Supplementary Material to "Estimation of Stochastic Volatility Models by Nonparametric Filtering"

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We here provide proofs of some technical results, including Lemma D.1 and Theorem 7.1.

Proof of $\sup_{t\in[0,T]}|B_t| = o_{a.s.}(T^{1/2}\log T)$ as $T\to\infty$ (Section 3.3). By the reflection principle of the BM (e.g., Section 2.6 of Karatzas and Shreve, 1991), we have $\Pr\left[\max_{t\leq T} B_t > \bar{C}\right] = 2 - 2\Phi(\bar{C}/\sqrt{T})$ for any $\bar{C}>0$, and $\Pr\left[\min_{t\leq T} B_t < \underline{C}\right] = 2\Phi(\underline{C}/\sqrt{T})$ for any $\underline{C}<0$, where Φ denotes the cumulative distribution function of the standard normal. Then, for any $\bar{c}>0$,

$$\begin{aligned}&\sum\nolimits_{J=1}^{\infty}\Pr\left[\max\nolimits_{t\leq J} B_t > \bar{c}J^{1/2}(\log J)\right] = 2\sum\nolimits_{J=1}^{\infty}\{1-\Phi(\bar{c}\log J)\}\\ \leq\ & 2\sum\nolimits_{J=1}^{\infty}\frac{1}{\bar{c}(\log J)}\exp\{-\bar{c}^2(\log J)^2/2\}<\infty,\end{aligned}$$

where we have used the inequality $1-\Phi(x)\leq(1/x)\exp\left\{-x^2/2\right\}$ (Problem 9.22, Ch. 2, Karatzas and Shreve, 1991). We can analogously show that for any $\underline{c}<0$,

$$\sum\nolimits_{J=1}^{\infty}\Pr\left[\min\nolimits_{t\leq J} B_t < \underline{c}J^{1/2}\sqrt{\log J}\right]\leq 2\sum\nolimits_{J=1}^{\infty}\{1-\Phi(|\bar{c}|\log J)\}<\infty.$$

These inequalities, together with the Borel-Cantelli lemma, imply that $\max_{t\leq J}|B_t| = o_{a.s.}(J^{1/2}\log J)$ for any integer $J\geq 1$. For any arbitrary real number $T\geq 1$, we set $J=\lceil T\rceil+1$, where $\lceil T\rceil$ denotes the integer part of T, and then obtain

$$\max\nolimits_{t\leq T}|B_t|\leq\max\nolimits_{t\leq\lceil T\rceil+1}|B_t| = o_{a.s.}((\lceil T\rceil+1)^{1/2}\log(\lceil T\rceil+1)) = o_{a.s.}\left(T^{1/2}\log T\right).$$

■

Lemma D.1. *Suppose that a stochastic process $\{\Gamma_t\}_{t\geq 0}$ on a probability space $(\Omega,\mathfrak{F},P)$ satisfies the condition:*

$$E\left[|\Gamma_t-\Gamma_s|^a\right]\leq C\,|t-s|^{1+b}, \tag{0.1}$$

for some positive constants a, b and C each of which is independent of s and t. Then, there exists a continuous modification $\{\tilde{\Gamma}_t\}_{t\geq 0}$ of $\{\Gamma_t\}_{t\geq 0}$, which is a.s. Hölder continuous with exponent d for every $d\in(0,b/a)$ with a coefficient $\vartheta:=\sum_{J=1}^{\infty}J^{2+d}(1/J!)^d$:

$$\Pr\left[\omega\in\Omega\,\middle|\,\exists\bar{\Delta}(\omega)\ \text{s.t.}\sup_{|t-s|\in(0,\bar{\Delta}(\omega));\ s,t\in[0,\infty)}\frac{|\tilde{\Gamma}_t(\omega)-\tilde{\Gamma}_s(\omega)|}{|t-s|^d}\leq\vartheta\right]=1. \tag{0.2}$$

Proof. The following arguments proceed along the lines of the proof of Theorem 2.2.8 in Karatzas and Shreve (1991) where s,t are supposed to take values in some finite interval $[0,T]$ ($T=\bar{T}$ fixed).

We first prove the global Hölder property of the process on an enlarging interval, i.e., $[0,T]$ where $T\to\infty$, and next show that it actually holds over the infinite interval $[0,\infty)$. For any $\varepsilon>0$, we have

$$\Pr\left[\left|\Gamma_t-\Gamma_s\right|\geq\varepsilon\right]\leq\frac{E\left[\left|\Gamma_t-\Gamma_s\right|^a\right]}{\varepsilon^a}\leq C\varepsilon^{-a}\left|t-s\right|^{1+b}$$

by Čebyšev's inequality, and thus $\Gamma_t\overset{P}{\to}\Gamma_s$ as $s\to t$. Setting $t=km/m!$, $s=(k-1)\,m/m!$ and $\varepsilon=(m/m!)^d$, we obtain $\Pr\left[\left|\Gamma_{km/m!}-\Gamma_{(k-1)m/m!}\right|\geq(m/m!)^d\right]\leq C\,(m/m!)^{(1+b-ad)}$ and consequently,

$$\Pr\left[\max_{1\leq k\leq m!}\left|\Gamma_{km/m!}-\Gamma_{(k-1)m/m!}\right|\geq(m/m!)^d\right]\leq Cm^{(1+b-ad)}\Big/(m!)^{b-ad}\,.$$

By the fact that $\sum_{m=1}^{\infty}m^{(1+b-ad)}/(m!)^{b-ad}$ exists and the Borel-Cantelli lemma, there exists a set $\Omega^*\in\mathfrak{F}$ with $\Pr\left(\Omega^*\right)=1$ such that

$$\forall\omega\in\Omega^*,\exists m^*\left(\omega\right),\ \forall m\geq m^*\left(\omega\right):\max_{1\leq k\leq m!}\left|\Gamma_{km/m!}-\Gamma_{(k-1)m/m!}\right|<(m/m!)^d \tag{0.3}$$

, where m^* is a positive and integer-valued random variable. For each integer $m\,(\geq1)$ and any integer $l\geq m$, consider the following sets, $E_l^m:=\{km/l!\mid k=0,1,\ldots,l!\}$ and $E^m:=\bigcup_{l=1}^{\infty}E_l^m$. The set E_l^m consists of $(l!+1)$ points in $[0,m]$, while E^m consists of infinitely many points in $[0,m]$. Note that E^m is dense in $[0,m]$ for any m. Now fix $\omega\,(\in\Omega^*)$ and $m\,(\geq m^*\left(\omega\right))$. We shall show that

$$\forall l>m,\forall t,s\in E_l^m\text{ with }\left|t-s\right|\in(0,m/m!):\left|\Gamma_t\left(\omega\right)-\Gamma_s\left(\omega\right)\right|\leq2\sum_{J=m+1}^{l}J^2\left[J+1/\left((J+1)!\right)\right]^d. \tag{0.4}$$

To show this, we use the inductive method. First, we prove that the claim is true for $l=m+1$. For any $s,t\in E_{m+1}^m$ with $\left|t-s\right|\in(0,m/m!)$, there exist some $k_1,k_2\in\{0,1,\ldots,(m+1)!\}$ with $0\leq k_2-k_1\leq m$ such that

$$\begin{aligned}\left|\Gamma_t\left(\omega\right)-\Gamma_s\left(\omega\right)\right|\leq&\left|\Gamma_{k_1(m+1)/(m+1)!}\left(\omega\right)-\Gamma_{(k_1+1)(m+1)/(m+1)!}\left(\omega\right)\right|\\&+\left|\Gamma_{(k_1+1)(m+1)/(m+1)!}\left(\omega\right)-\Gamma_{(k_1+2)(m+1)/(m+1)!}\left(\omega\right)\right|\\&+\cdots+\left|\Gamma_{(k_2-1)(m+1)/(m+1)!}\left(\omega\right)-\Gamma_{k_2(m+1)/(m+1)!}\left(\omega\right)\right|.\end{aligned}$$

Each term on the right-hand side is bounded by $m\left[(m+2)\,/\,(m+2)!\right]^d$, which is implied by the fact $E_{m+1}^m\subset E_{m+2}^{m+2}$ and the inequality eq. (0.3). Thus, by the triangle inequalities, we have $\left|\Gamma_t\left(\omega\right)-\Gamma_s\left(\omega\right)\right|\leq m^2\left[(m+2)\,/\,(m+2)!\right]^d$. Second, suppose that eq. (0.4) is valid for $l=m+1,\ldots,L-1$. For $s<t$, $(s,t\in E_L^m)$ with $\left|t-s\right|\in(0,m/m!)$, consider the numbers $s_1:=\min\left\{u\in E_{L-1}^m:u\geq s\right\}$ and $t_1:=\max\left\{u\in E_{L-1}^m:u\leq t\right\}$, and notice that $s_1,t_1\in E_{L-1}^m\subset E_L^m\subset E_{L+1}^{L+1}$; $s,t\in E_L^m\subset E_{L+1}^{L+1}$; $s_1-s<m/\left(L-1\right)!$; and $t-t_1<m/\left(L-1\right)!$. By the inequality eq. (0.3) with $m=L+1$, $\left|\Gamma_{s_1}\left(\omega\right)-\Gamma_s\left(\omega\right)\right|\leq mL\left((L+1)\,/\,(L+1)!\right)^d$ and $\left|\Gamma_t\left(\omega\right)-\Gamma_{t_1}\left(\omega\right)\right|\leq mL\left((L+1)\,/\,(L+1)!\right)^d$. There are two possible relationships among s,t,s_1 and t_1: (i) if $\left|t-s\right|\geq m/\left(L-1\right)!$, it holds that $s\leq s_1\leq t_1\leq t$ (with at least one inequality strict); (ii) if $\left|t-s\right|<m/\left(L-1\right)!$, either of $\left|t-s\right|<\left|s_1-t_1\right|=m/\left(L-1\right)!$ or $\left|s_1-t_1\right|=0$. Thus, we have $|t_1 - s_1| \leq \max\{m/(L-1)!, |t-s|\} \leq m/m!$, and use the induction assumption (0.4) with $l = L-1$:

$$|\Gamma_{t_1}(\omega) - \Gamma_{s_1}(\omega)| \leq 2\sum_{J=m+1}^{L-1} J^2\left[\frac{J+1}{(J+1)!}\right]^d. \tag{0.5}$$

Therefore,

$$|\Gamma_t(\omega) - \Gamma_s(\omega)| \leq 2mL\left((L+1)/(L+1)!\right)^d + 2\sum_{J=m+1}^{L-1} J^2\left[\frac{J+1}{(J+1)!}\right]^d < 2\sum_{J=m+1}^{L} J^2\left[\frac{J+1}{(J+1)!}\right]^d.$$

We have shown eq. (0.4) for any $l\,(> m)$, as desired.

We can now show that $\{\Gamma_t(\omega) \mid t \in E^m\}$ is uniformly Hölder in t for $\forall\omega \in \Omega^*$ for any m. Consider any numbers $s, t \in E^m$ with $m \geq m^*\,(= m^*(\omega))$ and $|t-s| < \bar{\Delta}(\omega) \equiv m^*/m^*!$. Note that $E^m \subseteq E^{m'}$ for $m \leq m'$. We can pick some $m'\,(\geq m)$ such that $s, t \in E^{m'}$ with $(m'+2)/(m'+2)! \leq t-s < (m'+1)/(m'+1)!$. Then, by eq. (0.4), we obtain

$$|\Gamma_t(\omega) - \Gamma_s(\omega)| \leq 2\sum_{J=m'+1}^{\infty} J^2\left[\frac{J+1}{(J+1)!}\right]^d \leq \left[(m'+2)/(m'+2)!\right]^d \times \sum_{J=1}^{\infty} J^{2+d}(1/J!)^d$$

and thus, $|\Gamma_t(\omega) - \Gamma_s(\omega)|/|t-s|^d \leq c_h$ where $\vartheta := \sum_{J=1}^{\infty} J^{2+d}(1/J!)^d$. Note that the existence of ϑ can be checked by d'Alembert's criterion for any $d\,(> 0)$.

We define $\{\tilde{\Gamma}_t\}_{t\geq 0}$ as follows. For $\omega \notin \Omega^*$, set $\tilde{\Gamma}_t(\omega) = 0$, $t \in [0, m]$. For $\omega \in \Omega^*$ and $t \in E^m$, set $\tilde{\Gamma}_t(\omega) = \Gamma_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0, m] \cap (E^m)^c$, choose a sequence $\{s_n\}_{n=1}^{\infty}$with $s_n\,(\in E^m) \to t$; by the uniform continuity and the fact that s_n is Cauchy, $\{\Gamma_{s_n}(\omega)\}_{n=1}^{\infty}$ is also Cauchy, whose limit depends of t but not on the particular sequence $\{s_n\}$; and thus let $\tilde{\Gamma}_t(\omega) = \lim_{s_n\to t}\Gamma_{s_n}(\omega)$. Thus, the resulting process $\{\tilde{\Gamma}_t\}_{t\in[0,m]}$ is continuous, and is also uniformly Hölder in $t \in [0, m]$. We will show $\{\tilde{\Gamma}_t\}$ is a modification of $\{\Gamma_t\}$: observe that for $t \in E^m$, $\tilde{\Gamma}_t = \Gamma_t$ a.s.; for $t \in [0, m] \cap (E^m)^c$ and $\{s_n\}$ with $s_n\,(\in E^m) \to t$, we have $\Gamma_{s_n} \to \Gamma_t$ in probability (by eq. (0.1)) and $\Gamma_{s_n} \to \tilde{\Gamma}_t$ a.s., which implies $\tilde{\Gamma}_t = \Gamma_t$ a.s.

Let $m = [T] + 1$ with $[T]$ denoting the largest integer less than or equal to T. Now, we have proved that for any $\omega \in \Omega^*$, there exist some $m^*(\omega)$ and $\bar{\Delta}(\omega)\,(\equiv m^*/m^*!)$ such that $\forall m \geq m^*(\omega)$

$$\sup_{\substack{|t-s|\in(0,\bar{\Delta}(\omega))\\ t,s\in[0,T]}} \left|\tilde{\Gamma}_t(\omega) - \tilde{\Gamma}_s(\omega)\right| \Big/ |t-s|^d \leq \sup_{\substack{|t-s|\in(0,\bar{\Delta}(\omega))\\ t,s\in[0,m]}} \left|\tilde{\Gamma}_t(\omega) - \tilde{\Gamma}_s(\omega)\right| \Big/ |t-s|^d \leq \vartheta,$$

which implies that $\Pr(\Omega_1) = 1$, where

$$\Omega_1 := \left\{\exists\bar{\Delta}(\omega),\ \exists T^*,\ \forall T\,(\geq T^*),\ \sup_{|t-s|\in(0,\bar{\Delta}(\omega));\ s,t\in[0,T]} \frac{\left|\tilde{\Gamma}_t(\omega)-\tilde{\Gamma}_s(\omega)\right|}{|t-s|^d} \leq \vartheta\right\}. \tag{0.6}$$

Note that

$$\Omega_1 \subset \left\{\exists\bar{\Delta}(\omega),\ \exists T^*,\ \sup_{|t-s|\in(0,\bar{\Delta}(\omega));\ s,t\in[0,T^*]} \frac{\left|\tilde{\Gamma}_t(\omega)-\tilde{\Gamma}_s(\omega)\right|}{|t-s|^d} \leq \vartheta\right\}$$

$$= \left\{\exists\bar{\Delta}(\omega),\ \exists T^*,\ \forall T\,(\leq T_*),\ \sup_{|t-s|\in(0,\bar{\Delta}(\omega));\ s,t\in[0,T]} \frac{\left|\tilde{\Gamma}_t(\omega)-\tilde{\Gamma}_s(\omega)\right|}{|t-s|^d} \leq \vartheta\right\} =: \Omega_2. \tag{0.7}$$

Since $\Pr(\Omega_1) = 1$, we then have $\Pr(\Omega_2) = 1$. For any events $E, F \in \mathfrak{F}$, we have the inequality: $\Pr(E \cap F) \geq \Pr(E) + \Pr(F) - 1$. With $E = \Omega_1$ and $F = \Omega_2$, we obtain $\Pr(\Omega_1 \cap \Omega_2) = 1$, which, together with

$$\Omega_1 \cap \Omega_2 = \left\{ \omega \in \Omega \,\middle|\, \exists \bar{\Delta}(\omega),\ \forall T, \sup_{|t-s| \in (0, \bar{\Delta}(\omega));\ s,t \in [0,T]} \frac{\left|\tilde{\Gamma}_t(\omega) - \tilde{\Gamma}_s(\omega)\right|}{|t-s|^d} \leq \vartheta \right\},$$

implies the desired result, eq. (0.2). ∎

Proof of Theorem 7.1. Let

$$\begin{aligned} U_1(j) &:= 2\partial_{\theta_1}\alpha(\sigma^2_{\tau_j};\theta_1^*)\left[\alpha(\sigma^2_{\tau_j};\theta_1^*)\delta - (\sigma^2_{\tau_{j+1}} - \sigma^2_{\tau_j})\right]; \\ U_2(j) &:= 2\partial_{\theta_2}\beta^2(\sigma^2_{\tau_j};\theta_2^*)[\beta^2(\sigma^2_{\tau_j};\theta_2^*)\delta - (\sigma^2_{\tau_{j+1}} - \sigma^2_{\tau_j})^2]. \end{aligned}$$

Then, we can then write

$$\begin{aligned} \hat{S}_k(\theta_k^*, \sigma^2) &= T^{-1}\sum_{j=1}^{N-1} U_k(j) \quad \text{and} \\ H_k^{*-1} E\left[\hat{S}_k(\theta_k^*, \sigma^2)\hat{S}_k(\theta_k^*, \sigma^2)^\star\right] H_k^{*-1} &= \mathcal{B}_{\theta_k}\mathcal{B}_{\theta_k}^\star + \mathcal{V}_{\theta_k} + \mathcal{C}_{\theta_k}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{B}_{\theta_k} &:= -H_k^{*-1}E\left[\hat{S}_k(\theta_k^*, \sigma^2)\right]; \\ \mathcal{V}_{\theta_k} &:= H_k^{*-1}\left(T^{-2}\sum_{j=1}^{N-1} E\left[(U_k(j) - E[U_k(j)])(U_k(j) - E[U_k(j)])^\star\right]\right)H_k^{*-1}; \\ \mathcal{C}_{\theta_k} &:= H_k^{*-1}\left(T^{-2}\sum_{1\leq i\neq j\leq N-1} E\left[(U_k(i) - E[U_k(i)])(U_k(j) - E[U_k(j)])^\star\right]\right)H_k^{*-1}. \end{aligned}$$

We below provide the proof for part (i) ($k = 1$) only. Part (ii) ($k = 2$) can be proved in the same way, and its proof is omitted. Let $\mathcal{L}$ be the differential operator defined by $\mathcal{L}f(x) = f'(x)\alpha(x) + f''(x)\beta^2(x)/2$ for any twice differentiable function f. We first consider the expression of $\mathcal{B}_{\theta_1}$:

$$\begin{aligned} E[U_k(j)] &= -\delta^2 E\left[\partial_{\theta_1}\alpha(\sigma_u^2;\theta_1^*)\mathcal{L}\alpha(\sigma_u^2)\right] + 2\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s}\int_{\tau_j}^{u} E\left[\mathcal{L}\partial_{\theta_1}\alpha(\sigma_v^2;\theta_1^*)\mathcal{L}\alpha(\sigma_u^2)\right] dvduds \\ &= -\delta^2 E\left[\partial_{\theta_1}\alpha(\sigma_t^2;\theta_1^*)\mathcal{L}\alpha(\sigma_t^2)\right][1 + O(\delta)], \qquad (0.8) \end{aligned}$$

uniformly over j, where we have applied the martingale property of stochastic integrals and Ito's lemma to $\alpha(\sigma_s^2) - \alpha(\sigma^2_{\tau_j})$ and $\partial_{\theta_1}\alpha(\sigma_u^2;\theta_1^*) - \partial_{\theta_1}\alpha(\sigma^2_{\tau_j};\theta_1^*)$, and the last equality holds since

$$E\left[\mathcal{L}\partial_{\theta_1}\alpha(\sigma_v^2;\theta_1^*)\mathcal{L}\alpha(\sigma_u^2)\right] \leq \left\{E\left[\left|\mathcal{L}\partial_{\theta_1}\alpha(\sigma_v^2;\theta_1^*)\right|^2\right]E\left[\left|\mathcal{L}\alpha(\sigma_u^2)\right|^2\right]\right\}^{1/2} \leq E\left[\left|\psi(\sigma_t^2)\right|^4\right] = O(1),$$

uniformly over any u and v, which follows from the moment conditions in Assumption C-SDR. Now, the above definition of $\mathcal{B}_{\theta_k}$ and eq. (0.8) implies eq. (7.3) of the main text. To find the expression of $\mathcal{V}_{\theta_k}$, first note that

$$\begin{aligned}
& E\left[U_1(j)\,U_1(j)^\bigstar\right] = 4E\Big[\partial_{\theta_1}\alpha(\sigma_t^2;\theta_1^*)\partial_{\theta_1}\alpha(\sigma_t^2;\theta_1^*)^\bigstar \\
& \times\left\{\alpha^2(\sigma_{\tau_j}^2)\delta^2 - 2\alpha(\sigma_{\tau_j}^2)\delta\left(\int_{\tau_j}^{\tau_{j+1}}\alpha\left(\sigma_s^2\right)ds + \int_{\tau_j}^{\tau_{j+1}}\beta\left(\sigma_s^2\right)dZ_s\right)\right. \\
& +2\int_{\tau_j}^{\tau_{j+1}}\left(\int_{\tau_j}^{s}\alpha(\sigma_u^2)du + \int_{\tau_j}^{s}\beta(\sigma_u^2)dZ_u\right)\alpha(\sigma_s^2)ds \\
& \left.+2\int_{\tau_j}^{\tau_{j+1}}\left(\int_{\tau_j}^{s}\alpha(\sigma_u^2)du + \int_{\tau_j}^{s}\beta(\sigma_u^2)dZ_u\right)\beta(\sigma_s^2)dZ_s + \int_{\tau_j}^{\tau_{j+1}}\beta^2(\sigma_s^2)ds\right\}\Big] \\
= \;& \underbrace{\delta E\left[\partial_{\theta_1}\alpha(\sigma_t^2;\theta_1^*)\partial_{\theta_1}\alpha(\sigma_t^2;\theta_1^*)^\bigstar\beta^2\left(\sigma_t^2\right)\right]}_{=\Omega_1^*}\left[1+O\left(\delta\right)\right],\text{ uniformly over } j
\end{aligned}$$

where Ito's lemma is applied to $(\sigma_{\tau_{j+1}}^2 - \sigma_{\tau_j}^2)^2$ in the first equality; and the second equality follows from arguments similar to those in deriving eq. (0.8). By the definition of $\mathcal{V}_{\theta_k}$ and the result that $E\left[U_k(j)\right] = O\left(\delta^2\right)$ uniformly over j,

$$\mathcal{V}_{\theta_k} = H_k^{*-1}\left(T^{-2}\sum\nolimits_{j=1}^{N-1}\left[\delta\Omega_1^*\left[1+o\left(1\right)\right] - O\left(\delta^4\right)\right]\right)H_k^{*-1} = T^{-1}H_k^{*-1}\Omega_1^*H_k^{*-1}\left[1+O\left(\delta\right)\right],$$

as claimed. To find the expression of $\mathcal{C}_{\theta_k}$, we write

$$\partial_{\theta_1}\alpha\left(\sigma_{\tau_j}^2;\theta_1^*\right)\left[\alpha(\sigma_{\tau_j}^2;\theta_1^*)\delta - (\sigma_{\tau_{j+1}}^2 - \sigma_{\tau_j}^2)\right] =: \Upsilon_1(j) + \Upsilon_2(j),$$

where $\Upsilon_1(j) := -\partial_{\theta_1}\alpha\left(\sigma_{\tau_j}^2;\theta_1^*\right)\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s}\mathcal{L}\alpha\left(\sigma_u^2\right)duds$ and

$$\Upsilon_2(j) := \partial_{\theta_1}\alpha\left(\sigma_{\tau_j}^2;\theta_1^*\right)\left\{\int_{\tau_j}^{\tau_{j+1}}\int_{\tau_j}^{s}\alpha'\left(\sigma_u^2\right)\beta\left(\sigma_u^2\right)dZ_uds - \int_{\tau_j}^{\tau_{j+1}}\beta\left(\sigma_s^2\right)dZ_s\right\}.$$

Then, by the martingale property of stochastic integrals, Fubini's theorem and the conditions in (C-SDR), $E\left[\Upsilon_1(i)\,\Upsilon_2(j)^\bigstar\right] = 0$ and $E\left[\Upsilon_2(i)\,\Upsilon_2(j)^\bigstar\right] = 0$ for $i \neq j$. Given the moment conditions in (C-SDR), we can show that $E\left[\Upsilon_1(i)\,\Upsilon_1(j)^\bigstar\right] = O\left(\delta^4\right)$ uniformly over any $i \neq j$, by using arguments analogous to those for $\mathcal{B}_{\theta_k}$ and $\mathcal{V}_{\theta_k}$. This, together with eq. (0.8),

$$E\left[\left(U_k(i) - E\left[U_k(i)\right]\right)\left(U_k(j) - E\left[U_k(j)\right]\right)^\bigstar\right] = E[\Upsilon_1(i)\,\Upsilon_1(j)^\bigstar] - E\left[U_k(i)\right]E\left[U_k(j)\right]^\bigstar = O\left(\delta^4\right),$$

which, together with the definition of $\mathcal{C}_{\theta_k}$, implies that $\mathcal{C}_{\theta_k} = O\left(\delta^2\right)$. This completes the proof. ■

References

Karatzas, I. and S.E. Shreve (1991) *Brownian Motion and Stochastic Calculus*. Second edition, Springer.