	**0.6**	99.4	1.4	**2.3**	96.3	0.5	**93.3**	6.1	0.1	**94.1**	5.8	0.0	**94.2**	5.7

IC(k,r) denotes the joint IC-based procedure for determining k and r; IC$(\hat{k}_{\text{IC}},r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained in the first step; IC$(1,r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k=1$; $Q^*_{r,\hat{k}_{\text{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained by an IC-based procedure in the first step.

TABLE 4.3: Determination of cointegration rank. VAR(2) model with rank $r_0 = 1$, Single Volatility Break [Case C]

		HQC(k, r)			BIC(k, r)			HQC$(\hat{k}_{\text{AIC}}, r)$			HQC$(\hat{k}_{\text{BIC}}, r)$			BIC$(\hat{k}_{\text{AIC}}, r)$			BIC$(\hat{k}_{\text{HQC}}, r)$		
γ	T	$r = 0$	1	2,3,4	$r = 0$	1	2,3,4	$r = 0$	1	2,3,4	$r = 0$	1	2,3,4	$r = 0$	1	2,3,4	$r = 0$	1	2,3,4
0.0	50	6.9	**28.6**	64.5	60.6	**31.9**	7.5	1.4	**14.4**	84.2	12.6	**40.8**	46.7	19.0	**43.1**	37.9	28.2	**39.4**	32.5
	100	4.8	**54.1**	41.1	46.8	**48.9**	4.4	8.3	**39.0**	52.7	4.3	**55.4**	40.2	53.8	**38.6**	7.7	44.2	**49.6**	6.2
	200	0.1	**65.9**	34.1	6.7	**90.1**	3.3	3.4	**56.9**	39.7	0.0	**66.1**	33.9	41.3	**55.1**	3.7	7.4	**89.2**	3.4
	400	0.0	**73.1**	26.9	0.0	**98.5**	1.5	0.1	**68.4**	31.5	0.0	**73.2**	26.8	9.4	**88.8**	1.8	0.0	**98.5**	1.5
0.3	50	4.8	**22.0**	73.2	51.6	**36.4**	12.0	0.9	**11.4**	87.8	11.3	**35.1**	53.6	14.2	**41.2**	44.7	17.6	**40.8**	41.6
	100	3.7	**48.7**	47.6	49.1	**43.7**	7.2	5.8	**36.6**	57.6	5.9	**46.2**	47.9	46.3	**43.4**	10.3	31.1	**58.4**	10.6
	200	0.1	**63.7**	36.3	9.7	**85.2**	5.1	1.8	**56.6**	41.6	0.1	**60.1**	39.8	29.5	**66.1**	4.5	4.1	**91.6**	4.2
	400	0.0	**72.0**	28.0	0.0	**98.3**	1.7	0.0	**67.8**	32.2	0.0	**71.8**	28.2	3.8	**94.2**	2.1	0.0	**98.3**	1.7
0.5	50	1.5	**16.0**	82.4	27.7	**50.0**	22.3	0.3	**8.3**	91.4	3.2	**26.3**	70.5	9.8	**37.3**	53.0	10.9	**38.1**	51.0
	100	0.9	**47.3**	51.9	15.5	**74.4**	10.1	3.9	**34.9**	61.1	0.5	**48.7**	50.8	37.1	**49.7**	13.2	14.8	**72.1**	13.2
	200	0.0	**61.7**	38.3	0.3	**95.1**	4.6	0.8	**54.7**	44.4	0.0	**61.9**	38.1	19.8	**74.6**	5.6	0.6	**94.3**	5.1
	400	0.0	**71.1**	28.9	0.0	**98.3**	1.7	0.0	**67.0**	33.0	0.0	**71.2**	28.8	1.4	**96.4**	2.3	0.0	**98.3**	1.7
0.9	50	0.0	**2.6**	97.4	2.8	**43.7**	53.5	0.0	**0.9**	99.1	0.1	**8.7**	91.2	1.5	**13.3**	85.2	0.8	**15.9**	83.3
	100	0.0	**27.8**	72.2	0.1	**77.8**	22.1	0.3	**17.7**	82.1	0.0	**30.2**	69.8	13.3	**55.5**	31.2	0.6	**71.2**	28.3
	200	0.0	**49.2**	50.8	0.0	**91.6**	8.4	0.0	**41.7**	58.3	0.0	**49.9**	50.1	3.1	**85.3**	11.6	0.0	**90.8**	9.2
	400	0.0	**63.2**	36.8	0.0	**97.1**	2.9	0.0	**58.4**	41.6	0.0	**63.3**	36.7	0.0	**96.2**	3.8	0.0	**97.0**	3.0

		HQC$(1, r)$			BIC$(1, r)$			$Q^*_{r,\hat{k}_{\text{AIC}},T}$			$Q^*_{r,\hat{k}_{\text{HQC}},T}$			$Q^*_{r,\hat{k}_{\text{BIC}},T}$		
0.0	50	13.3	**42.0**	44.7	61.1	**31.6**	7.3	81.4	**16.3**	2.3	66.8	**27.4**	5.8	84.4	**12.4**	3.2
	100	4.4	**55.5**	40.1	46.7	**49.0**	4.3	58.7	**33.3**	8.0	59.2	**34.2**	6.6	64.9	**30.7**	4.4
	200	0.0	**66.2**	33.8	6.6	**90.1**	3.3	34.2	**56.8**	9.0	11.6	**82.2**	6.2	11.6	**82.6**	5.8
	400	0.0	**73.2**	26.8	0.0	**98.5**	1.5	4.9	**87.6**	7.5	0.0	**94.4**	5.6	0.0	**94.6**	5.5
0.3	50	15.2	**39.1**	45.8	54.6	**34.8**	10.6	82.1	**15.7**	2.2	62.9	**30.9**	6.2	70.3	**24.2**	5.4
	100	8.9	**46.5**	44.7	56.7	**35.8**	7.5	54.6	**37.0**	8.4	45.8	**45.3**	8.9	54.3	**38.5**	7.2
	200	0.2	**53.5**	46.3	25.6	**63.5**	11.0	26.0	**64.4**	9.6	7.1	**85.8**	7.1	10.6	**80.4**	9.0
	400	0.0	**56.2**	43.8	0.0	**89.5**	10.5	2.2	**89.9**	7.9	0.0	**94.2**	5.8	0.0	**94.3**	5.7
0.5	50	8.5	**34.0**	57.4	32.8	**44.8**	22.5	82.2	**15.5**	2.3	62.2	**31.2**	6.6	53.9	**36.8**	9.4
	100	6.5	**35.4**	58.1	38.5	**43.9**	17.6	49.1	**41.8**	9.2	31.1	**59.6**	9.3	33.8	**58.7**	7.4
	200	0.4	**33.1**	66.5	26.5	**48.7**	24.8	17.6	**72.5**	9.9	1.2	**91.7**	7.1	1.0	**92.6**	6.4
	400	0.0	**32.2**	67.8	0.2	**68.2**	31.6	0.8	**91.1**	8.1	0.0	**94.1**	5.9	0.0	**94.2**	5.8
0.9	50	0.1	**6.9**	93.0	0.6	**17.2**	82.2	77.1	**19.5**	3.4	51.9	**39.2**	8.9	26.2	**61.1**	12.8
	100	0.1	**7.6**	92.3	1.0	**21.4**	77.6	37.0	**51.7**	11.3	2.9	**85.2**	11.8	0.8	**90.8**	8.4
	200	0.1	**1.1**	98.8	1.2	**19.4**	79.4	5.8	**83.5**	10.8	0.0	**92.1**	7.9	0.0	**93.0**	7.0
	400	0.0	**0.4**	99.6	0.5	**1.9**	97.6	0.0	**92.2**	7.8	0.0	**94.1**	5.9	0.0	**94.3**	5.7

IC(k, r) denotes the joint IC-based procedure for determining k and r; IC$(\hat{k}_{\text{IC}}, r)$ denotes the sequential IC-based procedure for determining k in the first step and r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained in the first step; IC$(1, r)$ denotes the semi-parametric IC-based approach by Cheng and Phillips (2009, 2012) for determining r by fixing $k = 1$; $Q^*_{r,\hat{k}_{\text{IC}},T}$ denotes the sequential wild bootstrap PLR test-based procedure for determining r in the second step based on the lag order $\hat{k}_{\text{IC}}(p)$ obtained by an IC-based procedure in the first step.