Supplement to: Adaptive Long Memory Testing under Heteroskedasticity

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This supplement provides additional results which are required for the proofs given in Appendix A of the paper "Adaptive Long Memory Testing under Heteroskedasticity". Throughout, let C denote a generic positive constant.

Lemma A. Let $\bar{\sigma}_t^2 = \sum_{i=1}^T w_{ti}\sigma_i^2$ and $\tilde{\sigma}_t^2 = \sum_{i=1}^T w_{ti}e_i^2$. Under Assumptions R, S, E and B,

(a) $\hat{\psi} \xrightarrow{p} \psi_0$;

(b) $\sqrt{T}\left(\hat{\psi} - \psi_0\right) \rightsquigarrow N\left(0, \nu^{-2}V_{\psi\psi}^{-1}\right)$;

(c) let $t = [Tr]$, for any fixed $r \in (0,1]$, $\frac{1}{Tb}\sum_{i=1}^T K_{ti} \to \int_{-\infty}^{\infty} K(z)\,dz = 1$;

(d) $\max_{t,i} w_{ti} = O(1/Tb)$;

(e) $\min_{1\leq t\leq T} \bar{\sigma}_t^2 \geq C > 0$;

(f) $\max_{1\leq t\leq T} E\left|\tilde{\sigma}_t^2 - \bar{\sigma}_t^2\right|^4 = O\left(1/(Tb)^2\right)$;

(g) $\max_t \left|\tilde{\sigma}_t^2 - \bar{\sigma}_t^2\right| = O_p\left(T^{-1/4}b^{-1/2}\right)$;

(h) $\left(\min_{1\leq t\leq T}\tilde{\sigma}_t^2\right)^{-1} = O_p(1)$;

(i) $\max_{1\leq t\leq T}\left|\hat{\sigma}_t^2 - \tilde{\sigma}_t^2\right| = O_p\left(1/\sqrt{T}b\right)$;

(j) $\left(\min_{1\leq t\leq T}\hat{\sigma}_t^2\right)^{-1} = O_p(1)$;

(k) $\sum_{t=1}^T\left(\hat{\sigma}_t^2 - \tilde{\sigma}_t^2\right)^2 = O_p\left(1/\sqrt{T}\right)$;

(l) $T^{-1}\sum_{t=1}^T\left|\bar{\sigma}_t^2 - \sigma_t^2\right| = o(1)$.

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Lemma B. Under Assumptions R, S, E and B,
(a)

$$T^{-1/2}\sum_{t=1}^{T}\left(\hat{s}_t\left(0,\psi_0\right)-s_t\left(0,\psi_0\right)\right)\xrightarrow{p}0,$$

where $\hat{s}_t\left(0,\psi_0\right)$ and $s_t\left(0,\psi_0\right)$ represent, respectively, the score vectors $\hat{s}_t\left(\gamma\right)$ and $s_t\left(\gamma\right)$ but evaluated at $\theta=0$ and $\psi=\psi_0$;
(b)

$$\sup_{\psi\in\Psi}T^{-1}\sum_{t=1}^{T}\left(\hat{h}_t\left(0,\psi\right)-h_t\left(0,\psi\right)\right)\xrightarrow{p}0,$$

where $\hat{h}_t\left(0,\psi\right)$ and $h_t\left(0,\psi\right)$ represent, respectively, the matrix of second derivatives of $\hat{l}_t\left(\gamma\right)$ and $l_t\left(\gamma\right)$ but evaluated at $\theta=0$;
(c) Let $\bar{\psi}=\arg\max_{\psi\in\Psi,\theta=0}L\left(\gamma\right).$ Then

$$\bar{\psi}\xrightarrow{p}\psi_0;$$

(d)

$$\tilde{\psi}\xrightarrow{p}\psi_0.$$

Proof of Lemma A

Part (a) The proof is given in Cavaliere, Nielsen and Taylor (2015a, hereafter CNT) Lemma A.1.

Part (b) This result follows from Cavaliere, Nielsen and Taylor (2015b) Lemma S.10, CNT Lemma A.2 and the standard Mean Value Theorem.

Parts (c) to (h) follows directly from Xu and Phillips (2008) Lemma A(c) to Lemma A(h).

Part (i) For $\hat{\psi},\psi_0\in\Psi$ and for some ψ^* in between $\hat{\psi}$ and ψ_0,

$$\begin{aligned}\hat{e}_i &= a\left(L;\hat{\psi}\right)y_i\\ &= a\left(L;\psi_0\right)y_i+\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right)\\ &= e_i+\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right)\end{aligned}$$

and so

$$\hat{e}_i^2=e_i^2+2e_i\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right)+\left(\hat{\psi}-\psi_0\right)'\left(a_\psi\left(L;\psi^*\right)y_i\right)\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right).$$

Denote $\hat{\psi}=\left(\hat{\psi}_1,\ldots,\hat{\psi}_{p+q}\right)'$, $\psi_0=\left(\psi_{0,1},\ldots,\psi_{0,p+q}\right)'$ and $\psi^*=\left(\psi_1^*,\ldots\psi_{p+q}^*\right)'$. We have $a_\psi\left(L;\psi\right)y_i=\left(z_{1,i}\left(\psi\right),\ldots,z_{p+q,i}\left(\psi\right)\right)'$ where $z_{k,i}\left(\psi\right)=\sum_{j=1}^{i-1}\varphi_{k,j}\left(\psi\right)e_{i-j}$ and $\varphi_{k,j}\left(\psi\right)$ decays exponentially in j for all $\psi\in\Psi$ under Assumption R and for $k=1,\ldots,p+q$.

We write

$$\hat{\sigma}_t^2-\tilde{\sigma}_t^2=\sum_{i=1}^{T}w_{ti}\left(\hat{e}_i^2-e_i^2\right)=A_{1,t}+A_{2,t},\tag{1}$$

where

$$A_{1,t} = 2\sum_{i=1}^{T} w_{ti}e_i\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right) = 2\sum_{j=1}^{p+q}\left(\hat{\psi}_j-\psi_{0,j}\right)\sum_{i=1}^{T} w_{ti}e_i z_{j,i}\left(\psi^*\right)$$

and

$$\begin{aligned} A_{2,t} &= \sum_{i=1}^{T} w_{ti}\left(\hat{\psi}-\psi_0\right)'\left(a_\psi\left(L;\psi^*\right)y_i\right)\left(a_\psi\left(L;\psi^*\right)y_i\right)'\left(\hat{\psi}-\psi_0\right) \\ &= \sum_{k=1}^{p+q}\sum_{l=1}^{p+q}\left(\hat{\psi}_k-\psi_{0,k}\right)\left(\hat{\psi}_l-\psi_{0,l}\right)\sum_{i=1}^{T} w_{ti}z_{k,i}\left(\psi^*\right)z_{l,i}\left(\psi^*\right). \end{aligned}$$

Now the required results that

$$\max_{1\le t\le T}\left|\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right| \le \max_{1\le t\le T}\left|A_{1,t}\right| + \max_{1\le t\le T}\left|A_{2,t}\right| = O_p\left(\frac{1}{\sqrt{Tb}}\right)$$

will follow from

$$\max_{1\le t\le T}\left|A_{1,t}\right| = O_p\left(\frac{1}{\sqrt{Tb}}\right) \tag{2}$$

and

$$\max_{1\le t\le T}\left|A_{2,t}\right| = O_p\left(\frac{1}{Tb}\right), \tag{3}$$

as we will now show.

First note that

$$\sup_{\psi\in\Psi} T^{-1}\sum_{i=1}^{T}\left|z_{k,i}\left(\psi\right)z_{l,i}\left(\psi\right)\right| = O_p\left(1\right) \quad \text{for } k,l=1,\ldots,p+q \tag{4}$$

and

$$\sup_{\psi\in\Psi} T^{-1}\sum_{i=1}^{T}\left|e_i z_{j,i}\left(\psi\right)\right| = O_p\left(1\right) \quad \text{for } j=1,\ldots,p+q. \tag{5}$$

We will only show (4) since (5) follows as a special case. To show (4) we will follow precisely the same lines as the proof given in Cavaliere, Nielsen and Taylor (2015b) Lemma S.10 equation (S.16) with $u_1 = u_2 = 0$ in their $M_T\left(u_1, u_2, \psi\right)$ notation. Following them, we write

$$\begin{aligned} T^{-1}\sum_{i=1}^{T}\left|z_{k,i}\left(\psi\right)z_{l,i}\left(\psi\right)\right| &\le T^{-1}\sum_{i=1}^{T}\sum_{m=1}^{i-1}\sum_{n=1}^{i-1}\left|\varphi_{k,m}\left(\psi\right)\right|\left|\varphi_{l,n}\left(\psi\right)\right|\left|e_{i-m}e_{i-n}\right| \\ &= \sum_{m=1}^{T-1}\sum_{n=1}^{T-1}\left|\varphi_{k,m}\left(\psi\right)\right|\left|\varphi_{l,n}\left(\psi\right)\right| T^{-1}\sum_{i=\max(m,n)+1}^{T}\left|e_{i-m}e_{i-n}\right|. \end{aligned} \tag{6}$$

Since $T^{-1}\sum_{i=\max(m,n)+1}^{T}\left|e_{i-m}e_{i-n}\right| = O_p\left(1\right)$ uniformly in m,n because $E\left|e_{i-m}e_{i-n}\right| \le \sup_r \sigma\left(r\right)^2 < \infty$ it thus follows that (see the treatment of Cavaliere, Nielsen and Taylor (2015b) equation (S.19))

$$\sup_{\psi\in\Psi}(6) = O_p\left(\sup_{\psi\in\Psi}\sum_{m=1}^{T-1}\sum_{n=1}^{T-1}\left|\varphi_{k,m}\left(\psi\right)\right|\left|\varphi_{l,n}\left(\psi\right)\right|\right) = O_p\left(1\right),$$

because $\sum_{j=1}^{\infty}\left|\varphi_{k,j}\left(\psi\right)\right| < \infty$ uniformly in $\psi\in\Psi$ under Assumption R.

To show (2), for $j=1,\ldots,p+q$ and since p and q are known fixed orders, we have

$$
\begin{aligned}
\max_t \left| 2\left(\hat{\psi}_j-\psi_{0,j}\right)\sum_{i=1}^T w_{ti}e_iz_{j,i}\left(\psi^*\right)\right| &\leq \max_t 2\left|\hat{\psi}_j-\psi_{0,j}\right|\sum_{i=1}^T w_{ti}\left|e_iz_{j,i}\left(\psi^*\right)\right| \\
&\leq 2T\left|\hat{\psi}_j-\psi_{0,j}\right|\left(\max_{t,i} w_{ti}\right)\sup_{\psi\in\Psi}T^{-1}\sum_{i=1}^T\left|e_iz_{j,i}\left(\psi\right)\right| \\
&= O_p\left(T^{1/2}\right)O\left(\frac{1}{Tb}\right)O_p\left(1\right)=O_p\left(\frac{1}{\sqrt{T}b}\right),
\end{aligned}
$$

by Lemma A(b,d) and (5).

To show (3), for $k,l=1,\ldots,p+q$ and given that p and q are known fixed orders, we have

$$
\begin{aligned}
&\max_t\left|\hat{\psi}_k-\psi_{0,k}\right|\left|\hat{\psi}_l-\psi_{0,l}\right|\sum_{i=1}^T w_{ti}\left|z_{k,i}\left(\psi^*\right)z_{l,i}\left(\psi^*\right)\right| \\
\leq\;& T\left|\hat{\psi}_k-\psi_{0,k}\right|\left|\hat{\psi}_l-\psi_{0,l}\right|\left(\max_{t,i}w_{ti}\right)\sup_{\psi\in\Psi}T^{-1}\sum_{i=1}^T\left|z_{k,i}\left(\psi\right)z_{l,i}\left(\psi\right)\right| \\
=\;& O_p\left(\frac{1}{Tb}\right),
\end{aligned}
$$

by (4). This concludes the proof for this part (i).

Part (j) This follows from Xu and Phillips (2008) Lemma A(j) and our Lemma A(i) above.

Part (k) Recall equation (1), we have

$$
\sum_{t=1}^T\left(\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right)^2=\sum_{t=1}^T\left(A_{1,t}+A_{2,t}\right)^2=\sum_{t=1}^T A_{1,t}^2+\sum_{t=1}^T A_{2,t}^2+2\sum_{t=1}^T A_{1,t}A_{2,t}
$$

and so the required result that $\sum_{t=1}^T\left(\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right)^2=O_p\left(1/\sqrt{T}\right)$ will follow if we show that:

$$
\begin{aligned}
\left|\sum_{t=1}^T A_{1,t}^2\right| &= O_p\left(\frac{1}{\sqrt{T}}\right); &(7)\\
\left|\sum_{t=1}^T A_{2,t}^2\right| &= O_p\left(\frac{1}{T}\right); &(8)\\
\left|\sum_{t=1}^T A_{1,t}A_{2,t}\right| &= O_p\left(\frac{1}{T^{3/4}}\right). &(9)
\end{aligned}
$$

To show (7), we write

$$
\begin{aligned}
\sum_{t=1}^T A_{1,t}^2 &= 4\sum_{t=1}^T\left(\sum_{k=1}^{p+q}\left(\hat{\psi}_k-\psi_{0,k}\right)\sum_{i=1}^T w_{ti}e_iz_{k,i}\left(\psi^*\right)\right)^2 \\
&\leq 4\sum_{t=1}^T\left(\sum_{k=1}^{p+q}\left(\hat{\psi}_k-\psi_{0,k}\right)^2\right)\sum_{k=1}^{p+q}\left(\sum_{i=1}^T w_{ti}e_iz_{k,i}\left(\psi^*\right)\right)^2 \\
&= 4\left(\sum_{k=1}^{p+q}T\left(\hat{\psi}_k-\psi_{0,k}\right)^2\right)\sum_{k=1}^{p+q}T^{-1}\sum_{t=1}^T\left(\sum_{i=1}^T w_{ti}e_iz_{k,i}\left(\psi^*\right)\right)^2.
\end{aligned}
$$

Since p and q are known fixed orders and $\left(\hat{\psi}_k - \psi_{0,k}\right) = O_p\left(T^{-1/2}\right)$ by Lemma A(b), equation (7) follows if, for $k = 1, ..., p+q$, we show that

$$\left| T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} e_i z_{k,i}\left(\psi^*\right) \right)^2 - T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} e_i z_{k,i}\left(\psi_0\right) \right)^2 \right| = O_p\left(\frac{1}{\sqrt{T}}\right) \tag{10}$$

and

$$T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} e_i z_{k,i}\left(\psi_0\right) \right)^2 = O_p\left(\frac{1}{Tb}\right). \tag{11}$$

To show (10), for some $\breve{\psi}$ in between $\hat{\psi}$ and ψ_0,

$$\begin{aligned} & \left| T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} e_i z_{k,i}\left(\psi^*\right) \right)^2 - T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} e_i z_{k,i}\left(\psi_0\right) \right)^2 \right| \\ \leq & \sum_{l=1}^{p+q} \left| T^{-1} \sum_{t=1}^{T} \sum_{i=1}^{T} \sum_{j=1}^{T} w_{ti} w_{tj} e_i e_j \frac{\partial}{\partial \psi_l} \left(z_{k,i}\left(\breve{\psi}\right) z_{k,j}\left(\breve{\psi}\right) \right) \right| \left|\psi_l^* - \psi_{0,l}\right| \\ \leq & \sum_{l=1}^{p+q} \sup_{\psi \in \Psi} \left| T^{-1} \sum_{t=1}^{T} \sum_{i=1}^{T} \sum_{j=1}^{T} w_{ti} w_{tj} e_i e_j \frac{\partial}{\partial \psi_l} \left(z_{k,i}\left(\psi\right) z_{k,j}\left(\psi\right) \right) \right| \left|\psi_l^* - \psi_{0,l}\right|. \end{aligned}$$

Recall that ψ^* lies in between $\hat{\psi}$ and ψ_0 (see part (i)) and Lemma A(b) implies that $\left|\psi_l^* - \psi_{0,l}\right| = O_p\left(T^{-1/2}\right)$ then (10) follows if, for $m, n = 0, 1$, we show that

$$\sup_{\psi \in \Psi} \left| T^{-1} \sum_{t=1}^{T} \sum_{i=1}^{T} \sum_{j=1}^{T} w_{ti} w_{tj} \left(e_i \frac{\partial^m}{\partial \psi_l^m} z_{k,i}\left(\psi\right) \right) \left(e_j \frac{\partial^n}{\partial \psi_l^n} z_{k,j}\left(\psi\right) \right) \right| = O_p\left(1\right). \tag{12}$$

We show only when $m = n = 0$ since the remaining cases may be shown in a similar fashion given that the coefficients of the linear processes resulting from the derivative wrt ψ still decay exponentially under Assumption R. To show (12), it suffices to show that, for $k, l = 1, \ldots, p+q$,

$$\sup_{\psi \in \Psi} T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} z_{k,i}\left(\psi\right) z_{l,i}\left(\psi\right) \right)^2 = O_p\left(1\right), \tag{13}$$

since (12) is a special case. To show (13), we will follow the same lines as those in the proof of Cavaliere, Nielsen and Taylor (2015b) Lemma S.10 equation (S.16). Following them, we write

$$T^{-1} \sum_{t=1}^{T} \left(\sum_{i=1}^{T} w_{ti} z_{k,i}\left(\psi\right) z_{l,i}\left(\psi\right) \right)^2 \tag{14}$$

$$\leq \sum_{m=1}^{T-1} \sum_{n=1}^{T-1} \sum_{a=1}^{T-1} \sum_{b=1}^{T-1} \left|\varphi_{k,m}\left(\psi\right)\right| \left|\varphi_{l,n}\left(\psi\right)\right| \left|\varphi_{k,a}\left(\psi\right)\right| \left|\varphi_{l,b}\left(\psi\right)\right| \times \tag{15}$$

$$T^{-1} \sum_{t=1}^{T} \sum_{i=\max(m,n)+1}^{T} \sum_{j=\max(a,b)+1}^{T} w_{ti} w_{tj} \left|e_{i-m} e_{i-n} e_{j-a} e_{j-b}\right|. \tag{16}$$

In (16), $T^{-1}\sum_{t=1}^{T}\sum_{i=\max(m,n)+1}^{T}\sum_{j=\max(a,b)+1}^{T} w_{ti}w_{tj}\left|e_{i-m}e_{i-n}e_{j-a}e_{j-b}\right| = O_p(1)$ uniformly in m,n,a,b because

$$
\begin{aligned}
& E\left|T^{-1}\sum_{t=1}^{T}\sum_{i=\max(m,n)+1}^{T}\sum_{j=\max(a,b)+1}^{T} w_{ti}w_{tj}e_{i-m}e_{i-n}e_{j-a}e_{j-b}\right| \\
\leq\; & T^{-1}\sum_{t=1}^{T}\sum_{i=\max(m,n)+1}^{T}\sum_{j=\max(a,b)+1}^{T} w_{ti}w_{tj}E\left|e_{i-m}e_{i-n}e_{j-a}e_{j-b}\right| \\
\leq\; & T^{-1}\sum_{t=1}^{T}\sum_{i=\max(m,n)+1}^{T}\sum_{j=\max(a,b)+1}^{T} w_{ti}w_{tj}\left(E\left(e_{i-m}^4\right)E\left(e_{i-n}^4\right)E\left(e_{j-a}^4\right)E\left(e_{j-b}^4\right)\right)^{1/4} \\
\leq\; & \tau_{0,0}\left(\sup_r \sigma(r)^2\right)^2 T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{T} w_{ti}\sum_{j=1}^{T} w_{tj} = \tau_{0,0}\left(\sup_r \sigma(r)^2\right)^2 < \infty.
\end{aligned}
$$

In (15), since $\sum_{m=1}^{\infty}\left|\varphi_{k,m}(\psi)\right| < \infty$ uniformly for all $\psi \in \Psi$ under assumption R, it thus follows that (see the treatment of Cavaliere, Nielsen and Taylor (2015b) equation (S.19))

$$
\sup_{\psi\in\Psi}(14) = O_p\left(\left(\sum_{m=1}^{T-1}\left|\varphi_{k,m}(\psi)\right|\right)^2\left(\sum_{n=1}^{T-1}\left|\varphi_{l,n}(\psi)\right|\right)^2\right) = O_p(1).
$$

To show (11), we write

$$
\begin{aligned}
& T^{-1}\sum_{t=1}^{T} E\left(\sum_{i=1}^{T} w_{ti}e_i z_{k,i}(\psi_0)\right)^2 \\
=\; & T^{-1}\sum_{t=1}^{T} E\left(\sum_{i=2}^{T} w_{ti}e_i \sum_{m=1}^{i-1}\varphi_{k,m}(\psi_0)e_{i-m}\right)^2 \\
\leq\; & T^{-1}\sum_{t=1}^{T}\sum_{i=2}^{T}\sum_{j=2}^{T} w_{ti}w_{tj}\sum_{m=1}^{i-1}\left|\varphi_{k,m}(\psi_0)\right|\sum_{n=1}^{j-1}\left|\varphi_{k,n}(\psi_0)\right|\left|E\left(e_{i-m}e_{j-n}e_i e_j\right)\right| \\
=\; & T^{-1}\sum_{t=1}^{T}\sum_{i=2}^{T} w_{ti}^2\sum_{m=1}^{i-1}\left|\varphi_{k,m}(\psi_0)\right|\sum_{n=1}^{i-1}\left|\varphi_{l,n}(\psi_0)\right|\left|E\left(e_{i-m}e_{i-n}E\left(e_i^2|\mathcal{F}_{i-1}\right)\right)\right| \\
\leq\; & \left(\sup_r \sigma(r)^2\right) T^{-1}\sum_{t=1}^{T}\sum_{i=2}^{T} w_{ti}^2\sum_{m=1}^{i-1}\left|\varphi_{k,m}(\psi_0)\right|\sum_{n=1}^{i-1}\left|\varphi_{l,n}(\psi_0)\right|\left|E\left(e_{i-m}e_{i-n}\right)\right| \\
=\; & \left(\sup_r \sigma(r)^2\right) T^{-1}\sum_{t=1}^{T}\sum_{i=2}^{T} w_{ti}^2\sum_{m=1}^{i-1}\varphi_{k,m}(\psi_0)^2 E\left(e_{i-m}^2\right) \\
\leq\; & \left(\sup_r \sigma(r)^2\right)^2\left(\max_{t,i} w_{ti}\right)\sum_{m=1}^{\infty}\varphi_{k,m}(\psi_0)^2 T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{T} w_{ti}\right) \\
\leq\; & \left(\sup_r \sigma(r)^2\right)^2\left(\max_{t,i} w_{ti}\right)\sum_{m=1}^{\infty}\varphi_{k,m}(\psi_0)^2 = O\left(\frac{1}{Tb}\right).
\end{aligned}
$$

To show (8), for $k, l = 1, \ldots, p+q$ and since p and q are known fixed orders, we write

$$\begin{aligned}
& \left(\hat{\psi}_k - \psi_{0,k}\right)^2 T\left(\hat{\psi}_l - \psi_{0,l}\right)^2 T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{T} w_{ti} z_{k,i}\left(\psi^*\right) z_{l,i}\left(\psi^*\right)\right)^2 \\
\leq \; & \left(\hat{\psi}_k - \psi_{0,k}\right)^2 T\left(\hat{\psi}_l - \psi_{0,l}\right)^2 \sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{T} w_{ti} z_{k,i}\left(\psi\right) z_{l,i}\left(\psi\right)\right)^2 \\
= \; & O_p\left(T^{-1}\right)
\end{aligned}$$

by Lemma A(b) and equation (13).

Finally (9) follows immediately from (7), (8) and Cauchy-Schwarz inequality.

Part (l) See Xu and Phillips (2008) Lemma A(l).

Proof of Lemma B

Part (a) We have

$$T^{-1/2}\sum_{t=1}^{T}\left(\hat{s}_{\theta,t}\left(0,\psi_0\right) - s_{\theta,t}\left(0,\psi_0\right)\right) = T^{-1/2}\sum_{t=1}^{T}\left(\hat{\sigma}_t^{-2} - \sigma_t^{-2}\right)\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t, \tag{17}$$

and

$$T^{-1/2}\sum_{t=1}^{T}\left(\hat{s}_{\psi,t}\left(0,\psi_0\right) - s_{\psi,t}\left(0,\psi_0\right)\right) = T^{-1/2}\sum_{t=1}^{T}\left(\hat{\sigma}_t^{-2} - \sigma_t^{-2}\right)\sum_{j=1}^{t-1}b_j\left(\psi_0\right)e_{t-j}e_t, \tag{18}$$

where $b_j\left(\psi_0\right)$ decays exponentially under Assumption R. We will only show (17) $\xrightarrow{p} 0$ since (18) $\xrightarrow{p} 0$ follows immediately from Xu and Phillips (2008). Let $\sigma = \left(\sigma_1, \sigma_2, \ldots, \sigma_T\right)'$ and define

$$B\left(\sigma\right) = T^{-1/2}\sum_{t=1}^{T}\sigma_t^{-2}\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t.$$

Following from Robinson (1987) and Xu and Phillips (2008), (17) $\xrightarrow{p} 0$ follows if we show the following three results:

$$B\left(\hat{\sigma}\right) - B\left(\tilde{\sigma}\right) \xrightarrow{p} 0, \tag{19}$$

$$B\left(\tilde{\sigma}\right) - B\left(\bar{\sigma}\right) \xrightarrow{p} 0, \tag{20}$$

$$B\left(\bar{\sigma}\right) - B\left(\sigma\right) \xrightarrow{p} 0. \tag{21}$$

Recall that

$$\tilde{\sigma}_t^2 = \sum_{i=1}^{T} w_{ti}e_i^2 \quad \text{and} \quad \bar{\sigma}_t^2 = \sum_{i=1}^{T} w_{ti}\sigma_i^2.$$

To show (19), first note that

$$T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right)^2 = O_p\left(1\right) \tag{22}$$

because

$$\begin{aligned}
E\left(\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right)^2 &= \sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j}E\left(e_{t-i}e_{t-j}E\left(e_t^2|\mathcal{F}_{t-1}\right)\right) \\
&\leq \sup_r \sigma\left(r\right)^2\sum_{i=1}^{t-1}\frac{1}{i^2}E\left(e_{t-i}^2\right) \leq \left(\sup_r \sigma\left(r\right)^2\right)^2\frac{\pi^2}{6} < \infty \quad \text{uniformly in } t.
\end{aligned}$$

Then,

$$
\begin{aligned}
\left|B(\hat{\sigma})-B(\tilde{\sigma})\right| &= \left|T^{-1/2}\sum_{t=1}^{T}\left(\hat{\sigma}_t^{-2}-\tilde{\sigma}_t^{-2}\right)\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right| \\
&= \left|T^{-1/2}\sum_{t=1}^{T}\frac{\left(\hat{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)}{\hat{\sigma}_t^{2}\tilde{\sigma}_t^{2}}\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right| \\
&\leq \left(\frac{1}{\min\hat{\sigma}_t^{2}}\right)\left(\frac{1}{\min\tilde{\sigma}_t^{2}}\right)\left(\sum_{t=1}^{T}\left(\hat{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right)^2\right)^{1/2} \\
&= O_p\left(\frac{1}{T^{1/2}}\right)=o_p(1),
\end{aligned}
$$

where the second last equality uses Lemma A(h,j,k) and (22).

To show (20), we have

$$
\begin{aligned}
& T^{-1/2}\sum_{t=1}^{T}\left(\tilde{\sigma}_t^{-2}-\bar{\sigma}_t^{-2}\right)\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t \\
= \; & T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)\bar{\sigma}_t^{-4}\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t+T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)^2\tilde{\sigma}_t^{-2}\bar{\sigma}_t^{-4}\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t, \qquad (23)
\end{aligned}
$$

where the last equality uses

$$
p^{-1}-q^{-1}=(q-p)\,q^{-2}+(q-p)^2\,p^{-1}q^{-2} \qquad (24)
$$

for any non-zero real numbers p and q (see Xu and Phillips (2008) equation (21) page 276).

We now show that the first term in (23) $\xrightarrow{p} 0$. To see this, let

$$
x_{t-1}=\sum_{i=1}^{t-1}i^{-1}e_{t-i}.
$$

Then $E\left(\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)\bar{\sigma}_t^{-4}x_{t-1}e_t\right)=0$ follows the same arguments as in Xu and Phillips (2008) equation (22). Now

$$
\begin{aligned}
& E\left(T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)\bar{\sigma}_t^{-4}x_{t-1}e_t\right)^2 \\
= \; & T^{-1}\sum_{t=1}^{T}E\left(\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)^2\bar{\sigma}_t^{-8}x_{t-1}^2e_t^2\right) \\
& +T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}E\left(\bar{\sigma}_{t-s}^{2}-\tilde{\sigma}_{t-s}^{2}\right)\left(\bar{\sigma}_t^{2}-\tilde{\sigma}_t^{2}\right)\bar{\sigma}_{t-s}^{-4}\bar{\sigma}_t^{-4}x_{t-s-1}x_{t-1}e_{t-s}e_t. \qquad (25)
\end{aligned}
$$

By Lemma A(e,f), the first term $\to 0$ since

$$
\begin{aligned}
& T^{-1}\sum_{t=1}^{T} E\left(\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^2 \bar{\sigma}_t^{-8} x_{t-1}^2 e_t^2\right) \\
= \; & T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j} E\left(\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^2 \bar{\sigma}_t^{-8} e_{t-i}e_{t-j}e_t^2\right) \\
\leq \; & \left(\frac{1}{\min_t \bar{\sigma}_t^2}\right)^4 T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j}\left(E\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^4 E\left(e_{t-i}^2 e_{t-j}^2 e_t^4\right)\right)^{1/2} \\
\leq \; & \left(\frac{1}{\min_t \bar{\sigma}_t^2}\right)^4 \left(\max_t E\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^4\right)^{1/2} T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j}\left(E\left(e_{t-i}^4 e_{t-j}^4\right)E\left(e_t^8\right)\right)^{1/4} \\
\leq \; & \left(\frac{1}{\min_t \bar{\sigma}_t^2}\right)^4 \left(\max_t E\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^4\right)^{1/2} T^{-1}\sum_{t=1}^{T}\sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j}\left(E\left(e_{t-i}^8\right)E\left(e_{t-j}^8\right)\right)^{1/8} E\left(e_t^8\right)^{1/4} \\
\leq \; & \left(\frac{1}{\min_t \bar{\sigma}_t^2}\right)^4 \left(\max_t E\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)^4\right)^{1/2} T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{t-1}\frac{1}{i}\right)^2 \\
= \; & O\left(\frac{(\log T)^2}{Tb}\right).
\end{aligned}
$$

As for (25), we have, by Lemma A(e),

$$
\begin{aligned}
& T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1} E\left(\left(\bar{\sigma}_{t-s}^2-\tilde{\sigma}_{t-s}^2\right)\left(\bar{\sigma}_t^2-\tilde{\sigma}_t^2\right)\bar{\sigma}_{t-s}^{-4}\bar{\sigma}_t^{-4} x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \\
= \; & T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj} E\left(\left(e_i^2-\sigma_i^2\right)\left(e_j^2-\sigma_j^2\right)\bar{\sigma}_{t-s}^{-4}\bar{\sigma}_t^{-4} x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \\
\leq \; & CT^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj} E\left(\left(e_i^2-\sigma_i^2\right)\left(e_j^2-\sigma_j^2\right) x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \\
= \; & CT^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj} E\left(e_i^2 e_j^2 x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \qquad (26) \\
& -CT^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj}\sigma_j^2 E\left(e_i^2 x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \\
& -CT^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj}\sigma_i^2 E\left(e_j^2 x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \\
& +CT^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T}\sum_{j=1}^{T} w_{(t-s)i}w_{tj}\sigma_i^2\sigma_j^2 E\left(x_{t-s-1}x_{t-1}e_{t-s}e_t\right).
\end{aligned}
$$

We will only show that (26) is zero since the remaining terms follow immediately. First, when

$i = j$ we write (26) as

$$
\begin{aligned}
& T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{T} w_{(t-s)i}w_{ti}E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t\right) \qquad (27)\\
= & \ T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=1}^{t-1} w_{(t-s)i}w_{ti}E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t\right)\\
& +T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1} w_{(t-s)t}w_{tt}E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t\right)\\
& +T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=t+1}^{T} w_{(t-s)i}w_{ti}E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t\right).
\end{aligned}
$$

The first term is zero since

$$E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t|\mathcal{F}_{t-1}\right) = e_i^4 x_{t-s-1}x_{t-1}e_{t-s}E\left(e_t|\mathcal{F}_{t-1}\right) = 0,$$

the second term is zero since $w_{tt} = 0$ and finally the third term is zero because

$$
\begin{aligned}
& T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=t+1}^{T} w_{(t-s)i}w_{ti}E\left(E\left(e_i^4 x_{t-s-1}x_{t-1}e_{t-s}e_t|\mathcal{F}_{t-1}\right)\right)\\
= & \ T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=t+1}^{T} w_{(t-s)i}w_{ti}E\left(E\left(E\left(x_{t-s-1}x_{t-1}e_{t-s}e_t e_i^4|\mathcal{F}_{i-1}\right)|\mathcal{F}_{t-1}\right)\right)\\
= & \ T^{-1}\sum_{t=2}^{T}\sum_{s=1}^{t-1}\sum_{i=t+1}^{T} w_{(t-s)i}w_{ti}E\left(e_i^4\right)E\left(x_{t-s-1}x_{t-1}e_{t-s}E\left(e_t|\mathcal{F}_{t-1}\right)\right) = 0.
\end{aligned}
$$

Now when $i \neq j$ in (26), the arguments follow similarly from (27). For example if $i > j$ and both i and j greater than t then

$$
\begin{aligned}
E\left(e_i^2 e_j^2 x_{t-s-1}x_{t-1}e_{t-s}e_t\right) &= E\left(E\left(E\left(E\left(e_i^2 e_j^2 x_{t-s-1}x_{t-1}e_{t-s}e_t|\mathcal{F}_{i-1}\right)|\mathcal{F}_{j-1}\right)|\mathcal{F}_{t-1}\right)\right)\\
&= 0.
\end{aligned}
$$

For the second term in (23), we have, by Lemma A(e,g,h) and (22),

$$
\begin{aligned}
& \left|T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2\right)^2 \tilde{\sigma}_t^{-2}\bar{\sigma}_t^{-4}\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right|\\
\leq & \left(\frac{1}{\min \bar{\sigma}_t^2}\right)^2\left(\frac{1}{\min \tilde{\sigma}_t^2}\right)\left|T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2\right)^2\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right|\\
\leq & \left(\frac{1}{\min \bar{\sigma}_t^2}\right)^2\left(\frac{1}{\min \tilde{\sigma}_t^2}\right)\left(\sum_{t=1}^{T}\left(\bar{\sigma}_t^2 - \tilde{\sigma}_t^2\right)^4\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t\right)^2\right)^{1/2}\\
= & \ O_p\left(\frac{1}{T^{1/2}b}\right) = o_p(1).
\end{aligned}
$$

To show (21), we have

$$B(\bar{\sigma}) - B(\sigma) = T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{-2} - \sigma_t^{-2}\right)\sum_{i=1}^{t-1}\frac{1}{i}e_{t-i}e_t.$$

Note that

$$E\left(\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)\sum_{i=1}^{t-1} i^{-1} e_{t-i} e_t | \mathcal{F}_{t-1}\right)=\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)\sum_{i=1}^{t-1} i^{-1} e_{t-i} E\left(e_t | \mathcal{F}_{t-1}\right)=0$$

so that $\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)\sum_{i=1}^{t-1} i^{-1} e_{t-i} e_t$ is a martingale difference sequence with respect to $\mathcal{F}_{t-1}$. Also note that

$$\sigma_t^2\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)^2 \leq \bar{\sigma}_t^{-4}\sigma_t^{-2}\left|\bar{\sigma}_t^2+\sigma_t^2\right| \cdot\left|\bar{\sigma}_t^2-\sigma_t^2\right| \leq C\left|\bar{\sigma}_t^2-\sigma_t^2\right|$$

from Xu and Phillips (2008) equation (23), page 277. Then

$$\begin{aligned}
& E\left(T^{-1/2}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)\sum_{i=1}^{t-1}\frac{1}{i} e_{t-i} e_t\right)^2 \\
= \; & T^{-1}\sum_{t=1}^{T}\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)^2\sum_{i=1}^{t-1}\sum_{j=1}^{t-1}\frac{1}{i}\frac{1}{j} E\left(e_{t-i} e_{t-j} E\left(e_t^2 | \mathcal{F}_{t-1}\right)\right) \\
= \; & T^{-1}\sum_{t=1}^{T}\sigma_t^2\left(\bar{\sigma}_t^{-2}-\sigma_t^{-2}\right)^2\sum_{i=1}^{t-1}\frac{1}{i^2} E\left(e_{t-i}^2\right) \\
\leq \; & C\sup_r \sigma(r)^2 T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|\sum_{i=1}^{t-1}\frac{1}{i^2} \\
\leq \; & C\sup_r \sigma(r)^2\frac{\pi^2}{6} T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right| \rightarrow 0,
\end{aligned}$$

by Lemma A(l).

Part (b) Recall $h_t(\gamma)$ given just above Harris and Kew (2016) equation (A.3) and define its partitioned matrix as

$$\begin{aligned}
h_t(\gamma) &= -\frac{1}{\sigma_t^2}\left(c_0(L;\gamma) e_t \cdot c_2(L;\gamma) e_t+\left(c_1(L;\gamma) e_t\right)\left(c_1(L;\gamma) e_t\right)'\right) \\
&= \left(\begin{array}{cc} h_{\theta\theta,t}(\gamma) & h_{\theta\psi,t}(\gamma) \\ h_{\psi\theta,t}(\gamma) & h_{\psi\psi,t}(\gamma)\end{array}\right).
\end{aligned}$$

We will only deal with the term $h_{\theta\theta,t}(\gamma)$ as the remaining terms are dealt with in the same fashion. Note that

$$h_{\theta\theta,t}(\gamma)=-\frac{1}{\sigma_t^2}\left(X_{0,t}(\gamma) X_{2,t-2}(\gamma)+X_{1,t-1}(\gamma)^2\right)$$

where $X_{0,t}(\gamma)=c_0(L;\gamma) e_t$, $X_{2,t-2}(\gamma)$ is the $(1,1)$-element of the matrix $c_2(L;\gamma) e_t$ and $X_{1,t-1}(\gamma)$ is the first element of the vector $c_1(L;\gamma) e_t$. Also $\hat{h}_{\theta\theta,t}(\gamma)$ is similarly defined by replacing σ_t^2 in $h_{\theta\theta,t}(\gamma)$ with $\hat{\sigma}_t^2$.

For clarity we write (θ,ψ) for γ and we will show that

$$\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left(\hat{h}_{\theta\theta,t}(0,\psi)-h_{\theta\theta,t}(0,\psi)\right) \xrightarrow{p} 0. \tag{28}$$

To show (28), we have

$$T^{-1}\sum_{t=1}^{T}\left(\hat{h}_{\theta\theta,t}(0,\psi)-h_{\theta\theta,t}(0,\psi)\right)=T^{-1}\sum_{t=1}^{T}\left(\hat{\sigma}_t^{-2}-\sigma_t^{-2}\right)\left(X_{0,t}(0,\psi) X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right).$$

First we will show that

$$\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left|X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right| = O_p(1). \tag{29}$$

To show (29) we write

$$\begin{aligned}
&\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left|X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right| \\
\leq\; &\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left|X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)\right| + \sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}X_{1,t-1}(0,\psi)^2 \\
\leq\; &\left(\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}X_{0,t}(0,\psi)^2\right)^{1/2}\left(\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}X_{2,t-2}(0,\psi)^2\right)^{1/2} + \sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}X_{1,t-1}(0,\psi)^2 .
\end{aligned}$$

Now from Cavaliere, Nielsen and Taylor (2015b) Lemma S.10,

$$\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}X_{0,t}(0,\psi)^2 = O_p(1)$$

follows from setting $k=l=u_1=u_2=0$ in their Lemma S.10 equation (S.16). Similarly for $X_{2,t-2}(0,\psi)^2$ with $k=l=2$ and $u_1=u_2=0$ and lastly for $X_{1,t-1}(0,\psi)^2$ with $k=l=1$ and $u_1=u_2=0$.

Second we define

$$A(\sigma,\psi) = T^{-1}\sum_{t=1}^{T}\sigma_t^{-2}\left(X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right).$$

Now, following Xu and Phillips (2008), (28) follows from showing the following results:

$$\sup_{\psi\in\Psi}\left|A(\hat{\sigma},\psi)-A(\tilde{\sigma},\psi)\right| \xrightarrow{p} 0; \tag{30}$$

$$\sup_{\psi\in\Psi}\left|A(\tilde{\sigma},\psi)-A(\bar{\sigma},\psi)\right| \xrightarrow{p} 0; \tag{31}$$

$$\sup_{\psi\in\Psi}\left|A(\bar{\sigma},\psi)-A(\sigma,\psi)\right| \xrightarrow{p} 0. \tag{32}$$

To show (30), by Lemma A(h,i,j) and equation (29) above, we write

$$\begin{aligned}
&\sup_{\psi\in\Psi}\left|A(\hat{\sigma},\psi)-A(\tilde{\sigma},\psi)\right| \\
=\; &\sup_{\psi\in\Psi}\left|T^{-1}\sum_{t=1}^{T}\frac{\left(\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right)}{\tilde{\sigma}_t^2\hat{\sigma}_t^2}\left(X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right)\right| \\
\leq\; &\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\frac{\left|\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right|}{\tilde{\sigma}_t^2\hat{\sigma}_t^2}\left|X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right| \\
\leq\; &\frac{\max_t\left|\hat{\sigma}_t^2-\tilde{\sigma}_t^2\right|}{\left(\min_t\tilde{\sigma}_t^2\right)\left(\min_t\hat{\sigma}_t^2\right)}\sup_{\psi\in\Psi} T^{-1}\sum_{t=1}^{T}\left|X_{0,t}(0,\psi)\,X_{2,t-2}(0,\psi)+X_{1,t-1}(0,\psi)^2\right| \\
=\; &O_p\left(\frac{1}{\sqrt{T}b}\right).
\end{aligned}$$

To show (31), using the same arguments as (30) but now using Lemma A(e,g,h) and (29), we have (31) $= O_p\left(T^{-1/4}b^{-1/2}\right)$.

We now show (32). To simplify notation let $X_t(\psi) = X_{0,t}(0,\psi)X_{2,t-2}(0,\psi) + X_{1,t-1}(0,\psi)^2$. We write

$$\begin{aligned}\sup_{\psi\in\Psi}\left|A(\bar{\sigma},\psi) - A(\sigma,\psi)\right| &= \sup_{\psi\in\Psi}\left|T^{-1}\sum_{t=1}^{T}\frac{\left(\bar{\sigma}_t^2-\sigma_t^2\right)}{\sigma_t^2\bar{\sigma}_t^2}X_t(\psi)\right| \\ &\le \left(\frac{1}{\inf_r \sigma(r)^2}\right)\left(\frac{1}{\min_t \bar{\sigma}_t^2}\right)\sup_{\psi\in\Psi}T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|\left|X_t(\psi)\right|.\end{aligned}$$

Then, given Lemma A(e), (32) follows if we show

$$\sup_{\psi\in\Psi}T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|\left|X_t(\psi)\right| = o_p(1). \tag{33}$$

To show (33), we first show pointwise convergence in ψ; that is

$$T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|\left|X_t(\psi)\right| \xrightarrow{p} 0. \tag{34}$$

To show this, we first note that

$$\begin{aligned}E\left|X_t(\psi)\right| &= E\left|X_{0,t}(0,\psi)X_{2,t-2}(0,\psi) + X_{1,t-1}(0,\psi)^2\right| \\ &\le \left(E\left(X_{0,t}(0,\psi)\right)^2\right)^{1/2}\left(E\left(X_{2,t-2}(0,\psi)\right)^2\right)^{1/2} + E\left(X_{1,t-1}(0,\psi)^2\right) \\ &< C < \infty \text{ uniformly in } t,\end{aligned} \tag{35}$$

because

$$E\left(X_{0,t}(0,\psi)\right)^2 = E\left(\sum_{j=0}^{t-1}\varphi_j(\psi)e_{t-j}\right)^2 = \sum_{j=0}^{t-1}\varphi_j(\psi)^2 E\left(e_{t-j}^2\right) \le \sup_r \sigma^2(r)\sum_{j=0}^{\infty}\varphi_j(\psi)^2 < C < \infty.$$

Similarly for $\sup_{1\le t\le T}E\left(X_{2,t-2}(0,\psi)\right)^2 < \infty$ and $\sup_{1\le t\le T}E\left(X_{1,t-1}(0,\psi)^2\right) < \infty$ because the coefficients of these linear processes are square summable. Now to show (34), we have

$$E\left|T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|\left|X_t(\psi)\right|\right| \le T^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right|E\left|X_t(\psi)\right| \le CT^{-1}\sum_{t=1}^{T}\left|\bar{\sigma}_t^2-\sigma_t^2\right| = o(1)$$

by Lemma A(l) and (35).

Finally, given (34), (33) follows if we show that $\left|T^{-1}\sum_{t=1}^{T}\left(\bar{\sigma}_t^2-\sigma_t^2\right)\sigma_t^{-2}\bar{\sigma}_t^{-2}X_t(\psi)\right|$ is stochastically equicontinuous. By the same argument that shows (29) and Cavaliere, Nielsen and Taylor (2015b) Lemma S.10, the stochastic equicontinuity holds since $\left(\bar{\sigma}_t^2-\sigma_t^2\right)\sigma_t^{-2}\bar{\sigma}_t^{-2}$ is a non-stochastic sequence and does not depend on ψ and hence does not change the steps of their proof of Lemma S.10 equation (S.16).

Part (c) For clarity we write (θ,ψ) for γ in what follows. Let

$$r(\psi) = \lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}E\left(\frac{e_t(0,\psi)}{\sigma_t}\right)^2 = \lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}E\left(\frac{a(L;\psi)\,a(L;\psi_0)^{-1}e_t}{\sigma_t}\right)^2$$

and

$$r_T(\psi) = T^{-1}\sum_{t=1}^{T}\left(\frac{e_t(0,\psi)}{\sigma_t}\right)^2 = T^{-1}\sum_{t=1}^{T}\left(\frac{a(L;\psi)\,a(L;\psi_0)^{-1}e_t}{\sigma_t}\right)^2.$$

Following the proof of CNT Lemma A.1, the required result that $\bar{\psi} \xrightarrow{p} \psi_0$ holds if we show

$$\sup_{\psi\in\Psi}|r_T(\psi) - r(\psi)| \xrightarrow{p} 0, \tag{36}$$

and

$$\inf_{\psi\in\Psi\cap\{\psi:\|\psi-\psi_0\|\geq\epsilon\}} r(\psi) > r(\psi_0) \quad \text{for all } \epsilon > 0, \tag{37}$$

where $\|x\|$ denote the usual Euclidean norm for any vector x.

We first define

$$r^{\#}(\psi) = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left(a(L;\psi)\,a(L;\psi_0)^{-1}\varepsilon_t\right)^2$$

and

$$r_T^{\#}(\psi) = T^{-1}\sum_{t=1}^{T}\left(a(L;\psi)\,a(L;\psi_0)^{-1}\varepsilon_t\right)^2.$$

To show (36), we write

$$\begin{aligned}
&\sup_{\psi\in\Psi}|r_T(\psi) - r(\psi)| \\
=\;& \sup_{\psi\in\Psi}\left|r_T(\psi) - r_T^{\#}(\psi) + r_T^{\#}(\psi) - r^{\#}(\psi) + r^{\#}(\psi) - r(\psi)\right| \\
\leq\;& \sup_{\psi\in\Psi}\left|r_T(\psi) - r_T^{\#}(\psi)\right| + \sup_{\psi\in\Psi}\left|r_T^{\#}(\psi) - r^{\#}(\psi)\right| + \sup_{\psi\in\Psi}\left|r^{\#}(\psi) - r(\psi)\right| \qquad (38)
\end{aligned}$$

For the first term in (38) we have

$$\begin{aligned}
&\sup_{\psi\in\Psi}\left|r_T(\psi) - r_T^{\#}(\psi)\right| \\
=\;& \sup_{\psi\in\Psi}\left|T^{-1}\sum_{t=1}^{T}\left(\left(\sum_{j=0}^{t-1}\varphi_j(\psi)\left(\frac{\sigma_{t-j}}{\sigma_t}\right)\varepsilon_{t-j}\right)^2 - \left(\sum_{j=0}^{t-1}\varphi_j(\psi)\,\varepsilon_{t-j}\right)^2\right)\right|. \qquad (39)
\end{aligned}$$

Since $\varphi_i(\psi)$ decays exponentially in i uniformly over all $\psi \in \Psi$ under Assumption R, (39) $\xrightarrow{p} 0$ follows by the same arguments leading to (A.6) in Harris and Kew (2016). The second term in (38) $\xrightarrow{p} 0$ follows from CNT Lemma A.1 equation (A.1). The third term $\xrightarrow{p} 0$ follows from the proof of (A.6) in Harris and Kew (2016).

Finally, equation (37) holds because the third term in (38) $\xrightarrow{p} 0$ and CNT Lemma A.1 equation (A.2).

Part (d). Define

$$\tilde{r}_T(\psi) = T^{-1}\sum_{t=1}^{T}\left(\frac{e_t(0,\psi)}{\hat{\sigma}_t}\right)^2.$$

Recall $r_T(\psi)$ and $r(\psi)$ given in part (c) above. The required result that $\tilde{\psi} \xrightarrow{p} \psi_0$ holds if we show

$$\sup_{\psi\in\Psi}|\tilde{r}_T(\psi) - r(\psi)| \xrightarrow{p} 0.$$

To see this, we write

$$\sup_{\psi\in\Psi}\left|\tilde{r}_T(\psi)-r_T(\psi)+r_T(\psi)-r(\psi)\right|\leq\sup_{\psi\in\Psi}\left|\tilde{r}_T(\psi)-r_T(\psi)\right|+\sup_{\psi\in\Psi}\left|r_T(\psi)-r(\psi)\right|.$$

The second term $\xrightarrow{p} 0$ because of (36). As for the first term,

$$\sup_{\psi\in\Psi}\left|\tilde{r}_T(\psi)-r_T(\psi)\right|=\sup_{\psi\in\Psi}\left|T^{-1}\sum_{t=1}^{T}\left(\left(\frac{e_t(0,\psi)}{\hat{\sigma}_t}\right)^2-\left(\frac{e_t(0,\psi)}{\sigma_t}\right)^2\right)\right|\xrightarrow{p}0,$$

by the same arguments that give (28) with $e_t(0,\psi)=X_{0,t}(\psi)$.

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