SUPPLEMENT TO "NEYMAN'S $C(\alpha)$ TEST FOR UNOBSERVED HETEROGENEITY"

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Appendix A. Power Comparison for Example 4.2

Here we conduct a power comparison between the $C(\alpha)$ test and the Breusch and Pagan (1979) LM test described in Example 4.2. Two different alternative β_i distributions are considered. The first one assumes β_i takes value 0 for $i = 1, \ldots, N/2$ and $cN^{-1/4}$ for $i = N/2+1, \ldots, N$. We let c take 51 distinct values equally spaced from 0 to $\sqrt{50}$. The second case assumes $\beta_i \sim \mathcal{N}(0, \sigma^2)$ with σ taking 21 distinct values from 0 to 1. For simplicity, we consider the case with dimension two, where both x covariates are standard normal variables. The sample size is fixed at 400 with 10000 replications. Figure 1 presents the power curve for the 10 % nominal level. In both cases, the $C(\alpha)$ test dominate the power curve of the LM test based on the usual χ^2 asymptotics.

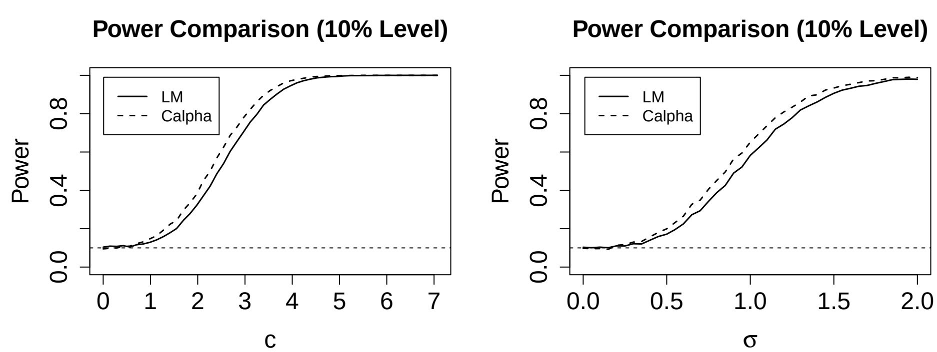

Figure 1. Power Comparison of Slope Heterogeneity Test for Linear Regression Model: The left figure corresponds to the first experiment and the right to the second. The dotted curve corresponds to the power curve of the $C(\alpha)$ test based on the mixture of χ^2 asymptotics and the solid curve for the LM test based on the χ^2_3 asymptotics.

Appendix B. Computational Details in Examples 4.3

Joint test for Gaussian panel data model. The information matrix for $(\xi, \theta) = (\xi_1, \xi_2, \mu_0, \sigma_0^2)$ is

$$I = \begin{pmatrix} I_{\xi\xi} & I_{\xi\theta} \\ I_{\theta\xi} & I_{\theta\theta} \end{pmatrix} = \frac{NT}{\sigma_0^4} \begin{pmatrix} 2T & \sigma_0^2 & 0 & 1 \\ \sigma_0^2 & (T+3)\sigma_0^4/2 & 0 & \sigma_0^2/2 \\ 0 & 0 & \sigma_0^2 & 0 \\ 1 & \sigma_0^2/2 & 0 & 1/2 \end{pmatrix}$$

We further find

$$I_{\xi.\theta} = I_{\xi\xi} - I_{\xi\theta} I_{\theta\theta}^{-1} I_{\theta\xi} = \begin{pmatrix} 2NT(T-1)/\sigma_0^4 & 0 \\ 0 & NT(T/2+1) \end{pmatrix}$$

and

$$I_{\xi\theta} I_{\theta\theta}^{-1} = \begin{pmatrix} 0 & 2 \\ 0 & \sigma_0^2 \end{pmatrix}.$$

As we have remarked in Section 3, the diagonality of $I_{\xi.\theta}$ provides much convenience for finding the optimal test statistics. Denote

$$T_n := \begin{pmatrix} t_{1n} \\ t_{2n} \end{pmatrix} = I_{\xi.\theta}^{-1/2} \begin{pmatrix} \sum_i v_{i1} - 2\sum_i v_{4i} \\ \sum_i v_{2i} - \sigma_0^2 \sum_i v_{4i} \end{pmatrix} = \begin{pmatrix} (2NT(T-1)/\sigma_0^4)^{-1/2} \left(\sum_i (\frac{\bar{y}_{i.} - \mu_0}{\sigma_0^2/T})^2 - NT/\sigma_0^2\right) \\ (NT(T/2+1))^{-1/2} \left(\sum_i (Z_i - T/2)^2 - NT/2\right) \end{pmatrix}$$

Replacing (μ_0, σ_0^2) by their MLEs yields the joint $C(\alpha)$ test.

Appendix C. Claim in Section 5

Here we provide the detail derivation for the claim in Section 4 that the reparameterization adopted in Chesher (1984) and Cox (1983) for heterogeneity test requires extra moment conditions on U for second derivative of log density with respect to the test parameter to be bounded.

Proposition 1. For iid random variable $Y_1, \dots, Y_n$ each with density function $\int p(y; \lambda_0 + \tau\sqrt{\eta} u_i) dF(u_i)$, where U_i is a random variable with zero mean and unit variance. The second-order derivative of the log density with respect to η evaluated under $\eta = 0$ is unbounded unless $\mathbb{E}(U^3) = 0$ and $\mathbb{E}(U^4) < \infty$.

Proof. Denote the log density as $l = \log \int p(y;\lambda_0 + \tau\sqrt{\eta}u_i)dF(u_i)$. The first order derivative with respect to η is

$$\nabla_\eta l\big|_{\eta=0} = \frac{\tau \int \nabla_\lambda p(y;\lambda_0) u dF(u)}{2\sqrt{\eta}\int p(y;\lambda_0)dF(u)} = \frac{\tau^2}{2}\mathbb{E}(U^2)\frac{\nabla_\lambda^2 p(y;\lambda_0)}{p(y;\lambda_0)}$$

The last step is obtained by applying the l'Hôpital's rule.

The second order derivative is

$$\begin{aligned}\nabla_\eta^2 l\big|_{\eta=0} &= \frac{\tau^2\sqrt{\eta}\int \nabla_\lambda^2 p(y;\lambda_0)u^2 dF(u) - \tau\int \nabla_\lambda p(y;\lambda_0)u dF(u)}{4\eta\sqrt{\eta}\int p(y;\lambda_0)dF(u)}\bigg|_{\eta=0} - \left(\nabla_\eta l\big|_{\eta=0}\right)^2 \\ &= \frac{\tau^3\int \nabla_\lambda^3 p(y;\lambda_0)u^3 dF(u)}{12\sqrt{\eta}\int p(y;\lambda_0)dF(u)}\bigg|_{\eta=0} - \left(\nabla_\eta l\big|_{\eta=0}\right)^2\end{aligned}$$

Provided that $\nabla_\lambda^3 p(y;\lambda_0)$ is not degenerately zero, $\nabla_\eta^2 l$ is unbounded unless $\mathbb{E}(U^3) = 0$ and $\mathbb{E}(U^4) < \infty$ so that we can apply l'Hôpital's rule again and get

$$\nabla_\eta^2 l\big|_{\eta=0} = \frac{\tau^4}{12}\left[\mathbb{E}(U^4)\frac{\nabla_\lambda^4 p(y;\lambda_0)}{p(y;\lambda_0)} - 3\mathbb{E}(U^2)^2\left(\frac{\nabla_\lambda^2 p(y;\lambda_0)}{p(y;\lambda_0)}\right)^2\right] < \infty$$

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References

Breusch, T. & A. Pagan (1979) A simple test for heteroscedasticity and random coefficient variation. *Econometrica* 47(5), 1287–1294.

Chesher, A. (1984) Testing for neglected heterogeneity. *Econometrica* 52(4), 865–872.

Cox, D. (1983) Some remarks on overdispersion. *Biometrika* 70(1), 269–274.