# Supplement to: Dynamic Panel Anderson-Hsiao Estimation with Roots Near Unity*


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This supplement provides detailed derivations and proofs of the results in the paper "Dynamic Panel Anderson-Hsiao Estimation with Roots Near Unity".

**Proof of Theorem 1.** Part (i) follows by the Lindeberg Lévy CLT

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix} \sum_{t=2}^{T}\Delta u_{it}y_{it-2} \\ \sum_{t=2}^{T}\Delta y_{it-1}y_{it-2} \end{pmatrix} \Rightarrow N\left(0, V_T\right), \tag{0.1}$$

with

$$V_T = \begin{pmatrix} \mathbb{E}\left(\sum_{t=2}^{T}\Delta u_{it}y_{it-2}\right)^2 & \mathbb{E}\left(\sum_{t=2}^{T}\Delta u_{it}y_{it-2}\right)\left(\sum_{t=2}^{T}\Delta y_{it-1}y_{it-2}\right) \\ \mathbb{E}\left(\sum_{t=2}^{T}\Delta u_{it}y_{it-2}\right)\left(\sum_{t=2}^{T}\Delta y_{it-1}y_{it-2}\right) & \mathbb{E}\left(\sum_{t=2}^{T}\Delta y_{it-1}y_{it-2}\right)^2 \end{pmatrix}. \tag{0.2}$$

To evaluate it is convenient to use partial summation

$$\sum_{t=2}^{T}\Delta u_{it}y_{it-2} = u_{iT}y_{iT-2} - u_{i1}y_{i0} - \sum_{t=3}^{T}u_{it-1}\Delta y_{it-2} \tag{0.3}$$

To compute $V_T$, note that

$$\mathbb{E}\left(\sum_{t=2}^{T}\Delta u_{it}y_{it-2}\right)^2 = \mathbb{E}\left\{\left(u_{iT}y_{iT-2} - u_{i1}y_{i0}\right) - \sum_{t=3}^{T}u_{it-1}u_{it-2}\right\}^2$$


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$$
\begin{aligned}
&= \mathbb{E}\left(u_{iT}y_{iT-2} - u_{i1}y_{i0}\right)^2 - 2\mathbb{E}\left\{\left(u_{iT}y_{iT-2} - u_{i1}y_{i0}\right)\left(\sum_{t=3}^{T} u_{it-1}u_{it-2}\right)\right\} + \mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}u_{it-2}\right)^2 \\
&= \mathbb{E}u_{iT}^2 y_{iT-2}^2 + \mathbb{E}u_{i1}^2 y_{i0}^2 + \sum_{t=3}^{T} \mathbb{E}\left(u_{it-1}^2 u_{it-2}^2\right) \\
&= \sigma^4 T_2 + 2\sigma^2 \mathbb{E}y_{i0}^2 + \sigma^4 T_2 = 2\sigma^4 T_2,
\end{aligned}
$$

the final line following if the initial condition $y_{i0} = 0$, which will be assumed in the calculations below. The large $n$ asymptotic results will continue to hold for $y_{i0} = O_p(1)$ even for finite $T$ with some obvious minor adjustments to the variance matrix expressions involving quantities of $O(1)$ in $T$. Next

$$
\begin{aligned}
\mathbb{E}\left(\sum_{t=2}^{T} \Delta y_{it-1}y_{it-2}\right)^2 &= \mathbb{E}\left(\sum_{t=2}^{T} u_{it-1}y_{it-2}\right)^2 = \sigma^2 \sum_{t=2}^{T} \mathbb{E}y_{it-2}^2 = \sigma^4 \sum_{t=2}^{T} (t-2) \\
&= \sigma^4 T_2 T_1/2,
\end{aligned}
$$

and, with $y_{i0} = 0$ (or up to $O(1)$ in $T$ if $y_{i0} \neq 0$)

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{t=2}^{T} \Delta u_{it}y_{it-2}\right)\left(\sum_{t=2}^{T} u_{it-1}y_{it-2}\right) = \mathbb{E}\left\{\left(\left(u_{iT}y_{iT-2} - u_{i1}y_{i0}\right) - \sum_{t=3}^{T} u_{it-1}u_{it-2}\right)\left(\sum_{t=2}^{T} u_{it-1}y_{it-2}\right)\right\} \\
&= -\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}u_{it-2}\right)\left(\sum_{t=2}^{T} u_{it-1}y_{it-2}\right) - \mathbb{E}\left(u_{i1}y_{i0}\right)^2 \\
&= -\mathbb{E}\left\{\sum_{t=3}^{T} \left(u_{it-1}u_{it-2}\right)\left(u_{it-1}y_{it-2}\right) + \sum_{s,t=3;s\neq t}^{T} \left(u_{it-1}u_{it-2}\right)\left(u_{is-1}y_{is-2}\right)\right\} \\
&= -\sum_{t=3}^{T} \mathbb{E}u_{it-1}^2 u_{it-2}^2 = -\sigma^4 \sum_{t=3}^{T} 1 = -\sigma^4 T_2.
\end{aligned}
$$

Then

$$
V_T = \begin{pmatrix} 2\sigma^4 T_2 & -\sigma^4 T_2 \\ -\sigma^4 T_2 & \sigma^4 T_2 T_1/2 \end{pmatrix} = \sigma^4 T_2 \begin{pmatrix} 2 & -1 \\ -1 & T_1/2 \end{pmatrix},
$$

as stated.

For Part (ii), simply write $\rho_{IV} - 1 = \frac{N_{nT}}{D_{nT}}$, and note from (i) that $(N_{nT}, D_{nT}) \underset{n\to\infty}{\Rightarrow}$

$\sigma^2 T_2^{1/2}\left(\xi_{N,T},\xi_{D,T}\right)$, where $\left(\xi_{N,T},\xi_{D,T}\right)$ is bivariate $N\left(0,\begin{bmatrix} 2 & -1 \\ -1 & T_1/2 \end{bmatrix}\right)$. Next, decompose $\xi_{N,T}$ as $\xi_{N,T}=\xi_{N.D,T}+\frac{-1}{T_1/2}\xi_{D,T}$ where $\xi_{N.D,T}\equiv N\left(0,2-\frac{(-1)^2}{T_1/2}\right)=N\left(0,2\left(1-\frac{1}{T_1}\right)\right)$ is independent of $\xi_{D,T}$, so that

$$\begin{pmatrix} \xi_{N.D,T} \\ \xi_{D,T} \end{pmatrix} \equiv N\left(0,\begin{bmatrix} 2\left(1-\frac{1}{T_1}\right) & 0 \\ 0 & T_1/2 \end{bmatrix}\right).$$

Combining these results, we have by joint weak convergence and continuous mapping that as $n\to\infty$ with $T$ fixed,

$$\begin{aligned} \rho_{IV}-1 &= \frac{N_T}{D_T} \underset{n\to\infty}{\Rightarrow} \frac{\xi_{N,T}}{\xi_{D,T}} = \frac{\xi_{N.D,T}-\frac{2}{T_1}\xi_{D,T}}{\xi_{D,T}} \quad (0.4) \\ &= -\frac{2}{T_1}+\frac{\xi_{N.D,T}}{\xi_{D,T}} = -\frac{2}{T_1}+\frac{2\left(1-\frac{1}{T_1}\right)^{1/2}}{T_1^{1/2}}\frac{\zeta_N}{\zeta_D} \\ &\equiv -\frac{2}{T_1}+2\frac{\left(1-\frac{1}{T_1}\right)^{1/2}}{T_1^{1/2}}\mathbb{C}, \quad (0.5) \end{aligned}$$

where $(\zeta_N,\zeta_D)\equiv N(0,I_2)$ and $\mathbb{C}$ is a standard Cauchy variate. Thus

$$\rho_{IV}-1 \underset{n\to\infty}{\Rightarrow} -\frac{2}{T_1}+2\frac{\left(1-\frac{1}{T_1}\right)^{1/2}}{T_1^{1/2}}\mathbb{C}, \quad (0.6)$$

yielding the stated result. ■

**Proof of Theorem 2.** By definition we have

$$\begin{aligned} \rho_{IV}-1 &= \frac{\sum_{i=1}^n\sum_{t=2}^T \Delta u_{it}y_{it-2}}{\sum_{i=1}^n\sum_{t=2}^T \Delta y_{it-1}y_{it-2}} \\ &= \frac{\sum_{i=1}^n\left\{(u_{iT}y_{iT-2}-u_{i1}y_{i0})-\sum_{t=3}^T u_{it-1}u_{it-2}\right\}}{\sum_{i=1}^n\sum_{t=2}^T u_{it-1}y_{it-2}}, \end{aligned}$$

and rescaling gives

$$\sqrt{T}\left(\rho_{IV}-1\right)=\frac{\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left\{\left(u_{iT}y_{iT-2}-u_{i1}y_{i0}\right)-\sum_{t=3}^{T}u_{it-1}u_{it-2}\right\}}{\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}u_{it-1}y_{it-2}}. \tag{0.7}$$

By partial summation

$$\sum_{t=1}^{T}u_{it}y_{it-1}=\sum_{t=1}^{T}u_{it}\left(\sum_{s=1}^{t-1}u_{is}+y_{i0}\right)=\frac{1}{2}\left\{\left(\sum_{t=1}^{T}u_{it}\right)^{2}-\sum_{t=1}^{T}u_{it}^{2}\right\}+\sum_{t=1}^{T}u_{it}y_{i0}.$$

Using the fact that $\mathbb{E}\left(u_{it}u_{is}u_{is-1}\right)=0$ for all $(t,s)$, we have by standard functional limit theory for $r\in[0,1]$

$$T^{-1/2}\sum_{t=1}^{\lfloor Tr\rfloor}\begin{bmatrix}u_{it}\\ u_{it}u_{it-1}\end{bmatrix}\Rightarrow\begin{bmatrix}B_{i}\left(r\right)\\ G_{i}\left(r\right)\end{bmatrix}\equiv BM\left(\begin{bmatrix}\sigma^{2}&0\\0&\sigma^{4}\end{bmatrix}\right),$$

where $B_i$ and $G_i$ are independent Brownian motions for all $i$. Then, since $y_{i0}=O_p\left(1\right)$ and $T^{-1}\sum_{t=1}^{T}u_{it}=o_p\left(1\right)$, we deduce the joint weak convergence (Phillips, 1987a, 1989)

$$\begin{bmatrix}T^{-1/2}\sum_{t=1}^{T}u_{it}\\ T^{-1}\sum_{t=1}^{T}u_{it}y_{it-1}\\ T^{-1}\sum_{t=1}^{T}u_{it}u_{it-1}\end{bmatrix}\underset{T\to\infty}{\Rightarrow}\begin{bmatrix}B_{i}\left(1\right)\\ \frac{1}{2}\left\{B_{i}\left(1\right)^{2}-\sigma^{2}\right\}\\ G_{i}\left(1\right)\end{bmatrix}=\begin{bmatrix}B_{i}\left(1\right)\\ \int_{0}^{1}B_{i}dB_{i}\\ G_{i}\left(1\right)\end{bmatrix}. \tag{0.8}$$

Since $u_{it}$ is *iid* over $t$ and $i$, it follows that $u_{iT}\Rightarrow u_{i\infty}$ as $T\to\infty$, where the limit variates $\{u_{i\infty}\}$ are independent over $i$ and have the same distribution as $u_{it}$. Note that $u_{iT}$ is independent of $\left(T^{-1/2}\sum_{t=1}^{T_1}u_{it},T^{-1}\sum_{t=1}^{T_1}u_{it}y_{it-1},T^{-1}\sum_{t=1}^{T_1}u_{it}u_{it-1}\right)$ and, hence, asymptotically independent of $\left(T^{-1/2}\sum_{t=1}^{T}u_{it},T^{-1}\sum_{t=1}^{T}u_{it}y_{it-1},T^{-1}\sum_{t=1}^{T}u_{it}u_{it-1}\right)$. It follows that $u_{i\infty}$ is independent of the vector of limit variates (0.8). We therefore have the combined weak convergence

$$\begin{bmatrix}T^{-1/2}\sum_{t=1}^{T}u_{it}\\ T^{-1}\sum_{t=1}^{T}u_{it}y_{it-1}\\ T^{-1/2}\sum_{t=1}^{T}u_{it}u_{it-1}\\ u_{iT}\end{bmatrix}\underset{T\to\infty}{\Rightarrow}\begin{bmatrix}B_{i}\left(1\right)\\ \frac{1}{2}\left\{B_{i}\left(1\right)^{2}-\sigma^{2}\right\}\\ G_{i}\left(1\right)\\ u_{i\infty}\end{bmatrix}=\begin{bmatrix}B_{i}\left(1\right)\\ \int_{0}^{1}B_{i}dB_{i}\\ G_{i}\left(1\right)\\ u_{i\infty}\end{bmatrix}. \tag{0.9}$$

Setting $G_i = G_i(1)$, the stated result

$$\sqrt{T}\left(\rho_{IV}-1\right) \underset{T\to\infty}{\Rightarrow} \frac{\sum_{i=1}^{n}\left\{u_{i\infty}B_i(1)-G_i\right\}}{\sum_{i=1}^{n}\frac{1}{2}\left\{B_i(1)^2-\sigma^2\right\}} \tag{0.10}$$

follows from (0.7) and (0.9) by continuous mapping.

For part (ii) we consider sequential asymptotics in which $T\to\infty$ is followed by $n\to\infty$. Observe that $u_{i\infty}B_i(1)-G_i$ is *iid* over $i$ with zero mean and variance

$$\mathbb{E}\left\{u_{i\infty}B_i(1)-G_i(1)\right\}^2 = \mathbb{E}\left(u_{i\infty}^2\right)\mathbb{E}\left(B_i(1)^2\right)+\mathbb{E}\left(G_i(1)^2\right)=2\sigma^4,$$

and is uncorrelated with $B_i(1)^2$. Since $\left\{B_i(1)^2-\sigma^2\right\}$ is *iid* with zero mean and variance $2\sigma^4$, application of the Lindeberg Lévy CLT as $n\to\infty$ gives

$$\begin{bmatrix} n^{-1/2}\sum_{i=1}^{n}\left\{u_{i\infty}B_i(1)-G_i\right\} \\ n^{-1/2}\sum_{i=1}^{n}\frac{1}{2}\left\{B_i(1)^2-\sigma^2\right\}\end{bmatrix} \underset{n\to\infty}{\Rightarrow} \begin{bmatrix} \left(2\sigma^4\right)^{1/2}\zeta_N \\ \left(\sigma^4/2\right)^{1/2}\zeta_D\end{bmatrix}, \tag{0.11}$$

where $(\zeta_N,\zeta_D)\equiv N(0,I_2)$. Hence,

$$\sqrt{T}\left(\rho_{IV}-1\right) \underset{(n,T)_{\text{seq}}\to\infty}{\Rightarrow} 2\mathbb{C}, \tag{0.12}$$

giving the required result. ■

**Proof of Theorem 3.** We proceed by examining a set of sufficient conditions for joint convergence limit theory developed in Phillips and Moon (1999). In particular, we consider conditions that suffice to ensure that sequential convergence as $(n,T)_{\text{seq}}\to\infty$ (i.e., $T\to\infty$ followed by $n\to\infty$) implies joint convergence $(n,T)\to\infty$ where there is no restriction on the diagonal path in which $n$ and $T$ pass to infinity.

We start by defining the vector of standardized components appearing in the numerator and denominator of $\rho_{IV}$

$$X_{nT} = \left(n^{-1/2}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left\{\left(u_{iT}y_{iT-2}-u_{i1}y_{i0}\right)-\sum_{t=3}^{T}u_{it-1}u_{it-2}\right\}, n^{-1/2}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=2}^{T}u_{it-1}y_{it-2}\right)\right)'. \tag{0.13}$$

From (0.9) and (0.11) we have the sequential convergence

$$
\begin{aligned}
X_{nT} \underset{T\to\infty}{\Rightarrow} X_n \quad &: = \left( n^{-1/2}\sum_{i=1}^{n}\left\{u_{i\infty}B_i\left(1\right)-G_i\left(1\right)\right\}, n^{-1/2}\sum_{i=1}^{n}\frac{1}{2}\left\{B_i\left(1\right)^2-\sigma^2\right\}\right)' \\
\underset{n\to\infty}{\Rightarrow} X \quad &: = \left( \left(2\sigma^4\right)^{1/2}\zeta_N, \left(\frac{\sigma^4}{2}\right)^{1/2}\zeta_D\right), \qquad (0.14)
\end{aligned}
$$

which in turn implies the sequential limit $\sqrt{T}\left(\rho_{IV}-1\right) \underset{(n,T)_{\text{seq}}\to\infty}{\Rightarrow} 2\mathbb{C}$ given in (0.12). By Lemma 6(b) of Phillips and Moon (1999), when $X_{nT} \underset{T\to\infty}{\Rightarrow} X_n \underset{n\to\infty}{\Rightarrow} X$ sequentially, joint weak convergence $X_{nT} \Rightarrow X$ as $(n,T)\to\infty$ holds if and only if

$$
\limsup_{n,T\to\infty}\left|\mathbb{E}f\left(X_{nT}\right)-\mathbb{E}f\left(X_n\right)\right| = 0 \qquad (0.15)
$$

for all bounded, continuous real functions $f$ on $\mathbb{R}^2$.

Simple primitive conditions sufficient for (0.15) to hold are available in the case where the components of the random quantity $X_{nT}$ involve averages of *iid* random variables as in the present case where we have $X_{nT} = n^{-1/2}\sum_{i=1}^{n} Y_{iT}$ with the $Y_{iT}$ independent over $i$. Component-wise we have

$$
X_{nT} = \left(X_{1nT}, X_{2nT}\right)' := \left(n^{-1/2}\sum_{i=1}^{n}Y_{1iT}, n^{-1/2}\sum_{i=1}^{n}Y_{2iT}\right),
$$

where $Y_{iT} = \left(Y_{1iT}, Y_{2iT}\right)'$ with

$$
\begin{aligned}
Y_{1iT} &= \frac{1}{\sqrt{T}}\left\{\left(u_{iT}y_{iT-2}-u_{i1}y_{i0}\right)-\sum_{t=3}^{T}u_{it-1}u_{it-2}\right\} \underset{T\to\infty}{\Rightarrow} Y_{1i} := u_{i\infty}B_i\left(1\right)-G_i\left(1\right), \\
Y_{2iT} &= \frac{1}{T}\sum_{t=2}^{T}u_{it-1}y_{it-2} \underset{T\to\infty}{\Rightarrow} Y_{2i} := \frac{1}{2}\left\{B_i\left(1\right)^2-\sigma^2\right\},
\end{aligned}
$$

for all $i$. The working probability space can be expanded as needed to ensure that the (limit) random quantities $Y_i := \left(Y_{1i}, Y_{2i}\right)'$ are defined in the same space for all $i$ so that averages involving $\sum_{i=1}^{n} Y_i$ are meaningful. In this framework we can use a result on joint convergence by Phillips and Moon (1999) – see lemma PM below – to verify condition (0.15). In what follows we use the notation of lemma PM.

We proceed to verify these conditions for $Y_{iT}$ and $Y_i$. First, $Y_{iT}$ is integrable since

$$\begin{aligned}
\mathbb{E}\left|u_{iT}y_{iT-2}\right| &\leq \left(\mathbb{E}\left|u_{iT}\right|^2\mathbb{E}\left|y_{iT-2}\right|^2\right)^{1/2} < \infty, \\
\mathbb{E}\left|\sum_{t=2}^{T}u_{it-1}u_{it-2}\right| &\leq T\mathbb{E}\left|u_{it-1}u_{it-2}\right| \leq T\mathbb{E}\left(u_{it}^2\right) < \infty, \\
\mathbb{E}\left|\sum_{t=2}^{T}u_{it-1}y_{it-2}\right| &\leq \sum_{t=2}^{T}\mathbb{E}\left|u_{it-1}y_{it-2}\right| \leq \sum_{t=2}^{T}\left(\mathbb{E}u_{it-1}^2\mathbb{E}y_{it-2}^2\right)^{1/2} < \infty.
\end{aligned}$$

To show (i) holds, observe that

$$\begin{aligned}
\mathbb{E}\left\|Y_{iT}\right\|^2 &= \mathbb{E}Y_{1iT}^2 + \mathbb{E}Y_{2iT}^2 \\
&= \frac{1}{T}\mathbb{E}\left\{u_{iT}y_{iT-2} - \sum_{t=3}^{T}u_{it-1}u_{it-2}\right\}^2 + \frac{1}{T^2}\mathbb{E}\left(\sum_{t=2}^{T}u_{it-1}y_{it-2}\right)^2 \\
&= 2\sigma^4\frac{T-2}{T} + \sigma^4\frac{1}{T^2}\sum_{t=2}^{T}(t-2) < \infty, \qquad (0.16)
\end{aligned}$$

when $y_{i0} = 0$, with obviously valid extension to the case where $y_{i0} = O_p(1)$ with finite second moments. Then

$$\limsup_{n,T\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|Y_{iT}\right\| = \limsup_{T\to\infty}\mathbb{E}\left\|Y_{iT}\right\| \leq \limsup_{T\to\infty}\left(\mathbb{E}\left\|Y_{iT}\right\|^2\right)^{1/2} < \infty,$$

as required. To show (ii) holds, simply observe that $\mathbb{E}Y_{iT} = \mathbb{E}Y_i = 0$. To show (iii) holds, note that

$$\limsup_{n,T\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|Y_{iT}\right\|\mathbf{1}\left\{\left\|Y_{iT}\right\| > n\epsilon\right\} = \limsup_{T\to\infty}\mathbb{E}\left\|Y_{iT}\right\|\mathbf{1}\left\{\left\|Y_{iT}\right\| > n\epsilon\right\} = 0, \text{ for all } \epsilon > 0,$$

since $\sup_T \mathbb{E}\left\|Y_{iT}\right\|^2 < \infty$ by virtue of (0.16). Finally, note that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|Y_i\right\|\mathbf{1}\left\{\left\|Y_i\right\| > n\epsilon\right\} = \limsup_{n\to\infty}\mathbb{E}\left\|Y_i\right\|\mathbf{1}\left\{\left\|Y_i\right\| > n\epsilon\right\} = 0,$$

since $\mathbb{E}\left\|Y_i\right\|^2 < \infty$, proving (iv). Hence, condition (0.15) holds and we have joint weak convergence

$$X_{nT} = n^{-1/2} \sum_{i=1}^{n} Y_{iT} \underset{n,T\to\infty}{\Rightarrow} X := \left( \left(2\sigma^4\right)^{1/2} \zeta_N, \left(\frac{\sigma^4}{2}\right)^{1/2} \zeta_D \right),$$

irrespective of the divergence rates of $n$ and $T$ to infinity. By continuous mapping, the required result follows for the GMM estimator so that $\sqrt{T}\left(\rho_{IV} - 1\right) \underset{n,T\to\infty}{\Rightarrow} 2\mathbb{C}$ jointly as $(n,T) \to \infty$ irrespective of the order and rates of divergence of the respective sample sizes.
■

**Lemma PM (Phillips and Moon, 1999, theorem 1)** *Suppose the $m \times 1$ random vectors $Y_{iT}$ are independent across $i$ for all $T$ and integrable. Assume that $Y_{iT} \Rightarrow Y_i$ as $T \to \infty$ for all $i$. Then, condition (0.15) holds if the following hold:*

*(i)* $\limsup_{n,T\to\infty} \frac{1}{n} \sum_{i=1}^{n} E\left\|Y_{iT}\right\| < \infty$,

*(ii)* $\limsup_{n,T\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\|\mathbb{E}Y_{iT} - \mathbb{E}Y_i\right\| < \infty$,

*(iii)* $\limsup_{n,T\to\infty} \frac{1}{n} \sum_{i=1}^{n} E\left\|Y_{iT}\right\| 1\left\{\left\|Y_{iT}\right\| > n\epsilon\right\} = 0$, *for all* $\epsilon > 0$

*(iv)* $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} E\left\|Y_i\right\| 1\left\{\left\|Y_i\right\| > n\epsilon\right\} = 0$, *for all* $\epsilon > 0$

**Proof of Theorem 4.** In case (i) $T$ is fixed as well as $c < 0$, which implies that $\rho = 1 + \frac{c}{\sqrt{T}}$ is fixed. So large $n$ asymptotics follow as in the (asymptotically) stationary case. By defintion we have $y_{it} = \alpha_i (1-\rho) + \rho y_{it-1} + u_{it} = -\frac{\alpha_i c}{\sqrt{T}} + \left(1 + \frac{c}{\sqrt{T}}\right) y_{it-1} + u_{it}$ and $\Delta y_{it} = \rho \Delta y_{it-1} + \Delta u_{it}$ so that $\Delta y_{it} = \alpha_i (1-\rho) + (\rho - 1) y_{it-1} + u_{it} = -\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-1} + u_{it}$. Then, as usual, $\mathbb{E}\left(u_{it} y_{it-2}\right) = \mathbb{E}\left(\Delta u_{it} y_{it-2}\right) = 0$ and orthogonality holds. When $y_{i0} = 0$, back substitution gives

$$y_{it} = \alpha_i \left(1 - \rho^t\right) + \sum_{j=0}^{t-1} \rho^j u_{it-j},$$

and $\mathbb{E}\left(y_{it}\right) = \alpha_i \left(1 - \rho^t\right)$, $\mathbb{V}\mathrm{ar}\left(y_{it}\right) = \sigma^2 \sum_{j=0}^{t-1} \rho^{2j} = \sigma^2 \frac{1-\rho^{2t}}{1-\rho^2}$, and $\mathbb{E}\left(y_{it}^2\right) = \sigma^2 \frac{1-\rho^{2t}}{1-\rho^2} +$

$\alpha_i^2 \left(1-\rho^t\right)^2$. Instrument relevance is determined by the magnitude of the moment

$$\begin{aligned}
\mathbb{E}\left(\Delta y_{it-1} y_{it-2}\right) &= \mathbb{E}\left(\left\{\alpha_i (1-\rho) + (\rho-1) y_{it-2} + u_{it-1}\right\} y_{it-2}\right) \\
&= \alpha_i^2 (1-\rho)\left(1-\rho^{t-2}\right) + (\rho-1)\left\{\sigma^2 \frac{1-\rho^{2(t-2)}}{1-\rho^2} + \alpha_i^2 \left(1-\rho^{t-2}\right)^2\right\} \\
&= -\sigma^2 \frac{1-\rho^{2(t-2)}}{1+\rho} - \alpha_i^2 (1-\rho)\left(1-\rho^{t-2}\right)\rho^{t-2} \qquad (0.17)
\end{aligned}$$

which is nonzero for $c < 0$ and zero when $c = 0$, corresponding to the unit root case ($\rho = 1$) considered earlier. Note that in the fully stationary case where initial conditions are in the infinite past so that $y_{i0} = \alpha_i + \sum_{j=0}^{\infty} \rho^j u_{i,-j}$ and $y_{it} = \alpha_i + \sum_{j=0}^{\infty} \rho^j u_{t-j}$ we have

$$\begin{aligned}
\mathbb{E}\left(\Delta y_{it-1} y_{it-2}\right) &= \alpha_i^2 (1-\rho) + (\rho-1)\,\mathbb{E}\left(y_{it}^2\right) = \alpha_i^2 (1-\rho) - (1-\rho)\left\{\frac{\sigma^2}{1-\rho^2} + \alpha_i^2\right\} \\
&= -\frac{\sigma^2}{1+\rho},
\end{aligned}$$

which corresponds with the leading term of (0.17) when $t \to \infty$ with $|\rho| < 1$.

Now consider the numerator and denominator of the centred and scaled GMM estimate

$$\sqrt{n}\left(\rho_{IV} - \rho\right) = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=2}^{T} \Delta u_{it} y_{it-2}}{\frac{1}{n} \sum_{i=1}^{n} \sum_{t=2}^{T} \Delta y_{it-1} y_{it-2}} =: \frac{N_{nT}}{D_{nT}}. \qquad (0.18)$$

First, noting that $\Delta y_{it-1} y_{it-2}$ is quadratic in $\alpha_i$, and using $T_j = T - j$ and $\sigma_\alpha^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \alpha_i^2$, the denominator of (0.18) takes the following form as $n \to \infty$

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} \sum_{t=2}^{T} \Delta y_{it-1} y_{it-2} \to_p \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(\sum_{t=2}^{T} \Delta y_{it} y_{it-2}\right) \\
= &\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{t=2}^{T} \left\{-\sigma^2 \frac{1-\rho^{2(t-2)}}{1+\rho} + \alpha_i^2 (1-\rho)\left(1-\rho^{t-2}\right)\left[1-\left(1-\rho^{t-2}\right)\right]\right\} \\
= &-\frac{\sigma^2}{1+\rho}\left[T_1 - \frac{1-\rho^{2T_1}}{1-\rho^2}\right] + \sigma_\alpha^2 (1-\rho)\left[T_1 - \frac{1-\rho^{T_1}}{1-\rho}\right] - \sigma_\alpha^2 (1-\rho)\left[T_1 - 2\frac{1-\rho^{T_1}}{1-\rho} + \frac{1-\rho^{2T_1}}{1-\rho^2}\right] \\
= &-\frac{\sigma^2}{1+\rho}\left[T_1 - \frac{1-\rho^{2T_1}}{1-\rho^2}\right] + \sigma_\alpha^2 (1-\rho)\left[\frac{1-\rho^{T_1}}{1-\rho} - \frac{1-\rho^{2T_1}}{1-\rho^2}\right] \\
= &-\frac{\sigma^2}{1+\rho}\left[T_1 - \frac{1-\rho^{2T_1}}{1-\rho^2}\right] + \sigma_\alpha^2 \left[1 - \rho^{T_1} - \frac{1-\rho^{2T_1}}{1+\rho}\right], \qquad (0.19)
\end{aligned}$$

which is again zero when $c = 0$ ($\rho = 1$). Turning to the numerator, we have $\mathbb{E}\left(\Delta u_{it} y_{it-2}\right) = 0$ by orthogonality and by a standard CLT argument for fixed $T$ as $n \to \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \sum_{t=2}^{T} \Delta u_{it} y_{it-2} \right) \Rightarrow N\left(0, v_T\right)$$

with

$$v_T = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( \sum_{t=2}^{T} \Delta u_{it} y_{it-2} \right)^2$$

We evaluate the above variance as follows. Using partial summation and $y_{i0} = 0$, we have

$$\sum_{t=2}^{T} \Delta u_{it} y_{it-2} = u_{iT} y_{iT-2} - u_{i1} y_{i0} - \sum_{t=3}^{T} u_{it-1} \Delta y_{it-2} = u_{iT} y_{iT-2} - \sum_{t=3}^{T} u_{it-1} \Delta y_{it-2}, \quad (0.20)$$

with variance

$$\begin{aligned} \mathbb{E} \left( u_{iT} y_{iT-2} - \sum_{t=3}^{T} u_{it-1} \Delta y_{it-2} \right)^2 &= \sigma^2 \mathbb{E} \left( y_{iT-2} \right)^2 + \sigma^2 \mathbb{E} \sum_{t=3}^{T} \left( \Delta y_{it-2} \right)^2 \\ &= \sigma^4 \frac{1 - \rho^{2T_2}}{1 - \rho^2} + \alpha_i^2 \sigma^2 \left( 1 - \rho^{T_2} \right)^2 + \sigma^2 \mathbb{E} \sum_{t=3}^{T} \left( \Delta y_{it-2} \right)^2 . \end{aligned}$$

Using $\mathbb{E}\left(y_{it}\right) = \alpha_i \left(1 - \rho^t\right)$, $\mathbb{V}\mathrm{ar}\left(y_{it}\right) = \sigma^2 \sum_{j=0}^{t-1} \rho^{2j} = \sigma^2 \frac{1-\rho^{2t}}{1-\rho^2}$, $\mathbb{E}\left(y_{it}^2\right) = \sigma^2 \frac{1-\rho^{2t}}{1-\rho^2} + \alpha_i^2 \left(1 - \rho^t\right)^2$, and $\Delta y_{it} = \alpha_i (1 - \rho) + (\rho - 1) y_{it-1} + u_{it}$, the final term $\sum_{t=3}^{T} \mathbb{E}\left(\Delta y_{it-2}\right)^2$ above is

$$\begin{aligned} & \sum_{t=3}^{T} \left\{ \alpha_i^2 (1-\rho)^2 + (1-\rho)^2 \mathbb{E}\left(y_{it-3}^2\right) + \sigma^2 - 2\alpha_i^2 (1-\rho)^2 \left(1 - \rho^{t-3}\right) \right\} \\ = \;& \sigma^2 T_2 - T_2 \alpha_i^2 (1-\rho)^2 + 2\alpha_i^2 (1-\rho)^2 \left( \frac{1 - \rho^{T_2}}{1 - \rho} \right) + (1-\rho)^2 \sum_{t=3}^{T} \mathbb{E}\left(y_{it-3}^2\right) \\ = \;& \sigma^2 T_2 - T_2 \alpha_i^2 (1-\rho)^2 + 2\alpha_i^2 (1-\rho) \left(1 - \rho^{T_2}\right) + (1-\rho)^2 \frac{\sigma^2}{1-\rho^2} \left[ T_2 - \frac{1 - \rho^{2T_2}}{1 - \rho^2} \right] \\ & + \alpha_i^2 (1-\rho)^2 \left[ T_2 - 2\frac{1 - \rho^{T_2}}{1 - \rho} + \frac{1 - \rho^{2T_2}}{1 - \rho^2} \right] \end{aligned}$$

$$= \sigma^2 T_2\left[1+\frac{(1-\rho)}{1+\rho}\right]-\frac{\sigma^2\left(1-\rho^{2T_2}\right)}{(1+\rho)^2}+2\alpha_i^2(1-\rho)\left(1-\rho^{T_2}\right)+\alpha_i^2(1-\rho)^2\left[-2\frac{1-\rho^{T_2}}{1-\rho}+\frac{1-\rho^{2T_2}}{1-\rho^2}\right]$$

$$= \frac{2\sigma^2 T_2}{1+\rho}-\frac{\sigma^2\left(1-\rho^{2T_2}\right)}{(1+\rho)^2}+\alpha_i^2(1-\rho)\frac{1-\rho^{2T_2}}{1+\rho}.$$

Then

$$\mathbb{E}\left(u_{iT}y_{iT-2}-\sum_{t=3}^{T}u_{it-1}\Delta y_{it-2}\right)^2=\sigma^4\frac{1-\rho^{2T_2}}{1-\rho^2}+\alpha_i^2\sigma^2\left(1-\rho^{T_2}\right)^2+\sigma^2\mathbb{E}\sum_{t=3}^{T}(\Delta y_{it-2})^2$$

$$= \sigma^4\frac{1-\rho^{2T_2}}{1-\rho^2}+\alpha_i^2\sigma^2\left(1-\rho^{T_2}\right)^2+\frac{2\sigma^4 T_2}{1+\rho}-\frac{\sigma^4\left(1-\rho^{2T_2}\right)}{(1+\rho)^2}+\alpha_i^2\sigma^2(1-\rho)\frac{1-\rho^{2T_2}}{1+\rho}$$

$$= \frac{2\sigma^4 T_2}{1+\rho}+\sigma^4\frac{2\rho\left(1-\rho^{2T_2}\right)}{(1-\rho^2)(1+\rho)}+\alpha_i^2\sigma^2\left(1-\rho^{T_2}\right)^2+\alpha_i^2\sigma^2(1-\rho)\frac{\left(1-\rho^{2T_2}\right)}{1+\rho}$$

$$= \frac{2\sigma^4 T_2}{1+\rho}+\sigma^4\frac{2\rho\left(1-\rho^{2T_2}\right)}{(1-\rho^2)(1+\rho)}+\alpha_i^2\sigma^2\left(1-\rho^{T_2}\right)\left[1-\rho^{T_2}+\frac{(1-\rho)\left(1+\rho^{T_2}\right)}{1+\rho}\right],$$

and

$$\omega_{NT}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(\sum_{t=2}^{T}\Delta u_{it}y_{it-2}\right)^2$$

$$= \frac{2\sigma^4 T_2}{1+\rho}+\sigma^4\frac{2\rho\left(1-\rho^{2T_2}\right)}{(1-\rho^2)(1+\rho)}+\sigma_\alpha^2\sigma^2\left(1-\rho^{T_2}\right)\left[1-\rho^{T_2}+\frac{(1-\rho)\left(1+\rho^{T_2}\right)}{1+\rho}\right] \quad (0.21)$$

From (0.19) we have

$$\omega_{DT}=\left\{-\frac{\sigma^2}{1+\rho}\left[T_1-\frac{1-\rho^{2T_1}}{1-\rho^2}\right]+\sigma_\alpha^2\left[1-\rho^{T_1}-\frac{1-\rho^{2T_1}}{1+\rho}\right]\right\}^2 =: v_{DT}^2 \quad (0.22)$$

which leads to the asymptotic variance

$$\omega_T^2 = \frac{\omega_{NT}}{\omega_{DT}}=\frac{\frac{2\sigma^4 T_2}{1+\rho}+\sigma^4\frac{2\rho\left(1-\rho^{2T_2}\right)}{(1-\rho^2)(1+\rho)}+\sigma_\alpha^2\sigma^2\left(1-\rho^{T_2}\right)\left[1-\rho^{T_2}+\frac{(1-\rho)\left(1+\rho^{T_2}\right)}{1+\rho}\right]}{\left\{-\frac{\sigma^2}{1+\rho}\left[T_1-\frac{1-\rho^{2T_1}}{1-\rho^2}\right]+\sigma_\alpha^2\left[1-\rho^{T_1}-\frac{1-\rho^{2T_1}}{1+\rho}\right]\right\}^2}$$

$$= \frac{2(1+\rho)}{T_1}+O\left(\frac{1}{T_1^{3/2}}\right), \quad (0.23)$$

giving the stated result for (i). The error magnitude as $T\to\infty$ in the asymptotic expansion (0.23) is justified as follows. Since $c<0$ is fixed we have $1-\rho^2=-2\frac{c}{\sqrt{T}}-\frac{c^2}{T}$ and

$$\frac{1-\rho^{2T}}{1-\rho^2}=\frac{1-\left[1+\frac{c}{\sqrt{T}}\right]^{2T}}{-2\frac{c}{\sqrt{T}}-\frac{c^2}{T}}\sim\frac{1-e^{c\frac{2T}{\sqrt{T}}}}{-2\frac{c}{\sqrt{T}}-\frac{c^2}{T}}\sim\frac{\sqrt{T}}{-2c}. \tag{0.24}$$

Then, by direct calculation as $T\to\infty$

$$\begin{aligned}\frac{\omega_{NT}}{\omega_{DT}} &= \frac{\frac{2\sigma^4T_2}{1+\rho}+\sigma^4\frac{2\rho}{(1+\rho)}\left(\frac{\sqrt{T_2}}{-2c}\right)+\sigma_\alpha^2\sigma^2\left(1-e^{cT_2/\sqrt{T}}\right)\left[1-e^{cT_2/\sqrt{T}}-c\frac{1+e^{cT_2/\sqrt{T}}}{\sqrt{T}(1+\rho)}\right]}{\left\{-\frac{\sigma^2}{1+\rho}\left[T_1-\left(\frac{\sqrt{T}}{-2c}\right)\left(1-e^{cT_1/\sqrt{T}}\right)\right]+\sigma_\alpha^2\left[1-e^{cT_1/\sqrt{T}}-\frac{1-e^{2cT_1/\sqrt{T}}}{1+\rho}\right]\right\}^2}\\ &= \frac{2\left(1+\rho\right)}{T_1}+O\left(\frac{1}{T_1^{3/2}}\right).\end{aligned} \tag{0.25}$$

The sequential limit theory (ii) follows directly from (i) and the asymptotic expansion (0.23) of $\omega_T^2$.

If $\rho=1+\frac{c}{T^\gamma}$ with $\gamma\in(0,1)$, it is clear that the above fixed $(T,c)$ limit theory as $n\to\infty$ continues to hold. Then, as $T\to\infty$, we have in place of (0.24)

$$\frac{1-\rho^{2T}}{1-\rho^2}=\frac{1-\left[1+\frac{c}{T^\gamma}\right]^{2T}}{-2\frac{c}{T^\gamma}-\frac{c^2}{T^\gamma}}\sim\frac{1-e^{2cT^{1-\gamma}}}{-2\frac{c}{T^\gamma}-\frac{c^2}{T^\gamma}}\sim\frac{T^\gamma}{-2c}$$

leading to

$$\frac{\omega_{NT}}{\omega_{DT}}=\frac{2\left(1+\rho\right)}{T_1}+O\left(\frac{1}{T_1^{2-\gamma}}\right)\sim\frac{4}{T_1}+O\left(\frac{1}{T_1^{2-\gamma}}\right).$$

It follows that (ii) continues to hold with the same convergence rate $\sqrt{nT}$ and same limit variance 4 for all $\gamma\in(0,1)$.

When $\gamma=1$, the sequential normal limit theory in (ii) still holds but the variance of the limiting distribution changes. Observe that in this case

$$\frac{1-\rho^{2T}}{1-\rho^2}=\frac{1-\left[1+\frac{c}{T}\right]^{2T}}{-2\frac{c}{T}-\frac{c^2}{T}}\sim\frac{1-e^{2c}}{-2\frac{c}{T}-\frac{c^2}{T}}\sim\frac{T\left(1-e^{2c}\right)}{-2c}.$$

Using (0.21) we then have the following limit behavior as $T \to \infty$

$$\frac{\omega_{NT}}{\omega_{DT}} \sim \frac{\sigma^4 T_2 + \sigma^4 T_2 \frac{\left(1-e^{2c}\right)}{-2c} + O(1)}{\left\{-\frac{\sigma^2 T_1}{2}\left[1 - \frac{\left(1-e^{2c}\right)}{-2c}\right] + O(1)\right\}^2} = \frac{4}{T_1} \frac{1 + \frac{\left(1-e^{2c}\right)}{-2c}}{\left[1 - \frac{\left(1-e^{2c}\right)}{-2c}\right]^2} \{1 + o(1)\},$$

so that

$$\omega_T^2 = \frac{-8c}{T_1} \frac{\left(1 - 2c - e^{2c}\right)}{\left(1 + 2c - e^{2c}\right)^2} \{1 + o(1)\}.$$

Hence

$$\sqrt{nT}\left(\rho_{IV} - \rho\right) \underset{(T,n)_{\text{seq}} \to \infty}{\Rightarrow} N\left(0, (-8c) \frac{\left(1 - 2c - e^{2c}\right)}{\left(1 + 2c - e^{2c}\right)^2}\right), \tag{0.26}$$

so the $\sqrt{nT}$ Gaussian limit theory holds but with a different variance when $\rho = 1 + \frac{c}{T}$. Observe that

$$(-8c) \frac{\left(1 - 2c - e^{2c}\right)}{\left(1 + 2c - e^{2c}\right)^2} \sim \frac{8}{c^2} \to \infty \quad \text{as } c \to 0,$$

indicating that the variance in (0.26) diverges and the $\sqrt{nT}$ convergence rate fails as the unit root is approached via $c \to 0$.

Next, examine the case where $\rho = 1 + \frac{c}{T^\gamma}$ with $\gamma > 1$ and $c < 0$, so that $\rho$ is in the immediate vicinity of unity, closer than the LUR case but still satisfying $\rho < 1$ for fixed $T$. In that case, we still have Gaussian limit theory as $n \to \infty$ because $|\rho| < 1$. To find the limit theory as $(T, n)_{\text{seq}} \to \infty$ we consider the behavior of the numerator and denominator of $\omega_T$. First, note that $\log\left[1 + \frac{c}{T^\gamma}\right]^{2T} = \frac{2c}{T^{\gamma-1}} - \frac{c^2}{T^{2\gamma-1}} + O\left(\frac{1}{T^{3\gamma-1}}\right)$ so that $\left[1 + \frac{c}{T^\gamma}\right]^{2T} = 1 + \frac{2c}{T^{\gamma-1}} - \frac{c^2}{T^{2\gamma-1}} + \frac{1}{2}\left(\frac{2c}{T^{\gamma-1}}\right)^2 + O\left(\frac{1}{T^{2\gamma-2}}\right)$ giving

$$\begin{aligned}
\frac{1-\rho^{2T}}{1-\rho^2} &= \frac{1 - \left[1 + \frac{c}{T^\gamma}\right]^{2T}}{-2\frac{c}{T^\gamma} - \frac{c^2}{T^{2\gamma}}} = \frac{1 - \left[1 + \frac{2c}{T^{\gamma-1}} - \frac{c^2}{T^{2\gamma-1}} + \frac{1}{2}\left(\frac{2c}{T^{\gamma-1}}\right)^2 + O\left(\frac{1}{T^{3(\gamma-1)}}\right)\right]}{-2\frac{c}{T^\gamma} - \frac{c^2}{T^{2\gamma}}} \\
&= \frac{-2cT + \frac{c^2}{T^{\gamma-1}} - \frac{2c^2}{T^{\gamma-2}} + o\left(\frac{1}{T^{\gamma-2}}\right)}{-2c\left\{1 + \frac{1}{2}\frac{c}{T^\gamma} + O\left(\frac{1}{T^{2\gamma}}\right)\right\}} = \left\{T + cT^{2-\gamma} + O\left(T^{1-\gamma}\right)\right\}\left\{1 + \frac{1}{2}\frac{c}{T^\gamma} + O\left(\frac{1}{T^{2\gamma}}\right)\right\}^{-1} \\
&= T + cT^{2-\gamma} + O\left(\frac{1}{T^{\gamma-1}}\right).
\end{aligned}$$

Using this result and $\omega_{DT} = v_{DT}^2$ with $v_{DT} = -\frac{\sigma^2}{1+\rho}\left[T_1 - \frac{1-\rho^{2T_1}}{1-\rho^2}\right] + \sigma_\alpha^2\left[1 - \rho^{T_1} - \frac{1-\rho^{2T_1}}{1+\rho}\right],$

we have

$$
\begin{aligned}
v_{DT} &= -\frac{\sigma^2}{1+\rho}\left[cT_1^{2-\gamma}\{1+o(1)\}\right]+\sigma_\alpha^2\left\{-\left[\frac{c}{T_1^{\gamma-1}}+\frac{T_1(T_1-1)}{2}\left(\frac{c}{T_1^\gamma}\right)^2+O\left(\frac{1}{T_1^{3(\gamma-1)}}\right)\right]\right.\\
&\quad\left.-\frac{-\frac{2c}{T_1^{\gamma-1}}-\frac{T_1(T_1-1)}{2}\left(\frac{c}{T_1^\gamma}\right)^2+O\left(\frac{1}{T_1^{3(\gamma-1)}}\right)}{2+\frac{2c}{T_1^{\gamma-1}}\{1+o(1)\}}\right\}\\
&= -\frac{c\sigma^2}{1+\rho}T_1^{2-\gamma}\{1+o(1)\}+\sigma_\alpha^2\left[-\frac{2c}{T_1^{\gamma-1}}-\frac{1}{2}\frac{c^2}{T_1^{2(\gamma-1)}}+O\left(\frac{1}{T_1^{3(\gamma-1)}}\right)\right.\\
&\quad\left.+\left\{\frac{c}{T_1^{\gamma-1}}+\frac{T_1(T_1-1)}{4}\left(\frac{c}{T_1^\gamma}\right)^2+O\left(\frac{1}{T_1^{3(\gamma-1)}}\right)\right\}\left\{1+\frac{c}{T_1^{\gamma-1}}\{1+o(1)\}\right\}^{-1}\right]\\
&= -\frac{c\sigma^2}{1+\rho}T_1^{2-\gamma}\{1+o(1)\}+\sigma_\alpha^2\left[-\frac{c}{T_1^{\gamma-1}}-\frac{1}{4}\frac{c^2}{T_1^{2(\gamma-1)}}+O\left(\frac{1}{T_1^{3(\gamma-1)}}\right)\right]\{1+o(1)\}\\
&= \left\{-\frac{c\sigma^2}{1+\rho}T_1^{2-\gamma}-c\sigma_\alpha^2\frac{c}{T_1^{\gamma-1}}\right\}\{1+o(1)\}=-\frac{c\sigma^2}{2}T_1^{2-\gamma}\{1+o(1)\},
\end{aligned}
$$

so that the denominator is $\omega_{DT}=\frac{c^2\sigma^4}{4}T_1^{4-2\gamma}\{1+o(1)\}$. The numerator $\omega_{NT}$ is

$$
\begin{aligned}
&\frac{2\sigma^4T_2}{1+\rho}+\sigma^4\frac{2\rho\left(1-\rho^{2T_2}\right)}{(1-\rho^2)(1+\rho)}+\sigma_\alpha^2\sigma^2\left(1-\rho^{T_2}\right)\left[1-\rho^{T_2}+\frac{(1-\rho)\left(1+\rho^{T_2}\right)}{1+\rho}\right]\\
=\;&\sigma^4T_2\left\{1-\frac{c}{T_1^{\gamma-1}}\right\}\{1+o(1)\}+\sigma^4\frac{\left(2+\frac{2c}{T^\gamma}\right)}{\left(2+\frac{c}{T^\gamma}\right)}\left\{T+cT^{2-\gamma}-\frac{1}{2}\frac{c}{T^{\gamma-1}}+O\left(\frac{T}{T^{2(\gamma-1)}}\right)\right\}\\
&+\sigma_\alpha^2\sigma^2\left\{-2cT_2-\frac{2T_2(2T_2-1)}{2}\frac{c^2}{T_2^\gamma}+O\left(\frac{T^3}{T^{2\gamma}}\right)\right\}\\
&\times\left\{-\left[\frac{c}{T_2^{\gamma-1}}+\frac{T_2(T_2-1)}{2}\left(\frac{c}{T_2^\gamma}\right)^2\{1+o(1)\}\right]-\frac{\frac{2c}{T^{\gamma-1}}\left[1+\frac{2c}{T_2^{\gamma-1}}+\frac{2T_2(2T_2-1)}{2}\left(\frac{c}{T_2^\gamma}\right)^2\{1+o(1)\}\right]}{2+\frac{2c}{T^{\gamma-1}}\{1+o(1)\}}\right\}\\
=\;&\left\{2\sigma^4T_2+2\sigma_\alpha^2\sigma^2(-c)T\left(\frac{-2c}{T_2^{\gamma-1}}\right)\right\}[1+o(1)]=2\sigma^4T_2[1+o(1)].
\end{aligned}
$$

Combining these results we obtain

$$\omega_T^2 = \frac{\omega_{NT}}{\omega_{DT}} = \frac{2\sigma^4 T_2 \left[1 + o(1)\right]}{\left\{\frac{c^2\sigma^4}{2} T_1^{4-2\gamma}\left\{1 + o(1)\right\}\right\}^2 \left\{1 + o(1)\right\}} = \frac{8T_2}{c^2 T_1^{2(2-\gamma)}} \left\{1 + o(1)\right\}.$$

It now follows that for $\rho = 1 + \frac{c}{T^\gamma}$ with $c < 0$ fixed and $\gamma > 1$

$$\sqrt{nT^{3-2\gamma}} \left(\rho_{IV} - \rho\right) \underset{(T,n)_{\text{seq}}\to\infty}{\Rightarrow} N\left(0, \frac{8}{c^2}\right).$$

Hence when $\rho$ is closer to unity than a local unit root, the $\sqrt{nT}$ rate of convergence is reduced to $\sqrt{n}T^{\frac{3-2\gamma}{2}}$. When $\gamma = \frac{3}{2}$ the rate of convergence is simply $\sqrt{n}$ and for $\gamma > \frac{3}{2}$ the large $n$ Gaussian asymptotic distribution $N\left(0, \omega_T^2\right)$ diverges as $T \to \infty$ because $\omega_T^2 = \frac{8T_2}{c^2 T_1^{2(2-\gamma)}} \left\{1 + o(1)\right\}$ diverges with $T$. In this event, sequential $(T,n)_{\text{seq}} \to \infty$ asymptotics fail. In effect, the convergence rate is slower than $\sqrt{n}$ and the non-Gaussian Cauchy limit theory cannot be captured in these $(T,n)_{\text{seq}}$ directional sequential asymptotics even though $\rho = 1 + \frac{c}{T^\gamma}$ with $\gamma > 1$ is in closer proximity to a unit root than the usual local unit root case with $\gamma = 1$. ■

**Proof of Theorem 5.** In the mildly integrated case where $\rho = 1 + \frac{c}{\sqrt{T}}$ we have $y_{it} = -\frac{\alpha_i c}{\sqrt{T}} + \left(1 + \frac{c}{\sqrt{T}}\right) y_{it-1} + u_{it}$ and $\Delta y_{it} = \rho \Delta y_{it-1} + \Delta u_{it}$ so that $\Delta y_{it} = -\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-1} + u_{it} = \alpha_i (1-\rho) + (\rho - 1) y_{it-1} + u_{it}$. By partial summation, as shown above, we have $\sum_{t=2}^T \Delta u_{it} y_{it-2} = u_{iT} y_{iT-2} - u_{i1} y_{i0} - \sum_{t=3}^T u_{it-1} \Delta y_{it-2}$, so that

$$\rho_{IV} - \rho = \frac{\sum_{i=1}^n \left\{\left(u_{iT} y_{iT-2} - u_{i1} y_{i0}\right) - \sum_{t=3}^T u_{it-1}\left[-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-3} + u_{it-2}\right]\right\}}{\sum_{i=1}^n \sum_{t=2}^T \left\{-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-2} + u_{it-1}\right\} y_{it-2}}. \quad (0.27)$$

Rescaling and using $y_{i0} = 0$ gives

$$\sqrt{T}\left(\rho_{IV} - \rho\right) = \frac{\sum_{i=1}^n \frac{1}{\sqrt{T}} \left\{u_{iT} y_{iT-2} - \sum_{t=3}^T u_{it-1}\left[-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-3} + u_{it-2}\right]\right\}}{\sum_{i=1}^n \frac{1}{T} \sum_{t=2}^T \left\{-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}} y_{it-2} + u_{it-1}\right\} y_{it-2}}. \quad (0.28)$$

Since $\frac{1}{\sqrt{T}} \sum_{t=2}^T u_{it-1} u_{it-2} \Rightarrow G_i \equiv N\left(0, \sigma^4\right)$, $\frac{1}{T} \sum_{t=2}^T u_{it-1} = o_p(1)$, and $\frac{1}{T} \sum_{t=2}^T u_{it-1} y_{it-3} =$

$o_p(1)$ – see (0.30) below – the numerator is

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left\{u_{iT}y_{iT-2}-\sum_{t=3}^{T}u_{it-1}\left[-\frac{\alpha_i c}{\sqrt{T}}+\frac{c}{\sqrt{T}}y_{it-3}+u_{it-2}\right]\right\}\\ = &-\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=3}^{T}u_{it-1}u_{it-2}+o_p(1)\Rightarrow-\sum_{i=1}^{n}G_i(1).\end{aligned}\tag{0.29}$$

Using Phillips and Magdalinos (2007, theorem 3.2) we find that

$$T^{-3/2}\sum_{t=2}^{T}y_{it}^2\to_p\frac{\sigma^2}{-2c},T^{-3/4}\sum_{t=2}^{T}y_{it-1}u_{it}\Rightarrow N\left(0,\frac{\sigma^4}{-2c}\right),\text{ and }T^{-3/2}\sum_{t=2}^{T}y_{it}=o_p(1).\tag{0.30}$$

The denominator of (0.28) therefore satisfies

$$\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\left\{-\frac{\alpha_i c}{\sqrt{T}}+\frac{c}{\sqrt{T}}y_{it-2}+u_{it-1}\right\}y_{it-2}\to_p\sum_{i=1}^{n}\left\{c\frac{\sigma^2}{-2c}\right\}.\tag{0.31}$$

Hence, using (0.29) and (0.31) we have $\sqrt{T}(\rho_{IV}-\rho)\underset{T\to\infty}{\Rightarrow}\frac{2}{n\sigma^2}\sum_{i=1}^{n}G_i=\frac{2}{n}\sum_{i=1}^{n}\zeta_i$ where $\zeta_i\sim_{iid}N(0,1)$. Then

$$\sqrt{nT}(\rho_{IV}-\rho)\underset{T\to\infty}{\Rightarrow}\frac{2}{\sqrt{n}}\sum_{i=1}^{n}\zeta_i\underset{n\to\infty}{\Rightarrow}N(0,4),\tag{0.32}$$

which gives (i) and then leads directly to the sequential limit $\sqrt{nT}(\rho_{IV}-\rho)\underset{(n,T)_{\text{seq}}\to\infty}{\Rightarrow}N(0,4)$. ■

**Proof of Theorem 6.** The proof follows the same lines as the proof of Theorem 3 above. As before, we define the vector of standardized components appearing in the numerator and denominator of $\sqrt{T}(\rho_{IV}-\rho)$ in (0.28)

$$X_{nT}=(X_{1nT},X_{2nT})':=\left(n^{-1/2}\sum_{i=1}^{n}Y_{1iT},n^{-1}\sum_{i=1}^{n}Y_{2iT}\right),$$

where $Y_{iT} = (Y_{1iT}, Y_{2iT})'$ with

$$\begin{aligned} Y_{1iT} &= \frac{1}{\sqrt{T}}\left\{u_{iT}y_{iT-2} - \sum_{t=3}^{T} u_{it-1}\left[-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}}y_{it-3} + u_{it-2}\right]\right\} \underset{T\to\infty}{\Rightarrow} Y_{1i} := -G_i\left(1\right), \\ Y_{2iT} &= \frac{1}{T}\sum_{t=2}^{T}\left\{-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}}y_{it-2} + u_{it-1}\right\} y_{it-2} \underset{T\to\infty}{\Rightarrow} Y_{2i} := c\frac{\sigma^2}{-2c}. \end{aligned}$$

From (0.29) and (0.31) we have the sequential convergence

$$\begin{aligned} X_{nT} \underset{T\to\infty}{\Rightarrow} X_n \quad &: = \left(-n^{-1/2}\sum_{i=1}^{n} G_i\left(1\right), n^{-1}\sum_{i=1}^{n}\left\{c\frac{\sigma^2}{-2c}\right\}\right)' \\ \underset{n\to\infty}{\Rightarrow} X \quad &: = \left(\sigma^2\zeta, -\frac{\sigma^2}{2}\right), \text{ where } \zeta = N\left(0,1\right), \end{aligned} \tag{0.33}$$

which in turn implies the sequential limit $\sqrt{nT}\left(\rho_{IV} - \rho\right) \underset{n\to\infty, T\to\infty}{\Rightarrow} N\left(0,4\right)$ given in (0.12). Since $X_{nT} \underset{T\to\infty}{\Rightarrow} X_n \underset{n\to\infty}{\Rightarrow} X$ sequentially, joint weak convergence $X_{nT} \Rightarrow X$ as $(n,T) \to \infty$ holds in the same manner as Theorem 3 with only minor definitional changes. First, $Y_{iT}$ is integrable just as before. To show Lemma A(i) holds, observe that

$$\begin{aligned} & \mathbb{E}\left\|Y_{iT}\right\|^2 = \mathbb{E}Y_{1iT}^2 + \mathbb{E}Y_{2iT}^2 \\ = & \frac{1}{T}\mathbb{E}\left\{u_{iT}y_{iT-2} - \sum_{t=3}^{T} u_{it-1}\left(-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}}y_{it-3} + u_{it-2}\right)\right\}^2 \\ & + \frac{1}{T^2}\mathbb{E}\left\{\sum_{t=2}^{T}\left(-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}}y_{it-2} + u_{it-1}\right) y_{it-2}\right\}^2 \\ = & \frac{\sigma^2}{T}\mathbb{E}y_{iT-2}^2 + \frac{1}{T}\mathbb{E}\left\{\sum_{t=3}^{T} u_{it-1}\left(-\frac{\alpha_i c}{\sqrt{T}} + \frac{c}{\sqrt{T}}y_{it-3} + u_{it-2}\right)\right\}^2 \\ = & \frac{\sigma^2}{T}\mathbb{E}y_{iT-2}^2 + \frac{1}{T}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}u_{it-2}\right)^2 + \frac{c^2\alpha_i^2}{T^2}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}\right)^2 + \frac{c^2}{T^2}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}y_{it-3}\right)^2 \\ & - \frac{2\alpha_i c}{T^{3/2}}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}\sum_{s=3}^{T} u_{is-2}\right) - \frac{2\alpha_i c^2}{T^2}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}\sum_{s=3}^{T} y_{is-3}\right) + \frac{2c}{T^{3/2}}\mathbb{E}\left(\sum_{t=3}^{T} u_{it-1}u_{it-2}\sum_{s=3}^{T} y_{is-3}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma^4}{-2c}\frac{\sqrt{T_2}}{T} + \sigma^4\frac{T_2}{T} + O\left(\frac{1}{T}\right) + \frac{c^2\sigma^2}{T^2}\sum_{t=3}^{T}\mathbb{E}y_{it-3}^2 + O\left(\frac{1}{\sqrt{T}}\right) + O\left(\frac{1}{T}\right) + O\left(\frac{1}{\sqrt{T}}\right) \\
&= \sigma^4\frac{T_2}{T} + o(1),
\end{aligned}
$$

since from (0.24) $\mathbb{E}\left(y_{it}^2\right) = \sigma^2\frac{1-\rho^{2t}}{1-\rho^2} + \alpha_i^2\left(1-\rho^t\right)^2 = \sigma^2\frac{\sqrt{t}}{-2c}\left\{1+o(1)\right\}$. Then $\mathbb{E}\left\|Y_{iT}\right\|^2 < \infty$ and we deduce that

$$
\limsup_{n,T\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\|Y_{iT}\right\| = \limsup_{T\to\infty}\mathbb{E}\left\|Y_{iT}\right\| \le \limsup_{T\to\infty}\left(\mathbb{E}\left\|Y_{iT}\right\|^2\right)^{1/2} < \infty,
$$

as required. Condition (ii) holds, as we again have $\mathbb{E}Y_{iT} = \mathbb{E}Y_i = 0$; and condition (iii) and (iv) hold because $\sup_T \mathbb{E}\left\|Y_{iT}\right\|^2 < \infty$ and $\mathbb{E}\left\|Y_i\right\|^2 < \infty$. We then have joint weak convergence

$$
X_{nT} = n^{-1/2}\sum_{i=1}^{n}Y_{iT} \underset{n,T\to\infty}{\Rightarrow} X := \left(\sigma^2\zeta, \frac{\sigma^2}{2}\right),
$$

irrespective of the divergence rates of $n$ and $T$ to infinity. By continuous mapping, the required result follows for the IV estimator so that $\sqrt{T}\left(\rho_{IV}-\rho\right) \underset{n,T\to\infty}{\Rightarrow} N(0,4)$ holds jointly as $(n,T)\to\infty$ irrespective of the order and rates of divergence. ■

**Proof of Theorem 7.** We have $\rho = 1 + \frac{c}{T^\gamma}$ for some fixed $c<0$ and let $T\to\infty$. In this case, $y_{it} = -\frac{\alpha_i c}{T^\gamma} + \left(1+\frac{c}{T^\gamma}\right)y_{it-1} + u_{it}$ and $\Delta y_{it} = \rho\Delta y_{it-1} + \Delta u_{it}$ so that $\Delta y_{it} = -\frac{\alpha_i c}{T^\gamma} + \frac{c}{T^\gamma}y_{it-1} + u_{it}$. As before, we have

$$
\sqrt{T}\left(\rho_{IV}-\rho\right) = \frac{\frac{1}{\sqrt{T}}\sum_{i=1}^{n}\left\{\left(u_{iT}y_{iT-2} - u_{i1}y_{i0}\right) - \sum_{t=3}^{T}u_{it-1}\left[-\frac{\alpha_i c}{T^\gamma} + \frac{c}{T^\gamma}y_{it-3} + u_{it-2}\right]\right\}}{\frac{1}{T}\sum_{i=1}^{n}\sum_{t=2}^{T}\left\{-\frac{\alpha_i c}{T^\gamma} + \frac{c}{T^\gamma}y_{it-2} + u_{it-1}\right\}y_{it-2}}. \tag{0.34}
$$

We use the following results from Phillips and Magdalinos (2007) and Magdalinos and Phillips (2009), which hold for all $\gamma\in(0,1)$,

$$
T^{-1-\gamma}\sum_{t=2}^{T}y_{it}^2 \to_p \frac{\sigma^2}{-2c}, T^{-(1+\gamma)/2}\sum_{t=2}^{T}y_{it-1}u_{it} \Rightarrow N\left(0,\frac{\sigma^4}{-2c}\right), \text{ and } T^{-1/2-\gamma}\sum_{t=2}^{T}y_{it} = O_p(1). \tag{0.35}
$$

Then, since $\frac{1}{\sqrt{T}}\sum_{t=2}^{T}u_{it-1}u_{it-2} \Rightarrow G_i \equiv N\left(0,\sigma^4\right)$, $\frac{1}{T^{1/2+\gamma}}\sum_{t=2}^{T}u_{it-1} = o_p(1)$, and $\frac{1}{T^{1/2+\gamma}}\sum_{t=2}^{T}u_{it-1}y_{it-3} =$

$o_p(1)$ when $\gamma \in (0,1)$, the numerator of (0.34) is

$$\begin{aligned} & \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \left\{ u_{iT} y_{iT-2} - \sum_{t=3}^{T} u_{it-1} \left[ -\frac{\alpha_i c}{T^{\gamma}} + \frac{c}{T^{\gamma}} y_{it-3} + u_{it-2} \right] \right\} \\ = & -\sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=3}^{T} u_{it-1} u_{it-2} + o_p(1) \Rightarrow -\sum_{i=1}^{n} G_i(1). \end{aligned}$$

Using (0.35), we find that the denominator of (0.34) satisfies

$$\sum_{i=1}^{n} \frac{1}{T} \sum_{t=2}^{T} \left\{ -\frac{\alpha_i c}{T^{\gamma}} + \frac{c}{T^{\gamma}} y_{it-2} + u_{it-1} \right\} y_{it-2} \to_p \sum_{i=1}^{n} \left\{ c \frac{\sigma^2}{-2c} \right\} = -\frac{\sigma^2 n}{2}.$$

Hence, as $T \to \infty$

$$\sqrt{T} \left( \rho_{IV} - \rho \right) \underset{T\to\infty}{\Rightarrow} \frac{-\sum_{i=1}^{n} G_i}{-\frac{\sigma^2}{2} n} = \frac{\sigma^2 \sum_{i=1}^{n} \zeta_i}{-\frac{\sigma^2}{2} n}, \quad \text{where } \zeta_i \sim_{iid} N(0,1).$$

Then, as $T \to \infty$ is followed by $n \to$, we have

$$\sqrt{nT} \left( \rho_{IV} - \rho \right) \underset{T\to\infty}{\Rightarrow} \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i \underset{n\to\infty}{\Rightarrow} N(0,4), \text{ for all } \gamma \in (0,1).$$

Next consider the case $\gamma = 1$. The numerator of (0.34) is then

$$\begin{aligned} & \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \left\{ u_{iT} y_{iT-2} - \sum_{t=3}^{T} u_{it-1} \left[ -\frac{\alpha_i c}{T} + \frac{c}{T} y_{it-3} + u_{it-2} \right] \right\} \\ = & \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \left\{ u_{iT} y_{iT-2} - \sum_{t=3}^{T} u_{it-1} u_{it-2} \right\} + o_p(1) \underset{T\to\infty}{\Rightarrow} u_{i\infty} K_{ci}(1) - \sum_{i=1}^{n} G_i(1), \end{aligned}$$

since by standard functional limit theory for near integrated processes (Phillips, 1987b) we have

$$\left( \frac{1}{T^{1/2}} y_{iT}, \frac{1}{T} \sum_{t=2}^{T} y_{it-1} u_{it} \right) \underset{T\to\infty}{\Rightarrow} \left( K_{ci}(r), \int_0^1 K_{ci} dB_i \right),$$

where $B_i(r) =: \sigma W_i(r)$ are *iid* Brownian motions with common variance $\sigma^2$, and $K_{ci}(r) =$

$\int_0^r e^{c(r-s)} dB_i(s) =: \sigma J_{ci}(r)$ is a linear diffusion. The denominator of (0.34) satisfies

$$\sum_{i=1}^{n} \frac{1}{T} \sum_{t=2}^{T} \left\{ -\frac{\alpha_i c}{T} + \frac{c}{T} y_{it-2} + u_{it-1} \right\} y_{it-2} \underset{T\to\infty}{\Rightarrow} \sum_{i=1}^{n} \left\{ c \int_0^1 K_{ci}(r)^2 dr + \int_0^1 K_{ci} dB_i \right\}.$$

Hence

$$\sqrt{T}(\rho_{IV} - \rho) \underset{T\to\infty}{\Rightarrow} \frac{\sum_{i=1}^{n} \{u_{i\infty} K_{ci}(1) - \sum_{i=1}^{n} G_i\}}{\sum_{i=1}^{n} \left\{ c \int_0^1 K_{ci}(r)^2 dr + \int_0^1 K_{ci}(r) dB_i \right\}} = \frac{\sum_{i=1}^{n} \left\{ \left(\sigma^{-1} u_{i\infty}\right) J_{ci}(1) - \zeta_i \right\}}{\sum_{i=1}^{n} \left\{ c \int_0^1 J_{ci}(r)^2 dr + \int_0^1 J_{ci} dW_i \right\}}, \tag{0.36}$$

where the $\zeta_i \sim_{iid} N(0,1)$ and are independent of the $W_i$ and $u_{i\infty}$ for all $i$. This gives the first part of (ii). Scaling the numerator and denominator of (0.36), noting that $\int_0^1 J_{ci}(r) dW_i$ has zero mean and finite variance, and using the independence of $\zeta_i, u_{i\infty}$, and $W_i$, we obtain

$$\begin{aligned} \sqrt{nT}(\rho_{IV} - \rho) \underset{T\to\infty}{\Rightarrow} & \quad \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \left(\sigma^{-1} u_{i\infty}\right) J_{ci}(1) - \zeta_i \right\}}{\frac{1}{n} \sum_{i=1}^{n} \left\{ c \int_0^1 J_{ci}(r)^2 dr + \int_0^1 J_{ci}(r) dW_i \right\}} \\ \underset{n\to\infty}{\Rightarrow} \frac{N\left(0, \frac{1-2c-e^{2c}}{-2c}\right)}{c\mathbb{E}\left(\int_0^1 J_{ci}(r)^2 dr\right)} & = N\left(0, -8c \frac{1-2c-e^{2c}}{\left(e^{2c}-1-2c\right)^2}\right), \end{aligned}$$

since, using results in Phillips (1987b), we have $\mathbb{E}\left(\int_0^1 J_{ci}(r)^2 dr\right) = \frac{e^{2c}-1-2c}{(2c)^2}$ and

$$\mathbb{E}\left\{ \left(\sigma^{-1} u_{i\infty}\right) J_{ci}(1) - \zeta_i \right\}^2 = \mathbb{E}\left(\sigma^{-1} u_{i\infty}\right)^2 \mathbb{E} J_{ci}(1)^2 + \mathbb{E}\zeta_i^2 = 1 + \frac{1-e^{2c}}{-2c} = \frac{1-2c-e^{2c}}{-2c}.$$

Hence, when $\gamma = 1$, we have

$$\sqrt{nT}(\rho_{IV} - \rho) \underset{(n,T)\to\infty}{\Rightarrow} N\left(0, (-8c) \frac{1-2c-e^{2c}}{\left(e^{2c}-1-2c\right)^2}\right) \tag{0.37}$$

From Lemma 2 of Phillips (1987b) we have

$$\left( (-2c) \int_0^1 J_{ci}(r)^2 dr, (-2c)^{1/2} \int_0^1 J_{ci}(r) dW_i \right) \underset{c\to 0}{\Rightarrow} (1, Z_i), \quad Z_i \sim_{iid} N(0,1,)$$

and $\frac{1-2c-e^{2c}}{-2c} = 2\{1+o(1)\}$ as $c \to 0$, so that

$$(-8c)\frac{1-2c-e^{2c}}{(e^{2c}-1-2c)^2} \sim (-8c)\frac{(-4c)}{\left\{\frac{1}{2}(2c)^2\right\}^2} = \frac{8}{c^2} \quad \text{for small } c \sim 0 \tag{0.38}$$

which explodes as $c \to 0$, consonant with the unit root case where we only have $\sqrt{T}$ convergence. Observe that both (0.37) and (0.38) correspond to earlier results with the reverse order of sequential convergence $(T,n)_{\text{seq}} \to \infty$.

Next suppose $\gamma > 1$ so that $\rho = 1 + \frac{c}{T^\gamma}$ is closer to unity than the LUR case with $\gamma = 1$. In this case, the numerator and denominator of (0.34) have the same limits as in the unit root case, viz.,

$$\begin{aligned} &\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left\{u_{iT}y_{iT-2} - \sum_{t=3}^{T}u_{it-1}\left[-\frac{\alpha_i c}{T^\gamma} + \frac{c}{T^\gamma}y_{it-3} + u_{it-2}\right]\right\} \\ = &\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\left\{u_{iT}y_{iT-2} - \sum_{t=3}^{T}u_{it-1}u_{it-2}\right\} + o_p(1) \underset{T\to\infty}{\Rightarrow} \sum_{i=1}^{n}\left\{u_{i\infty}B_i(1) - G_i\right\}, \end{aligned}$$

and

$$\sum_{i=1}^{n}\frac{1}{T}\sum_{t=2}^{T}\left\{-\frac{\alpha_i c}{T^\gamma} + \frac{c}{T^\gamma}y_{it-2} + u_{it-1}\right\}y_{it-2} \underset{T\to\infty}{\Rightarrow} \sum_{i=1}^{n}\left\{\int_0^1 B_i dB_i\right\}.$$

Then

$$\sqrt{T}(\rho_{IV} - \rho) \underset{T\to\infty}{\Rightarrow} \frac{\sum_{i=1}^{n}\{u_{i\infty}B_i(1) - G_i\}}{\sum_{i=1}^{n}\int_0^1 B_i dB_i} = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\left(\sigma^{-1}u_{i\infty}\right)W_i(1) - \zeta_i\right\}}{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^1 W_i dW_i} \underset{n\to\infty}{\Rightarrow} 2\mathbb{C},$$

since $\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\left(\sigma^{-1}u_{i\infty}\right)W_i(1) - \zeta_i\right\}, \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_0^1 W_i dW_i\right) \underset{n\to\infty}{\Rightarrow} N\left(0, \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}\right).$

■

# 1 References


Magdalinos, T. and P. C. B. Phillips (2009). "Limit theory for cointegrated systems with moderately integrated and moderately explosive regressors", 25, 482-526.

Phillips, P. C. B. (1987a). "Time Series Regression with a Unit Root," *Econometrica,* 55, 277–302.

Phillips, P. C. B. (1987b). "Towards a Unified Asymptotic Theory for Autoregression," *Biometrika* 74, 535–547.

Phillips, P. C. B. (1989). "Partially identified econometric models," *Econometric Theory* 5, 181–240.

Phillips, P. C. B., T. Magdalinos (2007), "Limit Theory for Moderate Deviations from a Unit Root," *Journal of Econometrics* 136, 115-130.

Phillips, P.C.B. and H.R. Moon (1999) : Linear Regression Limit Theory for Nonstationary Panel Data," *Econometrica,* 67, 1057-1111.