Supplement to "On the Power of Invariant Tests for Hypotheses on a Covariance Matrix"

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1 Comments on further results in Martellosio (2010)

Here we comment on problems in some results in Martellosio (2010) that have not been discussed in the main body of our article, referred to as PP in what follows. We also discuss if and how these problems can be fixed.

1.1 Comments on Lemmata D.2 and D.3 in Martellosio (2010)

Here we discuss problems with Lemmata D.2 and D.3 in Martellosio (2010) which are phrased in a spatial error model context. Correct versions of these lemmata, which furthermore are also not restricted to spatial regression models, have been given in Section 2.2.4 of PP. Both lemmata in Martellosio (2010) concern the quantity α^*, which is defined on p. 165 of Martellosio (2010) as follows:

> "For an exact invariant test of $\rho = 0$ against $\rho > 0$ in a SAR(1) model, α^* is the infimum of the set of values of $\alpha \in (0, 1]$ such that the limiting power does not vanish."

In this definition α denotes a generic symbol for the size of the test. Taken literally, the definition refers to one test only and hence does not make sense (as there is then only one associated value of α). From later usage of this definition in Martellosio (2010), it seems that the author had in mind a *family* of tests (rejection regions) like $\Phi_\kappa = \{y \in \mathbb{R}^n : T(y) > \kappa\}$, where T is a test

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statistic. Interpreting Martellosio's definition this way, it is clear that under the assumptions made in Martellosio (2010) (see Remark 2.1(ii) and Remark 2.3 in PP) his α^* coincides with $\alpha^*(T)$ defined in (23) in PP.

Lemma D.2 of Martellosio (2010), p. 181, then reads as follows:

> "Consider a model $G(X\beta, \sigma^2[(I-\rho W')(I-\rho W)]^{-1})$, where $G(\mu, \Gamma)$ denotes some multivariate distribution with mean μ and variance matrix Γ. When an invariant critical region for testing $\rho = 0$ against $\rho > 0$ is in form (9) [i.e., is of the form $\{y \in \mathbb{R}^n : T(y) > \kappa\}$ for some univariate test statistic T], and is such that $f_{\max}$ is not contained in its boundary, $\alpha^* = \Pr(T(\boldsymbol{z}) > T(f_{\max}); \boldsymbol{z} \sim G(0, I))$."

The statement of this lemma as well as its proof are problematic for the following reasons:

1. The lemma makes a statement about α^*, which is a quantity that depends not only on one specific critical region, but on a *family* of critical regions corresponding to a *family* of critical values κ against which the test statistic is compared. The critical region usually depends on κ and so does its boundary (cf. Proposition 2.11 in PP). Therefore, the assumption "... $f_{\max}$ is not contained in its [the invariant critical region's] boundary..." has little meaning in this context as it is not clear to which one of the many rejection regions the statement refers to. [Alternatively, if one interprets the statement of the lemma as requiring $f_{\max}$ not to be contained in the boundary of *every* rejection region in the family considered, this leads to a condition that typically will never be satisfied.]

2. The proof of the lemma is based on Corollary 1 in Martellosio (2010), the proof of which is incorrect as it is based on the incorrect Theorem 1 of Martellosio (2010).

3. The proof implicitly uses a continuity assumption on the cumulative distribution function of the test statistic under the null at the point $T(f_{\max})$ which is not satisfied in general.

Next we turn to Lemma D.3 in Martellosio (2010), which reads:

> "Consider a test that, in the context of a spatial error model with symmetric W, rejects $\rho = 0$ for small values of a statistic $\nu' B \nu$, where B is an $(n-k)\times(n-k)$ known symmetric matrix that does not depend on α, and ν is as defined in Section 2.2. Provided that $f_{\max} \notin \mathrm{bd}(\Phi)$, $\alpha^* = 0$ if and only if $Cf_{\max} \in E_1(B)$, and $\alpha^* = 1$ if and only if $Cf_{\max} \in E_{n-k}(B)$."

Here α refers to the size of the test, ν is given by $\mathrm{sign}(y_i)Cy/\|Cy\|$ for some fixed $i \in \{1, \ldots, n\}$, and Φ is not explicitly defined, but presumably denotes a rejection region corresponding to the test statistic $\nu' B \nu$. [Although the test statistic is not defined whenever $Cy = 0$, this does not pose a severe problem here since Martellosio (2010) considers only absolutely continuous distributions and since he assumes $k < n$; cf. Remark 2.13 in PP. Note furthermore that the factor $\mathrm{sign}(y_i)$ is irrelevant here.] Furthermore, $E_1(B)$ ($E_{n-k}(B)$) denotes the eigenspace corresponding to the smallest (largest) eigenvalue of B, and C in Martellosio (2010) stands for C_X. The statement of the lemma and its content are inadequate for the following reasons:

1. The proof of this lemma is based on Lemma D.2 of Martellosio (2010) which is invalid as discussed above.

2. Again, as in the statement of Lemma D.2 of Martellosio (2010), the author assumes that '... $f_{\max} \notin \mathrm{bd}(\Phi)$ ...', which is not meaningful, as the boundary typically depends on the critical value.

3. The above lemma in Martellosio (2010) requires W to be symmetric (although this is actually not used in the proof). Nevertheless, it is later applied to nonsymmetric weights matrices in the proof of Proposition 1 in Martellosio (2010).

As a point of interest we note that naively applying Lemma D.3 in Martellosio (2010) to the case where B is a multiple of the identity matrix I_{n-k} leads to the contradictory statement $0 = \alpha^* = 1$. However, in case B is a multiple of I_{n-k}, the test statistic degenerates, and thus the size of the test is 0 or 1, a case that is ruled out in Martellosio (2010) from the very beginning.

1.2 Comments on Proposition 1 and Lemma E.4 in Martellosio (2010)

Proposition 1 in Martellosio (2010) considers the pure SAR(1) model, i.e., $k = 0$ is assumed. This proposition reads as follows:

> "Consider testing $\rho = 0$ against $\rho > 0$ in a pure SAR(1) model. The limiting power of the Cliff-Ord test [cf. eq. (2) below] or of a test (8) [cf. eq. (1) below] is 1 irrespective of α [the size of the test] if and only if $f_{\max}$ is an eigenvector of W'."

We note that, while not explicit in the above statement, it is understood in Martellosio (2010) that $0 \leq \rho < \lambda_{\max}^{-1}$ is assumed. Similarly, the case $n = 1$ is not ruled out explicitly in the statement of the proposition, but it seems to be implicitly understood in Martellosio (2010) that $n \geq 2$ holds (note that in case $n = 1$ the test statistics degenerate and therefore the associated tests trivially have size equal to 0 or 1, depending on the choice of the critical value).

The test defined in equation (8) of Martellosio (2010) rejects for small values of

$$y'(I_n - \bar{\rho}W')(I_n - \bar{\rho}W)y/\|y\|^2, \tag{1}$$

where $0 < \bar{\rho} < \lambda_{\max}^{-1}$ is specified by the user. The argument in the proof of the proposition in Martellosio (2010) for this class of tests is incorrect for the following reasons:

1. The proof is based on Lemma D.3 in Martellosio (2010) which is incorrect as discussed in Section 1.1 above.

2. Even if Lemma D.3 in Martellosio (2010) were correct and could be used, this lemma would only deliver the result $\alpha^* = 0$ which does *not* imply, without a further argument, that the limiting power is equal to one for every size $\alpha \in (0, 1)$. By definition of α^*, $\alpha^* = 0$ only implies that the limiting power is *nonzero* for every size $\alpha \in (0, 1)$.

For the case of the Cliff-Ord test, i.e., the test rejecting for small values of

$$-y'Wy/\|y\|^2 = -0.5y'(W + W')y/\|y\|^2, \tag{2}$$

Martellosio (2010) argues that this can be reduced to the previously considered case, the proof of which is flawed as just shown. Apart from this, the reduction argument, which we now quote, has its own problems:

> "... By Lemma D.3 with $B = \Gamma^{-1}(\bar{\rho})$ [which equals $(I_n - \bar{\rho}W')(I_n - \bar{\rho}W)$], in order to prove that the limiting power of test (8) [cf. eq. (1) above] is 1 for any α [the size of the test], we need to show that $W' f_{\max} = \lambda_{\max} f_{\max}$ is necessary and sufficient for $f_{\max} \in E_n(\Gamma(\bar{\rho}))$. Clearly, if this holds for any $\bar{\rho} > 0$, it holds for $\bar{\rho} \to 0$ too, establishing also the part of the proposition regarding the Cliff-Ord test. ..."

The problem here is that it is less than clear what the precise mathematical "approximation" argument is. If we interpret it as deriving limiting power equal to 1 for the Cliff-Ord test from the corresponding result for tests of the form (8) and the fact that the Cliff-Ord test emerges as a limit of these tests for $\bar{\rho} \to 0$, then this involves an interchange of two limiting operations, namely $\rho \to \lambda_{\max}^{-1}$ and $\bar{\rho} \to 0$, for which no justification is provided. Alternatively, one could try to interpret the "approximation" argument as an argument that tries to derive $f_{\max} \in E_n(W + W')$ from $f_{\max} \in E_n(\Gamma(\bar{\rho}))$ for every $\bar{\rho} > 0$; of course, such an argument would need some justification which, however, is not provided. We note that this argument could perhaps be saved by using the arguments we provide in the proof of Proposition 4.10 in PP, but the proof of our correct version of Proposition 1 in Martellosio (2010), i.e., Proposition 4.9 in Section 4.1 of PP, is more direct and does not need such a reasoning. Furthermore, note that the proof of Proposition 4.9 in PP is based on our Proposition 2.26 in PP, which is a correct version of Lemma D.3 in Martellosio (2010) and which delivers not only the conclusion $\alpha^* = 0$, but the stronger conclusion that the limiting power is indeed equal to 1 for every size in $(0, 1)$.

We now turn to a discussion of Lemma E.4 of Martellosio (2010), which is again a statement about the Cliff-Ord test and tests of the form (8) in Martellosio (2010), but now in the context of the SEM (i.e., $k > 0$ is possible). The statement and the proof of the lemma suffer from the following shortcomings (again Lemma E.4 implicitly assumes that $0 \leq \rho < \lambda_{\max}^{-1}$ holds):

1. The proof of the lemma is based on Lemma D.3 in Martellosio (2010), which is incorrect (cf. the discussion in Section 1.1 above).

2. The proof uses non-rigorous arguments such as arguments involving a 'limiting matrix' with an infinite eigenvalue. Additionally, continuity of the dependence of eigenspaces on the underlying matrix is used without providing the necessary justification.

3. For the case of the Cliff-Ord test the same unjustified reduction argument as in the proof of Proposition 1 of Martellosio (2010) is used, cf. the preceding discussion.

For a correct version of Lemma E.4 of Martellosio (2010) see Proposition 4.10 in Section 4.1 of PP. As a point of interest we furthermore note that cases where the test statistics become degenerate (e.g., the case $n - k = 1$) are not ruled out explicitly in Lemma E.4 in Martellosio (2010); in these cases $\alpha^* = 1$ (and not $\alpha^* = 0$) holds.

1.3 Comments on Propositions 3, 4, and 5 in Martellosio (2010)

The proof of the part of Proposition 3 of Martellosio (2010) regarding point-optimal invariant tests seems to be correct except for the case where $\operatorname{span}(X)^{\perp}$ is contained in one of the eigenspaces of $\Sigma(\rho)$. In this case the test statistic of the form (8) in Martellosio (2010) is degenerate (see Section 4.3 of PP) and does not give the point-optimal invariant test (except in the trivial case where the size is 0 or 1, a case always excluded in Martellosio (2010)). However, this problem is easily fixed by observing that the point-optimal invariant test in this case is given by the randomized test $\varphi \equiv \alpha$, which is trivially unbiased. Two minor issues in the proof are as follows: (i) Lemma E.3 in Martellosio (2010) can only be applied as long as $\mathbf{z}_i^2 > 0$ for every $i \in H$. Fortunately, the complement of this event is a null-set allowing the argument to go through. (ii) The expression 'stochastically larger' in the paragraph following (E.4) should read 'stochastically smaller'. We also note that the assumption of Gaussianity can easily be relaxed to elliptical symmetry in view of G_X-invariance of the tests considered.

More importantly, the proof of the part of Proposition 3 of Martellosio (2010) concerning locally best invariant tests is highly deficient for at least two reasons: First, it is claimed that locally best invariant tests are of the form (7) in Martellosio (2010) with $Q = d\Sigma(\rho)/d\rho|_{\rho=0}$. While this is correct under regularity conditions (including a differentiability assumption on $\Sigma(\rho)$), such conditions are, however, missing in Proposition 3 of Martellosio (2010). Also, the case where $\operatorname{span}(X)^{\perp}$ is contained in one of the eigenspaces of $\Sigma(\rho)$ has to be treated separately, as then the locally best invariant test is given by the randomized test $\varphi \equiv \alpha$. Second, the proof uses once more an unjustified approximation argument in an attempt to reduce the case of locally best invariant tests to the case of point-optimal invariant tests. It is not clear what the precise nature of the approximation argument is. Furthermore, even if the approximation argument could be somehow repaired to deliver unbiasedness of locally best invariant tests, it is less than clear that strict unbiasedness could be obtained this way as strict inequalities are not preserved by limiting operations.

We next turn to the part of Proposition 4 of Martellosio (2010) regarding point-optimal invariant tests.[1] As in the case of Proposition 3 discussed above, the case where $\operatorname{span}(X)^{\perp}$ is contained in one of the eigenspaces of $\Sigma(\rho)$ has to be treated separately, and Gaussianity can be relaxed to elliptical symmetry. We note that the clause 'if and only if' in the last but one line of p. 185 of Martellosio (2010) should read 'if'. We also note that the verification of the first displayed inequality on p. 186 of Martellosio (2010) could be shortened (using Lemma E.3 (more precisely, the more general result referred to in the proof of this lemma) with $a_i = \lambda_i(W)/\tau_i(\rho)$, $b_i = \tau_i^2(\bar{\rho})$, and $p_i = \mathbf{z}_i^2/\tau_i^2(\rho)$ to conclude that the first display on p. 186 holds almost surely, and furthermore that it holds almost surely with equality if and only if all b_i or all a_i are equal, which is equivalent to all $\lambda_i(W)$ for $i \in H$ being equal).

Again, the proof of the part of Proposition 4 of Martellosio (2010) concerning locally best invariant tests is deficient as it is based on the same unjustified approximation argument mentioned before.

We next turn to Proposition 5 of Martellosio (2010). In the last of the three cases considered in this proposition, both test statistics are degenerate and hence the power functions are trivially constant equal to 0 or 1 (a case ruled out in Martellosio (2010)). More importantly, the proof of Proposition 5 is severely flawed for several reasons, of which we only discuss a few: First, the proof makes use of Corollary 1 of Martellosio (2010), the proof of which is based on the incorrect Theorem 1 in Martellosio (2010); it also makes use of Lemma E.4 and Proposition 4 of Martellosio (2010) which are incorrect as discussed before. Second, even if these results used in the proof were correct as they stand, additional problems would arise: Lemma E.4 only delivers $\alpha^* = 0$, and not the stronger conclusion that the limiting power equals 1, as would be required in the proof. Furthermore, Proposition 4 has Gaussianity of the errors as a hypothesis, while such an assumption is missing in Proposition 5.

We conclude by mentioning that a correct version of the part of Proposition 5 of Martellosio (2010) concerning tests of the form (8) in Martellosio (2010) can probably be obtained by substituting our Corollary 4.5 and Proposition 4.10 for Corollary 1 and Lemma E.4 of Martellosio (2010)

[1] While not explicit in the statement of this proposition, it is implicitly assumed that $0 \leq \rho < \lambda_{\max}^{-1}$ holds.

in the proof, but we have not checked the details. For the Cliff-Ord test this does not seem to work in the same way as the corresponding case of Proposition 4 of Martellosio (2010) is lacking a proof as discussed before.

2 Proofs of auxiliary results in PP

Proof of Lemma D.1: Let B be a Borel set in S^{n-1} and let $\chi : \mathbb{R}^n \backslash \{0\} \to S^{n-1}$ be given by $\chi(z) = z/\|z\|$. Then

$$\Pr(\mathbf{s} \in B) = \Pr(\mathbf{z} \in \chi^{-1}(B)) = \int_{\mathbb{R}^n \backslash \{0\}} \mathbf{1}_{\chi^{-1}(B)}(z)\, p(z)\, dz = \int_{(0,\infty)\times S^{n-1}} \mathbf{1}_{\chi^{-1}(B)}(rs)\, p(rs)\, dH(r,s)$$

where H is the pushforward measure of $\mu_{\mathbb{R}^n}$ (restricted to $\mathbb{R}^n \backslash \{0\}$) under the map $z \mapsto (\|z\|, z/\|z\|)$. But H is nothing else than the product of the measure on $(0,\infty)$ with Lebesgue density r^{n-1} and the surface measure $c\upsilon_{S^{n-1}}$ on S^{n-1} where c is given in the lemma (cf. Stroock (1999)). In view of Tonelli's theorem (observe all functions involved are nonnegative) and since $\mathbf{1}_{\chi^{-1}(B)}(rs) = \mathbf{1}_{\chi^{-1}(B)}(s) = \mathbf{1}_B(s)$ clearly holds for $s \in S^{n-1}$, we obtain

$$\Pr(\mathbf{s} \in B) = \int_{S^{n-1}} \mathbf{1}_B(s) \left(c \int_{(0,\infty)} p(rs)\, r^{n-1} d\mu_{(0,\infty)}(r) \right) d\upsilon_{S^{n-1}}(s),$$

which establishes the claims except for the last one. We next prove the final claim. First, observe that for every Borel set B in S^{n-1} we have $\upsilon_{S^{n-1}}(B) > 0$ if and only if $\mu_{\mathbb{R}^n}\left(\chi^{-1}(B)\right) > 0$. [This is seen as follows: Specializing what has been proved so far to the case where $\mathbf{z}$ follows a standard Gaussian distribution, shows that in this case $\mathbf{s}$ is uniformly distributed on S^{n-1}. Hence, $\upsilon_{S^{n-1}}(B) = \Pr(\mathbf{s} \in B) = \Pr(\mathbf{z} \in \chi^{-1}(B))$. But then the equivalence of the Gaussian measure with $\mu_{\mathbb{R}^n}$ establishes that $\upsilon_{S^{n-1}}(B) > 0$ if and only if $\mu_{\mathbb{R}^n}\left(\chi^{-1}(B)\right) > 0$.] Let now B satisfy $\upsilon_{S^{n-1}}(B) > 0$. Clearly, $\Pr(\mathbf{s} \in B) = \Pr(\mathbf{z} \in \chi^{-1}(B)) \geq \Pr(\mathbf{z} \in \chi^{-1}(B) \cap V)$ where V is an open neighborhood of the origin on which p is positive $\mu_{\mathbb{R}^n}$-almost everywhere. But then we must have $\mu_{\mathbb{R}^n}\left(\chi^{-1}(B) \cap V\right) > 0$, because $\mu_{\mathbb{R}^n}\left(\chi^{-1}(B)\right) > 0$ follows as a consequence of $\upsilon_{S^{n-1}}(B) > 0$ as just shown above and because $\chi^{-1}(B)$ can be written as a countable union of the sets $j\left(\chi^{-1}(B) \cap V\right)$ with $j \in \mathbb{N}$. By the assumption on p we can now conclude that $\Pr(\mathbf{z} \in \chi^{-1}(B) \cap V) > 0$ holds. Hence, we have established that $\Pr(\mathbf{s} \in B) > 0$ holds whenever $\upsilon_{S^{n-1}}(B) > 0$ is satisfied. ■

Proof of Lemma D.3: Part 1 is obvious. To prove Part 2 we denote the distribution of $\mathbf{z}/\|\mathbf{z}\|$ by G and the distribution of $\mathbf{r}$ by H. Because $\mathbf{z}/\|\mathbf{z}\|$ and $\mathbf{r}$ are independent, the joint distribution of $\mathbf{z}/\|\mathbf{z}\|$ and $\mathbf{r}$ on $S^{n-1} \times (0,\infty)$, equipped with the product σ-field, is given by the product measure $G \otimes H$. Therefore, the distribution of $\mathbf{z}^\dagger$ is the push-forward measure of $G \otimes H$ under the mapping $m(s,r) = rs$. Hence for every $A \in \mathcal{B}(\mathbb{R}^n)$ we have, using Tonelli's theorem and the fact that G and

H have densities g and h, respectively, that

$$\begin{aligned}
\Pr(\mathbf{z}^\dagger \in A) &= \int_{S^{n-1}\times(0,\infty)} \mathbf{1}_A(rs)d(G\otimes H)(s,r) = \int_{(0,\infty)}\int_{S^{n-1}} \mathbf{1}_A(rs)dG(s)dH(r) \\
&= \int_{(0,\infty)}\int_{S^{n-1}} \mathbf{1}_A(rs)g(s)d\upsilon_{S^{n-1}}(s)h(r)d\mu_{(0,\infty)}(r) \\
&= \int_{(0,\infty)} r^{n-1}\int_{S^{n-1}} \mathbf{1}_A(rs)g(s)r^{1-n}h\,(r)\,d\upsilon_{S^{n-1}}(s)d\mu_{(0,\infty)}(r) \\
&= \int_{(0,\infty)} r^{n-1}\int_{S^{n-1}} f\,(rs)\,d\upsilon_{S^{n-1}}(s)d\mu_{(0,\infty)}(r),
\end{aligned}$$

where for $x \in \mathbb{R}^n$ the function f is given by

$$f(x) = \begin{cases} \mathbf{1}_A(x)g(x/\|x\|)\|x\|^{1-n}h\,(\|x\|) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Since f is clearly a non-negative and Borel-measurable function, we can apply Theorem 5.2.2 in Stroock (1999) to see that

$$\begin{aligned}
\Pr(\mathbf{z}^\dagger \in A) &= \int_{(0,\infty)} r^{n-1}\int_{S^{n-1}} f(rs)d\upsilon_{S^{n-1}}(s)d\mu_{(0,\infty)}(r) \\
&= \int_{\mathbb{R}^n} c^{-1}f(x)d\mu_{\mathbb{R}^n}(x) = \int_{\mathbb{R}^n} \mathbf{1}_A(x)g^\dagger(x)d\mu_{\mathbb{R}^n}(x).
\end{aligned}$$

This establishes the second part of the lemma. To prove the third part denote by $D_{g^\dagger} \subseteq \mathbb{R}^n$, $D_g \subseteq S^{n-1}$ and $D_h \subseteq (0,\infty)$ the discontinuity points of $g^\dagger$, g, and h, respectively, which are measurable. Using Part 2 of the lemma we see that $x \neq 0$, $x/\|x\| \in \mathbb{R}^n\backslash D_g$, and $\|x\| \in \mathbb{R}^n\backslash D_h$ imply $x \in \mathbb{R}^n\backslash D_{g^\dagger}$. Therefore, negating the statement, we see that $\mathbf{1}_{D_{g^\dagger}}(x) \leq \mathbf{1}_{\{0\}}(x) + \mathbf{1}_{D_g}(x/\|x\|) + \mathbf{1}_{D_h}(\|x\|)$ must hold which implies

$$\mu_{\mathbb{R}^n}(D_{g^\dagger}) = \int_{\mathbb{R}^n} \mathbf{1}_{D_{g^\dagger}}(x)d\mu_{\mathbb{R}^n}(x) \leq \int_{\mathbb{R}^n} \mathbf{1}_{D_g}(x/\|x\|)d\mu_{\mathbb{R}^n}(x) + \int_{\mathbb{R}^n} \mathbf{1}_{D_h}(\|x\|)d\mu_{\mathbb{R}^n}(x). \tag{3}$$

Using again Theorem 5.2.2 in Stroock (1999) we see that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{1}_{D_g}(x/\|x\|)d\mu_{\mathbb{R}^n}(x) &= \int_{(0,\infty)} r^{n-1}\int_{S^{n-1}} \mathbf{1}_{D_g}(s)cd\upsilon_{S^{n-1}}(s)d\mu_{(0,\infty)}(r) \\
&= \int_{(0,\infty)} c\upsilon_{S^{n-1}}(D_g)r^{n-1}d\mu_{(0,\infty)}(r) = 0,
\end{aligned}$$

because $\upsilon_{S^{n-1}}(D_g) = 0$ holds by assumption. Similarly, we obtain

$$\int_{\mathbb{R}^n} \mathbf{1}_{D_h}(\|x\|)d\mu_{\mathbb{R}^n}(x) = \int_{S^{n-1}}\int_{(0,\infty)} r^{n-1}\mathbf{1}_{D_h}(r)d\mu_{(0,\infty)}(r)cd\upsilon_{S^{n-1}}(s) = 0,$$

because the inner integral is zero as a consequence of the assumption that $\mu_{(0,\infty)}(D_h) = 0$. Together with Equation (3) the last two displays establish $\mu_{\mathbb{R}^n}(D_{g^\dagger}) = 0$. To prove Part 4 denote by $Z_{g^\dagger} \subseteq \mathbb{R}^n$,

$Z_g \subseteq S^{n-1}$, and $Z_h \subseteq (0, \infty)$ the zero sets of $g^\dagger$, g, and h, respectively, which are obviously measurable. Replacing $D_{g^\dagger}$, D_g, and D_h with $Z_{g^\dagger}$, Z_g, and Z_h, respectively, in the argument used above then establishes Part 4. To prove the last part, we observe that g being constant $\upsilon_{S^{n-1}}$-almost everywhere implies that $\mathbf{z}/\|\mathbf{z}\|$ is uniformly distributed on S^{n-1}. Since $\mathbf{z}/\|\mathbf{z}\|$ is independent of $\mathbf{r}$, which is distributed as the square root of a χ^2 with n degrees of freedom, it is now obvious that $\mathbf{z}^\dagger$ is Gaussian with mean zero and covariance matrix I_n. ■

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