Supplementary material on "Limit Laws in Transaction-Level Asset Price Models"

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Lemma A.1. *If the durations $t_{i,k} - t_{i,k-1}$ form a stationary ergodic sequence with finite moment of order $2p+1$, if $\mathbb{P}(t_{i,1} > 0) = 1$ and if the associated point process has finite intensity, then*

$$\sup_{s \geq 0} \mathbb{E}[(t_{i,N_i(s)+1} - s)^p] < \infty \, .$$

Proof of Lemma A.1. We omit the index i. Let θ_t denote the shift operator and let $A(t)$ be the forward recurrence time. Then $A(s) = t_{N(s)+1} - s = t_1 \circ \theta_s$. Since the sequence $\{\tau_i\}$ is stationary under $\mathbb{P}$, there exists a probability law P^* such that N is a stationary ergodic point process under P^*, see Baccelli and Brémaud (2003, Section 1.3.5). Applying Baccelli and Brémaud (2003, Formula 1.3.3), we obtain

$$\begin{aligned}
\mathbb{E}[A^p(s)] &= \lambda^{-1}\mathbb{E}^* \left[\sum_{k=1}^{N(1)} t_1^p \circ \theta_s \circ \theta_{t_k} \right] = \lambda^{-1}\mathbb{E}^* \left[\sum_{k=1}^{N(1)} A^p(s+t_k) \right] \\
&= \lambda^{-1}\mathbb{E}^* \left[\sum_{k=1}^{N(1)} \{t_{N(s+t_k)+1} - s - t_k\}^p \right] \leq \lambda^{-1}\mathbb{E}^* \left[\sum_{k=1}^{N(1)} \{t_{N(s+1)+1} - s\}^p \right] \\
&= \lambda^{-1}\mathbb{E}^*[N(1)\{t_{N(s+1)+1} - s\}^p] \leq \lambda^{-1}\{\mathbb{E}^*[N(1)^2]\}^{1/2}\{\mathbb{E}^*[(t_{N(s+1)+1} - s)^{2p}]\}^{1/2} \, .
\end{aligned} \tag{A.1}$$

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Since N is stationary under P^*, the last term does not depend on s, and by the Ryll-Nardzewski inversion formula (Baccelli and Brémaud (2003, Formula 1.2.25)), we have

$$\mathbb{E}^*[(t_{N(s+1)+1} - s)^{2p}] = \mathbb{E}^*[(t_1 + 1)^{2p}] = \lambda\mathbb{E}[\int_0^{t_1} (t_1 + 1 - s)^{2p} \, \mathrm{d}s \leq \lambda\mathbb{E}[(1 + t_1)^{2p+1}]$$

By Baccelli and Brémaud (2003, Property 1.6.3), the point process N is stationary and ergodic under P^* since the sequence of durations τ_k is stationary and ergodic. Thus, By Daley and Vere-Jones (2003, Theorem 3.5.III), $\mathbb{E}^*[N(0,1)^2] < \infty$. Plugging the last two bounds into (A.1), we obtain that $\mathbb{E}[A^p(s)]$ is uniformly bounded. □

Lemma A.2. *Assume that there exists an increasing sequence $\{s_n, n \geq 0\}$ such that $s_0 = 0$ and*

(a) f is either constant or strictly increasing and differentiable on (s_n, s_{n+1}) and the jumps of f occur at some (but not necessarily all) of the s_n;

(b) if f is either constant or increasing on both intervals (s_n, s_{n+1}) and (s_{n+1}, s_{n+2}), then f has a jump at s_{n+1}.

Assume moreover that

- (minimum duration of trading and nontrading periods) *there exists $\delta_0 > 0$ such that $s_{n+1} - s_n \geq \delta_0$ for all $n \geq 0$;*
- (maximum duration of nontrading periods) *there exists C_0 such that for all $n \geq 0$, if f is constant on (s_n, s_{n+1}), then $s_{n+1} - s_n \leq C_0$;*
- (non stoppage of time during trading periods) *there exists $\delta_1 > 0$ such that for all $n \geq 0$, f is either constant on (s_n, s_{n+1}), or $f'(t) \geq \delta_1$ for all $t \in (s_n, s_{n+1})$.*

Let $\tilde{N}$ be a point process with event times $\{\tilde{t}_k\}$ and let N be the point process defined by $N(\cdot) = \tilde{N}(f(\cdot))$ with event times $\{t_k\}$. If $\sup_{s \geq 0} \mathbb{E}[(\tilde{t}_{\tilde{N}(s)+1} - s)^p] < \infty$, then $\sup_{s \geq 0} \mathbb{E}[(t_{N(s)+1} - s)^p] < \infty$.

Proof of Lemma A.2. Define the nondecreasing left-continuous inverse $f^{\leftarrow}$ of a nondecreasing càdlàg function f by

$$f^{\leftarrow}(u) = \inf\{t \mid f(t) \geq u\} \; .$$

Note first that $f^{\leftarrow}(u) \leq t$ if only if $u \leq f(t)$ and $f^{\leftarrow}(f(t)) \leq t$. Thus we see that

$$\begin{aligned} f^{\leftarrow}(\tilde{t}_n) \leq t &\Leftrightarrow \tilde{t}_n \leq f(t) \\ &\Leftrightarrow \tilde{N}(f(t)) \geq n \\ &\Leftrightarrow N(t) \geq n \; . \end{aligned}$$

This characterizes the sequence $\{t_n\}$, thus we obtain that $t_n = f^{\leftarrow}(\tilde{t}_n)$. The assumptions on f imply the following properties of $f^{\leftarrow}$.

- The jumps of $f^{\leftarrow}$ correspond to the intervals (s_n, s_{n+1}) where f is constant. More precisely, if f is constant on (s_n, s_{n+1}), then $f^{\leftarrow}$ has a jump at $f(s_n)$ of size $s_{n+1} - s_n$. Since $f^{\leftarrow}$ is left continuous, it holds that

$$f^{\leftarrow}(f(s_n)) = s_n \ , \quad \lim_{u \to f(s_n), u > f(s_n)} = s_{n+1} \ .$$

  Thus the jumps of $f^{\leftarrow}$ are of size C_0 at most.

- If f is increasing on an interval (s_n, s_{n+1}), then $f^{\leftarrow}$ is differentiable on $(f(s_n), f(s_n^-))$ and $(f^{\leftarrow})'(t) \leq \delta_1^{-1}$ for all $t \in (f(s_n), f(s_n^-))$.

- The jumps of f create no singularity in $f^{\leftarrow}$. If $f(s_n) > f(s_n^-)$, then $f^{\leftarrow}$ is constant on the interval $(f(s_n^-), f(s_n))$.

Let $\lceil x \rceil$ denote the smallest integer greater than or equal to the real number x. Then, for $0 \leq s \leq t$,

$$0 \leq f^{\leftarrow}(t) - f^{\leftarrow}(s) \leq C_0 \left\lceil \frac{t-s}{\delta_0} \right\rceil + \delta_1^{-1}(t-s) \ .$$

Thus, there exits constants c_1, c_2 such that for all $s \leq t$,

$$0 \leq f(t) - f(s) \leq c_1 + c_2(t-s) \ .$$

Consider now the forward recurrence time of the point process N. Then

$$\begin{aligned} 0 \leq t_{N(s)+1} - s &= f^{\leftarrow}(\tilde{t}_{\tilde{N}(s)+1}) - f^{\leftarrow}(f(s)) + f^{\leftarrow}(f(s)) - s \\ &\leq f^{\leftarrow}(\tilde{t}_{\tilde{N}(f(s))+1}) - f^{\leftarrow}(f(s)) \leq c_1 + c_2\{\tilde{t}_{\tilde{N}(f(s))+1} - f(s)\} \ . \end{aligned}$$

Thus, there exists constants c_3 and c_4 such that

$$\sup_{s \geq 0} \mathbb{E}[(t_{N(s)+1} - s)^p] \leq c_3 + c_4 \sup_{s \geq 0} \mathbb{E}[(\tilde{t}_{\tilde{N}(s)+1} - s)^p]$$

□

Lemma A.3. *Let $\{\epsilon_k\}$ be a sequence of i.i.d. positive random variables with finite mean μ_ϵ. Let $\{Y_k\}$ be a stationary standard Gaussian process such that*

$$\mathrm{cov}(Y_0, Y_k) = \ell(n) n^{2H-2} \tag{A.2}$$

for $H \in (1/2, 1)$ and ℓ a slowly varying function. For $k \geq 1$, define

$$\tau_k = \epsilon_k \mathrm{e}^{\sigma Y_k} \ .$$

Then the sequence $\{\tau_k\}$ is ergodic and Assumption 2.1 holds with $\lambda^{-1} = \mu_\epsilon \mathrm{e}^{\sigma^2/2}$. If $\mathbb{P}(\epsilon_1 > 0) = 1$ the Assumption 2.2 holds with $\mu = \lambda = \mu_\epsilon^{-1} \mathrm{e}^{-\sigma^2/2}$. If moreover $\mathbb{E}[\epsilon_1^q] < \infty$ for all $q \geq 1$, then (3.3) and (3.5) hold.

Remark A.1. If instead of (A.2) we assume that

$$\sum_{k=1}^{\infty} |\mathrm{cov}(Y_0, Y_k)| < \infty ,$$

then the moment requirement can be relaxed to $\mathbb{E}[\epsilon_1^3] < \infty$ to obtain (3.3) and $\mathbb{E}[\epsilon_1^5] < \infty$ to obtain (3.5).

Proof of Lemma A.3. Note first that $\mathbb{E}[\tau_k^p] < \infty$ as long as $\mathbb{E}[\epsilon_1^p] < \infty$. By Lemma A.1, in order to check condition (3.3), we must only prove that the induced point process has finite intensity, i.e. there exists $t > 0$ such that $\mathbb{E}[N(t)] < \infty$. See Baccelli and Brémaud (2003, Section 1.3.5). Note that

$$\mathbb{E}[N(x)] = \sum_{k=1}^{\infty} \mathbb{P}(N(x) \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(t_k \leq x) .$$

Thus, it suffices to prove that the series on the righthand side is summable. Denote $\mu = \mathbb{E}[\tau_k]$ and $\rho_n = \mathrm{cov}(Y_0, Y_n)$. Applying Deo et al. (2009, Proposition 1), we have

$$\mathbb{E}\left[\left|\sum_{k=1}^{n} \tau_k - n\mu\right|^p\right] = O(v_n^p)$$

with $v_n = n^H \ell(n)$. If $\mathbb{E}[\epsilon_1^p] < \infty$ for p such that $p(1-H) > 1$, for n such that $n\mu > x$, it holds that

$$\mathbb{P}(t_k \leq x) = O(x^{-1} v_k^p)$$

and this series is summable. □

Lemma A.4. *Assume that $\{\tau_k\}$ and $\{\xi_k\}$ are mutually independent stationary sequences such that $\mathbb{E}[\xi_k] = 0$, $\mathbb{E}[\tau_k^2] < \infty$ and $\mathbb{E}[\xi_k^2] < \infty$. Assume that the sequence of durations is weakly stationary and that $\mathrm{cov}(\tau_0, \tau_n) = 0(n^{-\delta})$ for some $\delta > 0$ and $\sup_{s\geq 0} \mathbb{E}[t_{N(s)+1} - s] < \infty$. Assume that $\mathrm{cov}(\xi_1, \xi_n) \sim cn^{2H-2}$, with $H \in (1/2, 1)$ and $c > 0$, and that*

$$n^{-H} \sum_{k=1}^{[n\cdot]} \xi_k \Rightarrow c' B_H$$

for some $c' > 0$. Then

$$n^{-H} \int_0^{Tt} \xi_{N(s)} \, \mathrm{d}s \Rightarrow c'' B_H(t)$$

for some $c'' > 0$.

Proof of Lemma A.4. Denote $\mathbb{E}[\tau_k] = \mu > 0$.

$$\begin{aligned}\int_0^T \xi_{N(s)} \, \mathrm{d}s &= \sum_{k=0}^{N(T)} \tau_{k+1}\xi_k - (t_{N(T)+1} - T)\xi_{N(T)+1} \\ &= \sum_{k=0}^{N(T)} (\tau_{k+1} - \mu)\xi_k + \mu \sum_{k=0}^{N(T)} \xi_k - (t_{N(T)+1} - T)\xi_{N(T)+1} \; .\end{aligned}$$

By independence of $\{\tau_k\}$ and $\{\xi_k\}$, we have (assuming without loss of generality that $2H - \delta > 1$),

$$\mathrm{var}\left(\sum_{k=0}^{n} (\tau_{k+1} - \mu)\xi_k \right) = O(n^{2H-\delta}) \; .$$

Thus, $n^{-H} \sum_{k=0}^{[n\cdot]} (\tau_{k+1} - \mu)\xi_k \Rightarrow 0$. Hence by the continuous mapping theorem, it also holds that $n^{-H} \sum_{k=0}^{N(T\cdot)} (\tau_{k+1} - \mu)\xi_k \Rightarrow 0$. By independence and by assumption, $(t_{N(t)+1} - T)\xi_{N(T)} = O_P(1)$. By the continuous mapping theorem, $n^{-H} \sum_{k=0}^{N(Tt)} \xi_k \Rightarrow c' B_H(\mu^{-1}t)$. □

Lemma A.5. *Let $\{\tau_k\}$, $\{V_k\}$ and $\{\zeta_k\}$ be sequences of random variables such that*

- *$\{\zeta_k\}$ is an i.i.d. sequence of zero-mean and unit variance random variables; $\{\tau_k\}$ and $\{V_k\}$ are sequences of positive random variables;*
- *the sequences $\{(\tau_k, V_k)\}$ and $\{\zeta_k\}$ are mutually independent;*
- *there exists $s > 0$ such that $n^{-1} \sum_{k=1}^{n} \tau_{k+1}^2 V_k^2 \xrightarrow{\mathbb{P}} s^2$;*
- *$\sup_{k \geq 0} \mathbb{E}[\tau_{k+1}^{2+\varepsilon} V_k^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$;*
- *$\sup_{s \geq 0} \mathbb{E}[t_{N(s)+1} - s] < \infty$.*

Define $\xi_k = \zeta_k V_k$. Then $T^{-1/2} \int_0^{T\cdot} \xi_{N(s)} \, \mathrm{d}s \Rightarrow cB$ for some $c > 0$.

Proof. Let $\mathcal{F}_k$ be the sigma-field generated by random variables $\{\tau_{j+1}, \zeta_j, V_j, j \leq k\}$. Then $\mathbb{E}[\xi_k \tau_{k+1} \mid \mathcal{F}_{k-1}] = \tau_{k+1} V_k \mathbb{E}[\zeta_k] = 0$. Thus, $\{\tau_{k+1}\xi_k\}$ is a martingale difference sequence. Under the stated assumptions, the martingale invariance principle Hall and Heyde (1980, Theorem 4.1) yields that $n^{-1/2} \sum_{k=1}^{[n\cdot]} \tau_{k+1}\xi_k \Rightarrow cB$ for some $c > 0$. As in the proof of Lemma A.4, denote $\mathbb{E}[\tau_k] = \mu > 0$ and write

$$\int_0^T \xi_{N(s)} \, \mathrm{d}s = \sum_{k=0}^{N(T)} \tau_{k+1}\xi_k + (t_{N(T)+1} - T)\xi_{N(T)} \; .$$

By the continuous mapping theorem, we have that $T^{-1/2} \sum_{k=1}^{N(T\cdot)} \tau_k \xi_{-1} \Rightarrow \lambda c B$. As previously, the last term is a negligible edge effect. This concludes the proof. □

Lemma A.6. *Let N be a stationary point process under P with intensity λ and let P^0 denote the Palm probability associated to P. Let $\gamma > 0$. Assume that there exist $\delta \in (0,1)$ and $q > 0$ such that*

$$\sup_{k\geq 1} k^{-q\delta} \mathbb{E}^0[|t_k - \lambda^{-1}k|^q] < \infty \; . \tag{A.3}$$

If (A.3) holds with $q \geq \gamma + 1$, then

$$\sup_{t\geq 2} \mathbb{E}\left[\left(\frac{N_i(t)}{t}\right)^{-\gamma} \mathbf{1}_{\{N_i(t)>0\}}\right] < \infty \; . \tag{A.4}$$

If (A.3) holds with $q > 1 + \gamma/(1-\delta)$, then $\mathbb{E}[N^\gamma(1)] < \infty$.

Proof. For $k \geq 2$, define $c_k = (k-1)^{-\gamma} - k^{-\gamma}$. Then, $\sum_{k=2}^\infty c_k = 1$ and applying summation by parts, we have

$$\begin{aligned}
\mathbb{E}[N^{-\gamma}(t)\mathbf{1}_{\{N(t)>0\}}] &= \sum_{k=1}^\infty k^{-\gamma}\mathbb{P}(N(t)=k) = \sum_{k=1}^\infty k^{-\gamma}\{\mathbb{P}(N(t)\geq k) - \mathbb{P}(N(t)\geq k+1)\} \\
&= \mathbb{P}(N(t)\geq 1) - \sum_{k=2}^\infty c_k \mathbb{P}(N(t)\geq k) \\
&= \mathbb{P}(t_1 \leq t) - \sum_{k=2}^\infty c_k \mathbb{P}(t_k \leq t) = -\mathbb{P}(t_1 > t) + \sum_{k=2}^\infty c_k \mathbb{P}(t_k > t) \; .
\end{aligned}$$

Without loss of generality, assume that the intensity of the point process is $\lambda = 1$. Then, by definition of c_k, we have, for $t \geq 2$,

$$t^\gamma \sum_{k\geq[t/2]+1} c_k \mathbb{P}(t_k > t) \leq t^\gamma([t/2])^{-\gamma} = O(1) \; .$$

For $k \leq [t/2]$, we have, by Markov's inequality,

$$\mathbb{P}(t_k > t) = \mathbb{P}(t_k - k > t - k) \leq \mathbb{P}(t_k - k > t/2) \leq ct^{-\gamma}\mathbb{E}[|t_k - k|^\gamma]$$

Applying the Ryll-Nardzewski inversion formula (Baccelli and Brémaud (2003, Formula 1.2.25)), we have

$$\mathbb{E}[|t_k - k|^\gamma] = \mathbb{E}^0[t_1|t_k - k|^\gamma] \leq \{\mathbb{E}^0[t_1^{1+\gamma}]\}^{1/(\gamma+1)}\{\mathbb{E}^0[|t_k - k|^{\gamma+1}|]\}^{\gamma/(\gamma+1)} \; .$$

Thus, applying Condition (A.3), we obtain that $\mathbb{P}(t_k > t) \leq c' t^{-\gamma} k^{\gamma\delta}$ and thus

$$t^\gamma \sum_{2\leq k\leq[t/2]} c_k \mathbb{P}(t_k > t) \leq c' \sum_{2\leq k\leq[t/2]} c_k k^{\gamma\delta} \leq c' \sum_{2\leq k\leq[t/2]} k^{-\gamma(1-\delta)-1} = O(1) \; .$$

This concludes the proof of (A.4). We now consider the positive moments of $N(1)$. Applying summation by part, we have

$$\mathbb{E}[N^\gamma(1)] = \sum_{k=1}^\infty \{k^\gamma - (k-1)^\gamma\} \mathbb{P}(N(1) \geq k) = \sum_{k=1}^\infty \{k^\gamma - (k-1)^\gamma\} \mathbb{P}(t_k \leq 1) \; .$$

For $k \geq 2$ and $q > 0$, we have, still assuming that $\lambda = 1$,

$$\mathbb{P}(t_k \leq 1) \leq \mathbb{P}(t_k - k \leq -k/2) \leq \mathbb{E}[|t_k - k|^q] k^{-q} \; .$$

Applying again the Ryll-Narzewski formula and Condition (A.3), we obtain, for $k \geq 2$,

$$\mathbb{P}(t_k \leq 1) \leq c k^{-q(1-\delta)} \; .$$

Thus,

$$\mathbb{E}[N^\gamma(1)] \leq 1 + c \sum_{k=1}^\infty \{k^\gamma - (k-1)^\gamma\} k^{-q(1-\delta)} \; .$$

The series is convergent as long as $q(1-\delta) > \gamma$. □

Proof of (3.4) for the LMSD model. Consider the LMSD model of Example 2.1. It is proved in Deo et al. (2009, Proposition 1) that (A.3) holds with $\delta = H_\tau$ if $\mathbb{E}[\epsilon_0^p] < \infty$ for all $p \geq 1$. Actually, a close inspection of the first lines of the proof shows that only q finite moments of ϵ_0 are needed. Thus (3.4) holds if $E^0[\epsilon_0^{9-4H}] < \infty$, and $\mathbb{E}[N^4(1)] < \infty$ if $E^0[\epsilon_0^q] < \infty$ for some $q > 1 + 4/(1 - H_\tau)$. □

Proof of (3.4) for the ACD model. Under the assumptions of Example 2.2, the sequence $\{\tau_k\}$ is geometrically β-mixing Carrasco and Chen (2002, Proposition 17). Denote $m = \mathbb{E}^0[\tau_1]$. The sequence $\{t_k\}$ is geometrically mixing, hence geometrically strong mixing. Thus, by Rio (2000, Theorem 2.5), for $q \geq 2$, if $E^0[\tau_1^{q+1+\epsilon}] < \infty$ for some $\epsilon > 0$, then $E^0[|t_n - mn|^{q+1}] = O(n^{(q+1)/2})$. Thus (A.3) holds with $\delta = 1/2$. □

References

Baccelli, F. and Brémaud, P. (2003). *Elements of queueing theory*, volume 26 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition.

Carrasco, M. and Chen, X. (2002). Mixing and moment properties of various GARCH and stochastic volatility models. *Econometric Theory*, 18(1):17–39.

Daley, D. J. and Vere-Jones, D. (2003). *An introduction to the theory of point processes. Vol. I: Elementary theory and methods. 2nd ed.* Probability and Its Applications. New York, NY: Springer.

Deo, R., Hurvich, C. M., Soulier, P., and Wang, Y. (2009). Conditions for the propagation of memory parameter from durations to counts and realized volatility. *Econometric Theory*, 25(3):764–792.

Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its application*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York. Probability and Mathematical Statistics.

Rio, E. (2000). *Théorie asymptotique des processus aléatoires faiblement dépendants*, volume 31 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin.