**Supplementary Material on "Noncausal Vector Autoregression"**

**Details on the discussion of nonidentifiability of noncausal VAR models in the Gaussian case in Section 2.1.** A practical complication with noncausal autoregressive models is that they cannot be identified by second order properties or Gaussian likelihood. In the univariate case this is explained, for example, in Brockwell and Davis (1987, p. 124-125)). To demonstrate the same in our multivariate case, note first that, by well-known results on linear filters (cf. Hannan (1970, p. 67)), the spectral density matrix of the process $y_t$ defined by (1) is given by

$$\begin{aligned} & (2\pi)^{-1} \Phi\left(e^{-i\omega}\right)^{-1} \Pi\left(e^{i\omega}\right)^{-1} \mathbb{C}\left(\epsilon_t\right) \Pi\left(e^{-i\omega}\right)'^{-1} \Phi\left(e^{i\omega}\right)'^{-1} \\ = \; & (2\pi)^{-1} \left[\Phi\left(e^{i\omega}\right)' \Pi\left(e^{-i\omega}\right)' \mathbb{C}\left(\epsilon_t\right)^{-1} \Pi\left(e^{i\omega}\right) \Phi\left(e^{-i\omega}\right)\right]^{-1}. \end{aligned}$$

In the latter expression, the matrix in the brackets is $2\pi$ times the spectral density matrix of a second order stationary process whose autocovariances are zero at lags larger than $r+s$. As is well known, this process can be represented as an invertible moving average of order $r+s$. Specifically, by a slight modification of Theorem 10' of Hannan (1970), we get the unique representation

$$\Phi\left(e^{i\omega}\right)' \Pi\left(e^{-i\omega}\right)' \mathbb{C}\left(\epsilon_t\right)^{-1} \Pi\left(e^{i\omega}\right) \Phi\left(e^{-i\omega}\right) = \left(\sum_{j=0}^{r+s} \mathcal{M}_j e^{-i\omega}\right)' \left(\sum_{j=0}^{r+s} \mathcal{M}_j e^{i\omega}\right),$$

where the $n \times n$ matrixes $\mathcal{M}_0, ..., \mathcal{M}_{r+s}$ are real with $\mathcal{M}_0$ positive definite, and the zeros of det $\left(\sum_{j=0}^{r+s} \mathcal{M}_j e^{i\omega}\right)$ lie outside the unique disc. Thus, the spectral density matrix of $y_t$ has the representation $(2\pi)^{-1} \left(\sum_{j=0}^{r+s} \mathcal{M}_j e^{ij\omega}\right)^{-1} \left(\sum_{j=0}^{r+s} \mathcal{M}_j e^{-ij\omega}\right)'^{-1}$, which is the spectral density matrix of a causal VAR($r+s$) process. Finally, note that a direct application of Hannan's (1970) Theorem 10' would give a representation with $\omega$ replaced by $-\omega$. That this modification is possible can be seen from the proof of the mentioned theorem (see the discussion starting in the middle of p. 64 of Hannan (1970)).

**Remaining part of the proof of Lemma 1.** Regarding $\boldsymbol{i}(\lambda)$, first notice that

$$\begin{aligned} \int_0^\infty \zeta^{n/2+1} f'\left(\zeta;\lambda_0\right) d\zeta &= \left(\zeta^{n/2+1} f\left(\zeta;\lambda\right)|_0^\infty - \frac{n+2}{2} \int_0^\infty \zeta^{n/2} f\left(\zeta;\lambda\right) d\zeta\right) \\ &= -\frac{n+2}{2} \cdot \frac{\Gamma\left(n/2\right)}{\pi^{n/2}} \mathbb{E}_\lambda\left(\rho_t^2\right), \end{aligned}$$

where we have used Assumptions 2(ii) and (iii), and the expression of the density of $\rho_t^2$

(see (10)). Proceeding as in the case of the first assertion yields

$$
\begin{aligned}
1 &= \left(\frac{2}{(n+2)\,\mathbb{E}_\lambda(\rho_t^2)}\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \zeta^{n/4+1/2}\frac{f'(\zeta;\lambda)}{\sqrt{f(\zeta;\lambda)}}\zeta^{n/4+1/2}\sqrt{f(\zeta;\lambda)}d\zeta\right)^2\\
&\le \left(\frac{2}{(n+2)\,\mathbb{E}_\lambda(\rho_t^2)}\right)^2\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \zeta^{n/2+1}\left(\frac{f'(\zeta;\lambda)}{f(\zeta;\lambda)}\right)^2 f(\zeta;\lambda)\,d\zeta\\
&\qquad\times\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \zeta^{n/2+1}f(\zeta;\lambda)\,d\zeta\\
&= \left(\frac{2}{(n+2)\,\mathbb{E}_\lambda(\rho_t^2)}\right)^2\cdot\boldsymbol{i}(\lambda)\cdot\mathbb{E}_\lambda\left(\rho_t^4\right)
\end{aligned}
$$

(see the definition of $\boldsymbol{i}(\lambda)$ in (12)). This shows the stated inequality and the condition for equality leads to the same condition as in the case of $\boldsymbol{j}(\lambda)$. Finally, in the Gaussian case, $\mathbb{E}_\lambda(\rho_t^2)=n$ and $\mathbb{E}_\lambda(\rho_t^4)=2n+n^2$, implying $\boldsymbol{i}(\lambda)=n(n+2)/4$. □

**Proof of the nonsingularity of the matrix $\boldsymbol{H}_1$ mentioned in Section 3.1.** To simplify notation we demonstrate the nonsingularity of $\boldsymbol{H}_1$ when $s=2$. From the definition of $\boldsymbol{H}_1$ it is not difficult to see that the possible singularity of $\boldsymbol{H}_1$ can only be due to a linear dependence of its last $n(r+2)$ rows and, furthermore, that it suffices to show the nonsingularity of the lower right hand corner $\boldsymbol{H}_1$ of order $n(r+2)\times n(r+2)$. This matrix reads as

$$
\boldsymbol{H}_1^{(2,2)} = \left[\begin{array}{cccccc|cc}
I_n & -\Phi_1 & -\Phi_2 & 0 & \cdots & \cdots & \cdots & 0\\
0 & \ddots & \ddots & \ddots & \ddots & & & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots\\
\vdots & \cdots & 0 & I_n & -\Phi_1 & -\Phi_2 & 0 & 0\\
\vdots & \cdots & 0 & 0 & I_n & -\Phi_1 & -\Phi_2 & 0\\
0 & \cdots & 0 & 0 & 0 & I_n & -\Phi_1 & -\Phi_2\\
\hline
-a_{nr}I_n & \cdots & \cdots & \cdots & \cdots & -a_1 I_n & I_n & 0\\
0 & -a_{nr}I_n & \cdots & \cdots & \cdots & \cdots & -a_1 I_n & I_n
\end{array}\right]
\overset{def}{=} \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{bmatrix},
$$

where the partition is as indicated. The determinant of $\boldsymbol{B}_{11}$ is evidently unity so that from the well-known formula for the determinant of a partitioned matrix it follows that we need to show the nonsingularity of the matrix $\boldsymbol{B}_{11\cdot 2}=\boldsymbol{B}_{22}-\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{12}$. The inverse of $\boldsymbol{B}_{11}$ depends on the coefficients of the power series representation of $L(z)=\Phi(z)^{-1}$ given by $L(z)=\sum_{j=0}^{\infty}L_j z^j$ where $L_0=I_n$ and, when convenient, $L_j=0$, $j<0$, will be used. Equating the coefficient matrices of powers of $z$ on both sides of the identity $L(z)\Phi(z)=I_n$ yields $L_j=L_{j-1}\Phi_1+L_{j-2}\Phi_2$. Using this identity it is readily seen that $\boldsymbol{B}_{11}^{-1}$ is an upper triangular matrix with $I_n$ on the diagonal and $L_j$, $j=1,...,nr-1$, on the diagonals above the main diagonal. This fact and straightforward but tedious calculations further show that

$$\boldsymbol{B}_{11\cdot 2} = \begin{bmatrix} I_n - \sum_{j=1}^{nr} a_j L_j & -\sum_{j=1}^{nr} a_j L_{j-1}\Phi_2 \\ -\sum_{j=1}^{nr} a_j L_{j-1} & I_n - \sum_{j=2}^{nr} a_j L_{j-2}\Phi_2 \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} - \sum_{j=1}^{nr} a_j \begin{bmatrix} L_j & L_{j-1}\Phi_2 \\ L_{j-1} & L_{j-2}\Phi_2 \end{bmatrix}.$$

Next define the companion matrix

$$\mathbf{\Phi} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ I_n & 0 \end{bmatrix}$$

and note that the latter condition in (2) implies that the eigenvalues of $\mathbf{\Phi}$ are smaller than one in absolute value. Also, the matrices $L_j$ and $L_{j-1}$ $(j \geq 0)$ can be obtained from the upper and lower left hand corners of the matrix $\mathbf{\Phi}^j$, respectively. Using these facts, the identity $L_j = L_{j-1}\Phi_1 + L_{j-2}\Phi_2$, and properties of the powers $\mathbf{\Phi}^j$ it can further be seen that

$$\boldsymbol{B}_{11\cdot 2} = I_{2n} - \sum_{j=1}^{nr} a_j \mathbf{\Phi}^j = \mathbf{P}\left( I_{2n} - \sum_{j=1}^{nr} a_j \mathbf{D}^j \right)\mathbf{P}^{-1},$$

where the latter equality is based on the Jordan decomposition of $\mathbf{\Phi}$ so that $\mathbf{\Phi} = \mathbf{PDP}^{-1}$. Thus, the determinant of $\boldsymbol{B}_{11\cdot 2}$ equals the determinant of the matrix in parentheses in its latter expression. Because $\mathbf{D}^j$ is an upper triangular matrix having the $j$th powers of the eigenvalues of $\mathbf{\Phi}$ on the diagonal this determinant is a product of quantities of the form $1 - \sum_{j=1}^{nr} a_j \nu^j$ where $\nu$ signifies an eigenvalue of $\mathbf{\Phi}$. By the latter condition in (2) the eigenvalues of $\mathbf{\Phi}$ are smaller than one in absolute value whereas the former condition in (2) implies that the zeros of $a(z)$ lie outside the unit disc. Thus, the nonsingularity of $\boldsymbol{B}_{11\cdot 2}$, and hence that of $\boldsymbol{H}_1^{(2,2)}$ and $\boldsymbol{H}_1$ follows.

We note that in the case $s = 1$ the preceding proof simplifies because then we need to show the nonsingularity of the matrix obtained from $\boldsymbol{H}_1^{(2,2)}$ by deleting its last $n$ rows and columns and setting $\Phi_2 = 0$. In place of $\boldsymbol{B}_{11\cdot 2}$ we then have $I_n - \sum_{j=1}^{nr} a_j \Phi_1^j$ and, because now the eigenvalues of $\Phi_1$ are smaller than one in absolute value, the preceding argument applies without the need to use a companion matrix. □

**Remaining parts of the proof of Lemma 4.** For the case $t = k$, $i = j \neq 0$ we have by independence $\mathbb{E}(\varepsilon_{t-i} \otimes e_{0t}) = \mathbb{E}(\varepsilon_{t-i}) \otimes \mathbb{E}(e_{0t}) = 0$. Thus, by the definition $\varepsilon_t = \Sigma_0^{-1/2}\epsilon_t$ and (B.3), (A.3), and arguments used in the previous case,

$$\mathbb{C}(\varepsilon_{t-i} \otimes e_{0t}, \varepsilon_{t-i} \otimes e_{0t}) = \mathbb{E}\left(\rho_{t-i}^2\right)\mathbb{E}\left[\rho_t^2\left(h_0\left(\rho_t^2\right)\right)^2\right]\left[\mathbb{E}\left(\upsilon_{t-i}\upsilon_{t-i}'\right) \otimes \mathbb{E}\left(\upsilon_t\upsilon_t'\right)\right].$$

The stated result is obtained from this by using definitions and $\mathbb{E}(\upsilon_t\upsilon_t') = n^{-1}I_n$.

In the case $t \neq k$, $i = t - k$, and $j = k - t$ we have $i \neq 0 \neq j$ and, as in the preceding case, $\mathbb{E}(\varepsilon_k \otimes e_{0t}) = 0$. We also note that $\varepsilon_t \otimes e_{0k} = K_{nn}(e_{0k} \otimes \varepsilon_t)$ (see Result 9.2.2(3) in

Lütkepohl (1996)). As before, we now obtain

$$\begin{aligned}\mathbb{C}\left(\varepsilon_k \otimes e_{0t}, \varepsilon_t \otimes e_{0k}\right) &= \mathbb{C}\left(\varepsilon_k \otimes e_{0t}, K_{nn}\left(e_{0k} \otimes \varepsilon_t\right)\right) \\ &= \mathbb{E}\left[\left(\rho_k \upsilon_k \otimes \rho_t h_0\left(\rho_t^2\right) \upsilon_t\right)\left(\rho_k h_0\left(\rho_k^2\right) \upsilon_k' \otimes \rho_t \upsilon_t'\right)\right] K_{nn}' \\ &= \left\{\mathbb{E}\left[\rho_t^2 h_0\left(\rho_t^2\right)\right]\right\}^2 \left\{\mathbb{E}\left(\upsilon_k \upsilon_k'\right) \otimes \mathbb{E}\left(\upsilon_t \upsilon_t'\right)\right\} K_{nn}' \\ &= \frac{1}{4} K_{nn},\end{aligned}$$

where the last equality follows from (B.1), the symmetry of the commutation matrix $K_{nn}$, and the fact $\mathbb{E}\left(\upsilon_t \upsilon_t'\right) = n^{-1} I_n$.

Finally, in the last case the stated results follows from independence. □

**Remaining parts of the proof of Step 1 of Proposition 2**. We consider the different blocks of $\mathcal{I}_{\theta\theta}\left(\theta_0\right)$ separately and, to simplify notation, we set $N = T - s - r$. In what follows, frequent use will be made of the identity $\left(f'\left(\epsilon_t' \Sigma_0^{-1} \epsilon_t; \lambda_0\right) / f\left(\epsilon_t' \Sigma_0^{-1} \epsilon_t; \lambda_0\right)\right) \Sigma_0^{-1} \epsilon_t = \Sigma_0^{-1/2} e_{0t}$ (see (A.1)). For convenience we also present expressions of the scores evaluated at $\theta_0$ (see Appendix A and note that in the first three expressions (A.1), (A.4), and (A.5) have been used):

$$\frac{\partial}{\partial \vartheta_1} g_t\left(\theta_0\right) = -2 \sum_{i=1}^{r} \frac{\partial}{\partial \vartheta_1} \pi_i'\left(\vartheta_{10}\right)\left(u_{0,t-i} \otimes I_n\right) \Sigma_0^{-1/2} e_{0t} \qquad \text{(S.1)}$$

$$\frac{\partial}{\partial \vartheta_2} g_t\left(\theta_0\right) = 2 \sum_{j=1}^{s} \frac{\partial}{\partial \vartheta_2} \phi_j'\left(\vartheta_{20}\right) \sum_{i=0}^{r}\left(y_{t+j-i} \otimes \Pi_{i0}'\right) \Sigma_0^{-1/2} e_{0t} \qquad \text{(S.2)}$$

$$\frac{\partial}{\partial \sigma} g_t\left(\theta_0\right) = -D_n'\left(\Sigma_0^{-1} \otimes \Sigma_0^{-1}\right)\left(\epsilon_t \otimes \Sigma_0^{1/2} e_{0t} + \frac{1}{2} \operatorname{vec}\left(\Sigma_0\right)\right) \qquad \text{(S.3)}$$

$$\frac{\partial}{\partial \lambda} g_t\left(\theta_0\right) = \frac{1}{f\left(\epsilon_t' \Sigma_0^{-1} \epsilon_t; \lambda_0\right)} \frac{\partial}{\partial \lambda} f\left(\epsilon_t' \Sigma_0^{-1} \epsilon_t; \lambda_0\right). \qquad \text{(S.4)}$$

*Block* $\mathcal{I}_{\vartheta_1\vartheta_1}\left(\theta_0\right)$. Set $U_{t-1}\left(\vartheta_2\right) = \left[\left(u_{t-1}\left(\vartheta_2\right) \otimes I_n\right)' \ \cdots \ \left(u_{t-r}\left(\vartheta_2\right) \otimes I_n\right)'\right]'$ and $U_{t-1}\left(\vartheta_{20}\right) = U_{0,t-1}$. Using (S.1) we then have

$$\frac{\partial}{\partial \vartheta_1} g_t\left(\theta_0\right) = -2 \nabla_1\left(\vartheta_{10}\right)' U_{0,t-1} \Sigma_0^{-1/2} e_{0t}.$$

From the definitions and (3) it can be seen that $U_{0,t-1}$ and $e_{0t}$ are independent and, as noticed after the definition of $\mathcal{I}_{\vartheta_1\vartheta_1}\left(\theta_0\right)$, $C_{11}\left(\theta_0; a, b\right) = \boldsymbol{j}_0 \mathbb{E}\left(u_{0,t-a} u_{0,t-b}'\right)$. Thus, (B.4), the preceding equation, and straightforward calculation yield $\mathbb{E}\left(\partial g_t\left(\theta_0\right) / \partial \vartheta_1\right) = 0$ and, furthermore,

$$\mathbb{C}\left(N^{-1/2} \sum_{t=r+1}^{T-s} \frac{\partial}{\partial \vartheta_1} g_t\left(\theta_0\right)\right) = \nabla_1\left(\vartheta_{10}\right)' C_{11}\left(\theta_0\right) \nabla_1\left(\vartheta_{10}\right) = \mathcal{I}_{\vartheta_1\vartheta_1}\left(\theta_0\right).$$

*Block* $\mathcal{I}_{\vartheta_2\vartheta_2}\left(\theta_0\right)$*, justification of the final step.* We need to show that

$$\sum_{k=-\infty}^{\infty} \sum_{i,j=0}^{r}\left(\Psi_{k+a-i,0} \Sigma_0^{1/2} \otimes \Pi_{i0}' \Sigma_0^{-1/2}\right) K_{nn}\left(\Sigma_0^{1/2} \Psi_{-k+b-j,0}' \otimes \Sigma_0^{-1/2} \Pi_{j0}\right) = 0.$$

To see this, notice that $(\Psi_{k+a-i,0}\Sigma_0^{1/2}\otimes\Pi_{i0}'\Sigma_0^{-1/2})K_{nn}=K_{nn}(\Pi_{i0}'\Sigma_0^{-1/2}\otimes\Psi_{k+a-i,0}\Sigma_0^{1/2})$ (see Lütkepohl (1996), Result 9.2.2 (5)(a)). Thus, the left hand side of the preceding equality can be written as

$$\begin{aligned}
K_{nn}\sum_{k=-\infty}^{\infty}\sum_{i,j=0}^{r}\left(\Pi_{i0}'\Psi'_{-k+b-j,0}\otimes\Psi_{k+a-i,0}\Pi_{j0}\right) &= K_{nn}\sum_{l=-\infty}^{\infty}\sum_{j=0}^{r}\left(\sum_{i=0}^{r}\Pi_{i0}'\Psi'_{-l+a+b-j-i,0}\otimes\Psi_{l,0}\Pi_{j0}\right)\\
&= K_{nn}\sum_{l=-\infty}^{\infty}\sum_{j=0}^{r}\left(L'_{l-a-b+j,0}\otimes\Psi_{l,0}\Pi_{j0}\right)\\
&= K_{nn}\sum_{k=0}^{\infty}\left(L'_{k,0}\otimes\sum_{j=0}^{r}\Psi_{k+a+b-j,0}\Pi_{j0}\right)\\
&= 0.
\end{aligned}$$

Here the second and fourth equalities are obtained from (B.6) (because $a,\, b>0$).

*Block* $\mathcal{I}_{\vartheta_1\vartheta_2}(\theta_0)$. Let $a\in\{1,...,r\}$ and $b\in\{1,...,s\}$. Using (3) and (5), and the definitions of $A_0(k,i)$ and $B_0(k)$ ($B_0(k)=0$ for $k<0$) given in the treatment of Block $\mathcal{I}_{\vartheta_2\vartheta_2}(\theta_0)$ we consider

$$\begin{aligned}
&\mathbb{C}\left(\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1/2}e_{0t},\sum_{i=0}^{r}\left(y_{k+b-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1/2}e_{0k}\right)\\
=\;&\sum_{c=0}^{\infty}\sum_{d=-\infty}^{\infty}\sum_{i=0}^{r}B_0(c)\,\mathbb{C}\left((\varepsilon_{t-a-c}\otimes e_{0t}),(\varepsilon_{k+b-i-d}\otimes e_{0k})\right)A_0(d,i)'\\
=\;&\frac{\boldsymbol{\tau}_0}{4}\sum_{c=a}^{\infty}\sum_{i=0}^{r}B_0(c-a)\,A_0(c+b-i,i)'\,\mathbf{1}(t=k)\\
&+\frac{1}{4}\sum_{i=0}^{r}B_0(t-k-a)\,K_{nn}A_0(k-t+b-i,i)'\,\mathbf{1}(t\neq k),
\end{aligned}$$

where the latter equality is based on Lemma 4. Summing over $t,k=r+1,...,T-s-r$, multiplying by $-4/N$, and letting $T$ tend to infinity yields the matrix $C_{12}(a,b;\theta_0)$ (see (S.1) and (S.2), and the definition of $\mathcal{I}_{\vartheta_1\vartheta_2}(\theta_0)$). Thus,

$$\begin{aligned}
C_{12}(a,b;\theta_0) &= -\boldsymbol{\tau}_0\sum_{c=a}^{\infty}\sum_{i=0}^{r}B_0(c-a)\,A_0(c+b-i,i)'\\
&\quad-\sum_{c=a}^{\infty}\sum_{i=0}^{r}B_0(c-a)\,K_{nn}A_0(-c+b-i,i)'.
\end{aligned}$$

It is easy to see that the first term on the right hand side equals the first term on the right hand side of the defining equation of $C_{12}(a,b;\theta_0)$. To show the same for the second term, we need to show that

$$-K_{nn}\left(\Psi'_{b-a,0}\otimes I_n\right)=-\sum_{c=a}^{\infty}\sum_{i=0}^{r}\left(M_{c-a,0}\Sigma_0^{1/2}\otimes\Sigma_0^{-1/2}\right)K_{nn}\left(\Sigma_0^{1/2}\Psi'_{-c+b-i,0}\otimes\Sigma_0^{-1/2}\Pi_{i0}\right).$$

Using Result 9.2.2 (5)(a) in Lütkepohl (1996) and the convention $M_{j0} = 0$, $j < 0$, we can write the right hand side as

$$
\begin{aligned}
-K_{nn} \sum_{c=-\infty}^{\infty} \sum_{i=0}^{r} \left( \Psi'_{-c+b-i,0} \otimes M_{c-a,0} \Pi_{i0} \right) &= -K_{nn} \sum_{k=-\infty}^{\infty} \left( \Psi'_{k0} \otimes \sum_{i=0}^{r} \Pi_{i0} M_{-k-a+b-i,0} \right) \\
&= K_{nn} \left( \Psi'_{b-a,0} \otimes I_n \right).
\end{aligned}
$$

Here the latter equality can be justified by using the identity $\Pi(z) M(z) = I_n$ to obtain an analog of (B.6) with $\Psi_{j-i,0}$ and $L_{-j0}$ replaced by $M_{j-i,0}$ and 0, respectively.

The preceding derivations and the definitions show that the covariance matrix of the scores of $\vartheta_1$ and $\vartheta_2$ divided by $N$ converges to $\mathcal{I}_{\vartheta_2 \vartheta_1}(\theta_0)$.

*Block* $\mathcal{I}_{\sigma\sigma}(\theta_0)$. First note that, by (S.3) and independence of $\epsilon_t$, we only need to show that $\mathbb{E}(\partial g_t(\theta_0)/\partial\sigma) = 0$ and $\mathbb{C}(\partial g_t(\theta_0)/\partial\sigma) = \mathcal{I}_{\sigma\sigma}(\theta_0)$. These facts can be established by observing

$$
\frac{\partial}{\partial\sigma} g_t(\theta_0) = -D_n'(\Sigma_0^{-1/2} \otimes \Sigma_0^{-1/2})(\varepsilon_t \otimes e_{0t} + \frac{1}{2}\text{vec}(I_n)).
$$

Thus, the desired results are obtained by using Lemma 4 (case $t = k$ and $i = j = 0$), and arguments in its proof.

*Block* $\mathcal{I}_{\lambda\lambda}(\theta_0)$. As in the preceding case, it suffices to show that $\mathbb{E}(\partial g_t(\theta_0)/\partial\lambda) = 0$ and $\mathbb{C}(\partial g_t(\theta_0)/\partial\lambda) = \mathcal{I}_{\lambda\lambda}(\theta_0)$. For the former, conclude from (S.4), the definition $\varepsilon_t = \Sigma_0^{-1/2}\epsilon_t$, and (B.3) that

$$
\begin{aligned}
\mathbb{E}_{\lambda_0}\left( \frac{\partial}{\partial\lambda} g_t(\theta_0) \right) &= \mathbb{E}_{\lambda_0}\left( \frac{1}{f(\rho_t^2;\lambda_0)} \cdot \frac{\partial}{\partial\lambda} f(\rho_t^2;\lambda) \Big|_{\lambda=\lambda_0} \right) \\
&= \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \zeta^{n/2-1} \frac{\partial}{\partial\lambda} f(\zeta;\lambda) \Big|_{\lambda=\lambda_0} d\zeta \\
&= \frac{\pi^{n/2}}{\Gamma(n/2)} \frac{\partial}{\partial\lambda} \int_0^\infty \zeta^{n/2-1} f(\zeta;\lambda)\, d\zeta |_{\lambda=\lambda_0} \\
&= 0.
\end{aligned}
$$

Here the second equality is based on the expression of the density function of $\rho_t^2$ (see (10)), the third one on Assumption 4(i), and the fourth one on the identity

$$
\int_0^\infty \zeta^{n/2-1} f(\zeta;\lambda)\, d\zeta = \frac{\Gamma(n/2)}{\pi^{n/2}} \int f(x'x;\lambda)\, dx = \frac{\Gamma(n/2)}{\pi^{n/2}},
$$

which can be obtained as in Fang et al. (1990, p. 35).

That $\mathbb{C}(\partial g_t(\theta_0)/\partial\lambda) = \mathcal{I}_{\lambda\lambda}(\theta_0)$ is an immediate consequence of Assumption 4(ii), (S.4), (B.3), and the expression of the density function of $\rho_t^2$.

*Blocks* $\mathcal{I}_{\vartheta_1\sigma}(\theta_0)$ *and* $\mathcal{I}_{\vartheta_1\lambda}(\theta_0)$. That these blocks are zero follows from (A.1), (S.1), (S.3), (S.4), independence of $\epsilon_t$, and the fact that $u_{0,t-i}$ $(i > 0)$ is independent of $\epsilon_t$ and has zero mean (see (3)).

*Block* $\mathcal{I}_{\vartheta_2\sigma}(\theta_0)$. Consider the covariance matrix (cf. the derivation of $\mathcal{I}_{\vartheta_2\vartheta_2}(\theta_0)$)

$$
\begin{aligned}
& \mathbb{C}\left(\sum_{i=0}^{r}\left(y_{t+a-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1/2}e_{0t},\frac{\partial}{\partial\sigma}g_k(\theta_0)\right) \\
= & -\sum_{c=-\infty}^{\infty}\sum_{i=0}^{r}A_0(c,i)\,\mathbb{C}\left((\varepsilon_{t+a-i-c}\otimes e_{0t}),(\varepsilon_k\otimes e_{0k})\right)(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n \\
= & -\sum_{i=0}^{r}A_0(a-i,i)\,D_nJ_0D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n\mathbf{1}(t=k).
\end{aligned}
$$

Here the former equality is based on (5), the definition on $A_0(c,i)$, and the expression of $\partial g_t(\theta_0)/\partial\sigma$ given in the case of block $\mathcal{I}_{\sigma\sigma}(\theta_0)$. The latter equality is due to Lemma 4. The stated expression of $\mathcal{I}_{\vartheta_2\sigma}(\theta_0)$ is a simple consequence of this and (S.2) and (S.3).

*Block* $\mathcal{I}_{\vartheta_2\lambda}(\theta_0)$. Similarly to the preceding case we consider the covariance matrix

$$
\begin{aligned}
& \mathbb{C}\left(\sum_{i=0}^{r}\left(y_{t+a-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1/2}e_{0t},\frac{\partial}{\partial\lambda}g_k(\theta_0)\right) \\
= & \sum_{c=-\infty}^{\infty}\sum_{i=0}^{r}A_0(c,i)\,\mathbb{C}\left((\varepsilon_{t+a-i-c}\otimes e_{0t}),\frac{\partial}{\partial\lambda}g_k(\theta_0)\right) \\
= & \sum_{c=-\infty}^{\infty}\sum_{i=0}^{r}A_0(c,i)\,\mathbb{E}\left[\left(\rho_{t+a-i-c}\upsilon_{t+a-i-c}\otimes\rho_th_0\left(\rho_t^2\right)\upsilon_t\right)\frac{1}{f_0\left(\rho_k^2\right)}\frac{\partial}{\partial\lambda'}f\left(\rho_k^2;\lambda_0\right)\right] \\
= & \sum_{c=-\infty}^{\infty}\sum_{i=0}^{r}A_0(c,i)\,\mathbb{E}\left(\upsilon_{t+a-i-c}\otimes\upsilon_t\right)\mathbb{E}\left[\rho_{t+a-i-c}\rho_th_0\left(\rho_t^2\right)\frac{1}{f_0\left(\rho_{k0}^2\right)}\frac{\partial}{\partial\lambda'}f\left(\rho_k^2;\lambda_0\right)\right].
\end{aligned}
$$

Here the first equality is justified by (5) whereas the remaining ones are obtained from (S.4), (B.3), (A.3), the independence of the processes $\rho_t$ and $\upsilon_t$, and the fact that $\partial g_t(\theta_0)/\partial\lambda$ has zero mean. Thus, because $\mathbb{E}(\upsilon_{t+a-i-c}\otimes\upsilon_t)=n^{-1}\mathrm{vec}(I_n)\,\mathbf{1}(c=a-i)$,

$$
\begin{aligned}
& \mathbb{C}\left(\sum_{i=0}^{r}\left(y_{t+a-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1/2}e_{0t},\frac{\partial}{\partial\lambda}g_k(\theta_0)\right) \\
= & \frac{1}{n}\sum_{i=0}^{r}A_0(a-i,i)\,\mathrm{vec}(I_n)\,\mathbb{E}\left(\rho_t^2\frac{h_0\left(\rho_t^2\right)}{f_0\left(\rho_t^2;\right)}\frac{\partial}{\partial\lambda'}f\left(\rho_t^2;\lambda_0\right)\right)\mathbf{1}(t=k),
\end{aligned}
$$

which in conjunction with (B.6) gives the desired result $\mathcal{I}_{\vartheta_2\lambda}(\theta_0)=0$.

*Block* $\mathcal{I}_{\sigma\lambda}(\theta_0)$. The employed arguments are similar to those in the cases of blocks $\mathcal{I}_{\sigma\sigma}(\theta_0)$ and $\mathcal{I}_{\lambda\lambda}(\theta_0)$. By the independence of $\epsilon_t$ it suffices to consider (see (S.3))

$$
\mathbb{C}\left(\frac{\partial}{\partial\sigma}g_t(\theta_0),\frac{\partial}{\partial\lambda}g_t(\theta_0)\right)=-D_n'\left(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2}\right)\mathbb{E}\left[(\varepsilon_t\otimes e_{0t})\frac{\partial}{\partial\lambda'}g_t(\theta_0)\right],
$$

where the expectation equals (see (B.3), (A.3), and (S.4))

$$
\mathbb{E}\left[\left(\rho_t\upsilon_t\otimes\rho_th_0\left(\rho_t^2\right)\upsilon_t\right)\frac{1}{f_0\left(\rho_t^2\right)}\frac{\partial}{\partial\lambda'}f\left(\rho_t^2;\lambda_0\right)\right]=\mathbb{E}\left(\upsilon_t\otimes\upsilon_t\right)\mathbb{E}\left[\rho_t^2\frac{h_0\left(\rho_t^2\right)}{f_0\left(\rho_t^2;\right)}\frac{\partial}{\partial\lambda'}f\left(\rho_t^2;\lambda_0\right)\right].
$$

Because $\mathbb{E}(\upsilon_t \otimes \upsilon_t) = n^{-1}\text{vec}(I_n) = n^{-1}D_n\text{vech}(I_n)$, the stated expression of $\mathcal{I}_{\sigma\lambda}(\theta_0)$ follows from the definitions and the expression of the density function of $\rho_t^2$ (see (10)).

Thus, we have completed the derivation of $\mathcal{I}_{\theta\theta}(\theta_0)$.

**Remaining part of the proof of Step 2 of Proposition 2.** To show that the infinite dimensional matrix $[\underline{G}_0(1) : \underline{G}_0(2) : \cdots]$ is of full row rank, first note that the first block of rows is readily seen to be of full row rank. Indeed, using the definition of $\underline{B}_0(k)$ it is straightforward to see that the matrix $[\underline{B}_0(1) : \cdots : \underline{B}_0(r)]$ $(rn^2 \times rn^2)$ is upper triangular with diagonal blocks $\Sigma_0^{-1/2} \otimes \Sigma_0^{-1/2}$ and, therefore, of full row rank. The last two blocks of rows are also linearly independent because the covariance matrix of $(x_{3t}, x_{4t})$ equals that of the scores of $\sigma$ and $\lambda$, which is positive definite by Assumption 5(ii). It is furthermore obvious that these two blocks of rows are linearly independent of the first block of rows. Thus, from the definition of $\underline{G}_0(k)$ it can be seen that it suffices to show that the infinite dimensional matrix $[\underline{A}_0(-1) : \underline{A}_0(-2) : \cdots]$ is of full row rank. We shall demonstrate that the matrix $[\underline{A}_0(-1) : \cdots : \underline{A}_0(-r-s)]$ $(sn^2 \times s(s+r)n^2)$ is of full row rank. For simplicity, we do this in the special case $s = 2$.

Consider the matrix product

$$
\begin{aligned}
&[\underline{A}_0(-1) : \cdots : \underline{A}_0(-r-2)] \begin{bmatrix} \Sigma_0^{-1/2}\Pi_{00} \otimes \Sigma_0^{1/2} & 0 \\ \vdots & \Sigma_0^{-1/2}\Pi_{00} \otimes \Sigma_0^{1/2} \\ \Sigma_0^{-1/2}\Pi_{r0} \otimes \Sigma_0^{1/2} & \vdots \\ 0 & \Sigma_0^{-1/2}\Pi_{r0} \otimes \Sigma_0^{1/2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=0}^{r}\left(\sum_{i=0}^{r} \Psi_{-j-i,0}\Pi_{i0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(\sum_{i=0}^{r} \Psi_{-1-j-i,0}\Pi_{i0} \otimes \Pi_{j0}'\right) \\ \sum_{j=0}^{r}\left(\sum_{i=0}^{r} \Psi_{1-j-i,0}\Pi_{i0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(\sum_{i=0}^{r} \Psi_{-j-i,0}\Pi_{i0} \otimes \Pi_{j0}'\right) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=0}^{r}\left(-L_{j0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(-L_{j+1,0} \otimes \Pi_{j0}'\right) \\ \sum_{j=0}^{r}\left(-L_{j-1,0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(-L_{j0} \otimes \Pi_{j0}'\right) \end{bmatrix},
\end{aligned}
$$

where the equalities follow from the definitions and from (B.6) by direct calculation. We shall show below that the last expression, a square matrix of order $2n^2 \times 2n^2$, is nonsingular. Assume this for the moment and note that the latter matrix in the product presented above is of full column rank $2n^2$ (because $\Pi_{00} = -I_n$ ). Thus, as the rank of a matrix product cannot exceed the ranks of the factors of the product, it follows that the matrix $[\underline{A}_0(-1) : \cdots : \underline{A}_0(-r-2)]$ has to be of full row rank $2n^2$.

To show the aforementioned nonsingularity, it clearly suffices to show the nonsingularity of the matrix

$$
\begin{aligned}
&\begin{bmatrix} \sum_{j=0}^{r}\left(-L_{j0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(-L_{j+1,0} \otimes \Pi_{j0}'\right) \\ \sum_{j=0}^{r}\left(-L_{j-1,0} \otimes \Pi_{j0}'\right) & \sum_{j=0}^{r}\left(-L_{j0} \otimes \Pi_{j0}'\right) \end{bmatrix} \begin{bmatrix} I_{n^2} & -\Phi_{10} \otimes I_n \\ 0 & I_{n^2} \end{bmatrix} \\
&= \begin{bmatrix} I_n & L_{10} - \Phi_{10} \\ 0 & I_n \end{bmatrix} \otimes I_n - \sum_{j=1}^{r}\left(\begin{bmatrix} L_{j0} & L_{j+1,0} - L_{j0}\Phi_{10} \\ L_{j-1,0} & L_{j,0} - L_{j-1,0}\Phi_{10} \end{bmatrix} \otimes \Pi_{j0}'\right) \\
&= \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} \otimes I_n - \sum_{j=1}^{r}\left(\begin{bmatrix} L_{j0} & L_{j-1,0}\Phi_{20} \\ L_{j-1,0} & L_{j-2,0}\Phi_{20} \end{bmatrix} \otimes \Pi_{j0}'\right).
\end{aligned}
$$

As in the proof of the nonsingularity of the matrix $\boldsymbol{H}_1$, we have here used the identity $L_{j0} = L_{j-1,0}\Phi_{10} + L_{j-2,0}\Phi_{20}$ with $L_{00} = I_n$ and $L_{j0} = 0$, $j < 0$, as well as direct calculation. In the same way as in that proof, we can now show the nonsingularity of the last matrix by using the fact that this matrix can be expressed as

$$I_{n^2} \otimes I_n - \sum_{j=1}^{r} \left(\mathbf{\Phi}_0^j \otimes \Pi'_{j0}\right) = \left(\mathbf{P}_0 \otimes I_n\right) \left(I_{n^2} \otimes I_n - \sum_{j=1}^{r} \left(\mathbf{D}_0^j \otimes \Pi'_{j0}\right)\right) \left(\mathbf{P}_0^{-1} \otimes I_n\right),$$

where $\mathbf{\Phi}_0$ is the companion matrix corresponding the matrix polynomial $I_n - \Phi_{10}z - \Phi_{20}z^2$ and $\mathbf{\Phi}_0 = \mathbf{P}_0\mathbf{D}_0\mathbf{P}_0^{-1}$ is its Jordan decomposition (cf. the aforementioned previous proof). The determinant of the matrix on the right hand side of the preceding equation is a product of determinants of the form $\det\left(I_n - \sum_{j=1}^{r} \Pi'_{j0}\nu^j\right)$ where $\nu$ signifies an eigenvalue of $\mathbf{\Phi}_0$. These determinants are nonzero because, by the latter condition in (2), the eigenvalues of $\mathbf{\Phi}_0$ are smaller than one in absolute value whereas the former condition in (2) implies that the zeros of $\det \Pi(z)$ lie outside the unit disc. This completes the proof of the positive definiteness of $\mathcal{I}_{\theta\theta}(\theta_0)$.

**Proof of Lemma 2**. In the same way as in the proof of Step 1 of Proposition 2 we consider the different blocks of $\mathcal{I}_{\theta\theta}(\theta_0)$ separately. For simplicity, we again suppress the subscript from the expectation operator and denote $\mathbb{E}(\cdot)$ instead of $\mathbb{E}_{\theta_0}(\cdot)$.

*Block* $\mathcal{I}_{\vartheta_1\vartheta_1}(\theta_0)$. Using the independence of $u_{0,t-i}$ $(i > 0)$ and $e_{0t}$ along with (B.4) it can be seen that the first term of the expression of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_1'$ (see Appendix A) evaluated at $\theta_0$ has zero expectation. Thus, it suffices to consider the expectation of the second term. To this end, recall the notation $\varepsilon_t = \Sigma_0^{-1/2}\epsilon_t$ and define

$$W^{(1)}_{\vartheta_1\vartheta_1}(a,b) = 2\mathbb{E}\left[h_0\left(\varepsilon_t'\varepsilon_t\right)\left(u_{0,t-a}u'_{0,t-b} \otimes \Sigma_0^{-1}\right)\right],$$

$$W^{(2)}_{\vartheta_1\vartheta_1}(a,b) = 4\mathbb{E}\left[\frac{f_0''\left(\varepsilon_t'\varepsilon_t\right)}{f_0\left(\varepsilon_t'\varepsilon_t\right)}\left(u_{0,t-a}u'_{0,t-b} \otimes \Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\right)\right],$$

and

$$W^{(3)}_{\vartheta_1\vartheta_1}(a,b) = -4\mathbb{E}\left[\left(h_0\left(\varepsilon_t'\varepsilon_t\right)\right)^2\left(u_{0,t-a}u'_{0,t-b} \otimes \Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\right)\right].$$

Using these definitions in conjunction with (A.6), (A.2), and (A.4) we can write the aforementioned expectation as

$$\begin{aligned}
& -2\sum_{a=1}^{r}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,\mathbb{E}\left[\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1/2}\frac{\partial}{\partial\vartheta_1'}e_t(\theta_0)\right] \\
= & -2\sum_{a=1}^{r}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,\mathbb{E}\left[h_0\left(\varepsilon_t'\varepsilon_t\right)\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1}\frac{\partial}{\partial\vartheta_1'}\epsilon_t(\vartheta_0)\right] \\
& -4\sum_{a=1}^{r}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,\mathbb{E}\left[\frac{f_0''\left(\varepsilon_t'\varepsilon_t\right)}{f_0\left(\varepsilon_t'\varepsilon_t\right)}\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\frac{\partial}{\partial\vartheta_1'}\epsilon_t(\vartheta_0)\right] \\
& +4\sum_{a=1}^{r}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,\mathbb{E}\left[\left(h_0\left(\varepsilon_t'\varepsilon_t\right)\right)^2\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\frac{\partial}{\partial\vartheta_1'}\epsilon_t(\vartheta_0)\right] \\
= & \sum_{a,b=1}^{r}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\left[W^{(1)}_{\vartheta_1\vartheta_1}(a,b)+W^{(2)}_{\vartheta_1\vartheta_1}(a,b)+W^{(3)}_{\vartheta_1\vartheta_1}(a,b)\right]\frac{\partial}{\partial\vartheta_1'}\pi_b(\vartheta_{10}).
\end{aligned}$$

We need to show that the last expression equals $-\mathcal{I}_{\vartheta_1\vartheta_1}(\theta_0)$, which follows if $\sum_{i=1}^{3} W^{(i)}_{\vartheta_1\vartheta_1}(a,b) = -C_{11}(a,b) \otimes \Sigma_0^{-1}$. To see this, conclude from the definitions, (B.3), and the fact $\mathbb{C}(\upsilon_t) = n^{-1} I_n$ that

$$W^{(1)}_{\vartheta_1\vartheta_1}(a,b) + W^{(2)}_{\vartheta_1\vartheta_1}(a,b) = 2\left[\mathbb{E}\left(h_0\left(\rho_t^2\right)\right) + \frac{2}{n}\mathbb{E}\left(\rho_t^2 \frac{f_0''\left(\rho_t^2\right)}{f_0\left(\rho_t^2\right)}\right)\right]\left(\mathbb{E}\left(u_{0,t-a}u'_{0,t-b}\right) \otimes \Sigma_0^{-1}\right).$$

Using definitions and the expression of the density of $\rho_t^2$ (see (10)) yields

$$\begin{aligned}
&\mathbb{E}\left(h_0\left(\rho_t^2\right)\right) + \frac{2}{n}\mathbb{E}\left(\rho_t^2 \frac{f_0''\left(\rho_t^2\right)}{f_0\left(\rho_t^2\right)}\right) && \text{(S.5)}\\
&= \frac{\pi^{n/2}}{\Gamma(n/2)}\left(\int_0^\infty \zeta^{n/2-1} f_0'(\zeta)\, d\zeta + \frac{2}{n}\int_0^\infty \zeta^{n/2} f_0''(\zeta)\, d\zeta\right)\\
&= \frac{\pi^{n/2}}{\Gamma(n/2)}\left(\int_0^\infty \zeta^{n/2-1} f_0'(\zeta)\, d\zeta + \frac{2}{n}\zeta^{n/2} f_0'(\zeta)\,|_0^\infty - \int_0^\infty \zeta^{n/2-1} f_0'(\zeta)\, d\zeta\right)\\
&= 0,
\end{aligned}$$

where the last two equalities are justified by Assumption 6(i). Thus, we can conclude that $W^{(1)}_{\vartheta_1\vartheta_1}(a,b) + W^{(2)}_{\vartheta_1\vartheta_1}(a,b) = 0$.

Regarding $W^{(3)}_{\vartheta_1\vartheta_1}(a,b)$, use again (B.3) and the fact $\mathbb{C}(\upsilon_t) = n^{-1} I_n$ to obtain

$$\begin{aligned}
W^{(3)}_{\vartheta_1\vartheta_1}(a,b) &= -\frac{4}{n}\mathbb{E}\left[\rho_t^2\left(h_0\left(\rho_t^2\right)\right)^2\right]\mathbb{E}\left(u_{0,t-a}u'_{0,t-b}\right) \otimes \Sigma_0^{-1}\\
&= -\boldsymbol{j}_0\mathbb{E}\left(u_{0,t-a}u'_{0,t-b}\right) \otimes \Sigma_0^{-1},
\end{aligned}$$

by the definitions of $h_0(\cdot)$ and $\boldsymbol{j}_0$ (see (11)). Thus, because $\boldsymbol{j}_0\mathbb{E}\left(u_{0,t-a}u'_{0,t-b}\right) = C_{11}(a,b)$, we have $\sum_{i=1}^{3} W^{(i)}_{\vartheta_1\vartheta_1}(a,b) = C_{11}(a,b) \otimes \Sigma_0^{-1}$, as desired.

*Block* $\mathcal{I}_{\vartheta_2\vartheta_2}(\theta_0)$. The first term on the right hand side of the expression of $\partial^2 g_t(\theta)/\partial\vartheta_2\partial\vartheta_2'$ (see Appendix A) evaluated at $\theta_0$ has zero expectation by arguments entirely similar to those used to show that the expectation of $\partial g_t(\theta_0)/\partial\vartheta_2$ is zero (see the proof of Proposition 2, Block $\mathcal{I}_{\vartheta_2\vartheta_2}(\theta_0)$). Thus, it suffices to consider the second term for which we first note that

$$\begin{aligned}
\mathbb{E}\left(\rho_t^4 \frac{f_0''\left(\rho_t^2\right)}{f_0\left(\rho_t^2\right)}\right) &= \frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty \zeta^{n/2+1} f_0''(\zeta)\, d\zeta\\
&= \frac{\pi^{n/2}}{\Gamma(n/2)}\left(\zeta^{n/2+1} f_0'(\zeta)\Big|_0^\infty - \frac{n+2}{2}\int_0^\infty \zeta^{n/2} f_0'(\zeta)\, d\zeta\right)\\
&= n(n+2)/4, && \text{(S.6)}
\end{aligned}$$

where the last equality is justified by Assumption 6(i) and (B.1).

Next define

$$W^{(1)}_{\vartheta_2\vartheta_2}(a,b) = 2\mathbb{E}\left[h_0\left(\varepsilon_t'\varepsilon_t\right)\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j} \otimes \Pi_{i0}'\Sigma_0^{-1}\Pi_{j0}\right)\right],$$

$$W^{(2)}_{\vartheta_2\vartheta_2}(a,b) = 4\mathbb{E}\left[\frac{f_0''(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j}\otimes\Pi'_{i0}\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right]$$

and

$$W^{(3)}_{\vartheta_2\vartheta_2}(a,b) = -4\mathbb{E}\left[\left(h_0(\varepsilon_t'\varepsilon_t)\right)^2\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j}\otimes\Pi'_{i0}\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right].$$

Using these definitions in conjunction with (A.5) and (A.6) the expectation of the second term on the right hand side of the expression of $\partial^2 g_t(\theta)/\partial\vartheta_2\partial\vartheta_2'$ (see Appendix A) evaluated at $\theta_0$ can be written as

$$\begin{aligned}
& 2\sum_{a=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_a'(\vartheta_{20})\,\mathbb{E}\left[\sum_{i=0}^{r}\left(y_{t+a-i}\otimes\Pi'_{i0}\right)\Sigma_0^{-1/2}\frac{\partial}{\partial\vartheta_2'}e_t(\theta_0)\right]\\
= \; & 2\sum_{a,b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_a'(\vartheta_{20})\,\mathbb{E}\left[\frac{f_0'(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j}\otimes\Pi'_{i0}\Sigma_0^{-1}\Pi_{j0}\right)\right]\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20})\\
& +4\sum_{a,b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_a'(\vartheta_{20})\,\mathbb{E}\left[\frac{f_0''(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j}\otimes\Pi'_{i0}\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right]\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20})\\
& -4\sum_{a,b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_a'(\vartheta_{20})\,\mathbb{E}\left[\left(\frac{f_0'(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\right)^2\sum_{i,j=0}^{r}\left(y_{t+a-i}y'_{t+b-j}\otimes\Pi'_{i0}\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right]\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20})\\
= \; & \sum_{a,b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_a'(\vartheta_{20})\left[W^{(1)}_{\vartheta_2\vartheta_2}(a,b)+W^{(2)}_{\vartheta_2\vartheta_2}(a,b)+W^{(3)}_{\vartheta_2\vartheta_2}(a,b)\right]\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20}).
\end{aligned}$$

Thus, to show that the last expression equals $-\mathcal{I}_{\vartheta_2\vartheta_2}(\theta_0)$ it suffices to show that $\sum_{i=1}^{3}W^{(i)}_{\vartheta_2\vartheta_2}(a,b) = -C_{22}(a,b,;\theta_0)$. To this end, first note that, by (5),

$$\begin{aligned}
W^{(1)}_{\vartheta_2\vartheta_2}(a,b) &= 2\sum_{i,j=0}^{r}\sum_{c,d=-\infty}^{\infty}\mathbb{E}\left[h_0(\varepsilon_t'\varepsilon_t)\left(\Psi_{c0}\epsilon_{t+a-i-c}\epsilon'_{t+b-j-d}\Psi'_{d0}\otimes\Pi'_{i0}\Sigma_0^{-1}\Pi_{j0}\right)\right]\\
&= \frac{2}{n}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left(h_0\left(\rho_t^2\right)\right)\sum_{i,j=0}^{r}\sum_{\substack{c=-\infty\\ c\neq 0}}^{\infty}A_0(c+a-i,i)\,A_0(c+b-j,j)\\
&\qquad -\sum_{i,j=0}^{r}A_0(a-i,i)\,A_0(b-j,j),
\end{aligned}$$

where, as before, $\Psi_{k0}\Sigma_0^{1/2}\otimes\Pi'_{i0}\Sigma_0^{-1/2} = A_0(k,i)$. The latter equality is a straightforward consequence of (B.3), (B.1), and the fact $\mathbb{C}(\upsilon_t) = n^{-1}I_n$.

For $W^{(2)}_{\vartheta_2\vartheta_2}(a,b)$ one obtains from (5)

$$\begin{aligned}
W^{(2)}_{\vartheta_2\vartheta_2}(a,b) &= 4\sum_{i,j=0}^{r}\sum_{c,d=-\infty}^{\infty}\mathbb{E}\left[\frac{f_0''(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\left(\Psi_{c0}\epsilon_{t+a-i-c}\epsilon_{t+b-j-d}'\Psi_{d0}'\otimes\Pi_{i0}'\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right] \\
&= \frac{4}{n^2}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left(\rho_t^2\frac{f_0''(\rho_t^2)}{f_0(\rho_t^2)}\right)\sum_{i,j=0}^{r}\sum_{\substack{c=-\infty\\c\neq 0}}^{\infty}A_0(c+a-i,i)\,A_0(c+b-j,j) \\
&\quad +4\mathbb{E}\left(\rho_t^4\frac{f_0''(\rho_t^2)}{f_0(\rho_t^2)}\right)\sum_{i,j=0}^{r}A_0(a-i,i)\,\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)A_0(b-j,j),
\end{aligned}$$

where the latter equality is again obtained from (B.3) and the fact $\mathbb{C}(\upsilon_t)=n^{-1}I_n$. From (S.5) and (S.6) we can now conclude that

$$\begin{aligned}
W^{(1)}_{\vartheta_2\vartheta_2}(a,b)+W^{(2)}_{\vartheta_2\vartheta_2}(a,b) &= -\sum_{i,j=0}^{r}A_0(a-i,i)\,A_0(b-j,j) \\
&\quad +n(n+2)\sum_{i,j=0}^{r}A_0(a-i,i)\,\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)A_0(b-j,j).
\end{aligned}$$

Next, arguments similar to those already used give

$$\begin{aligned}
W^{(3)}_{\vartheta_2\vartheta_2}(a,b) &= -4\sum_{i,j=0}^{r}\sum_{c,d=-\infty}^{\infty}\mathbb{E}\left[\left(h_0(\varepsilon_t'\varepsilon_t)\right)^2\left(\Psi_{c0}\epsilon_{t+a-i-c}\epsilon_{t+b-j-d}'\Psi_{d0}'\otimes\Pi_{i0}'\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{j0}\right)\right] \\
&= -\frac{4}{n^2}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left[\rho_t^2\left(h_0(\rho_t^2)\right)^2\right]\sum_{i,j=0}^{r}\sum_{\substack{c=-\infty\\c\neq 0}}^{\infty}A_0(c+a-i,i)\,A_0(c+b-j,j) \\
&\quad -4\mathbb{E}\left[\rho_t^4\left(h_0(\rho_t^2)\right)^2\right]\sum_{i,j=0}^{r}A_0(a-i,i)\,\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)A_0(b-j,j) \\
&= -\boldsymbol{\tau}_0\sum_{i,j=0}^{r}\sum_{\substack{c=-\infty\\c\neq 0}}^{\infty}A_0(c+a-i,i)\,A_0(c+b-j,j) \\
&\quad -4\sum_{i,j=0}^{r}A_0(a-i,i)\,D_nJ_0D_n'A_0(b-j,j).
\end{aligned}$$

Here the last equality follows from the definitions of $\boldsymbol{\tau}_0$, $\boldsymbol{\imath}_0$, and $J_0$ (in the term involving $J_0$ (B.6) has also been used).

From the preceding derivations we find that

$$\begin{aligned}
\sum_{i=1}^{3}W^{(i)}_{\vartheta_2\vartheta_2}(a,b) &= -\boldsymbol{\tau}_0\sum_{i,j=0}^{r}\sum_{\substack{c=-\infty\\c\neq 0}}^{\infty}A_0(c+a-i,i)\,A_0(c+b-j,j) \\
&\quad -\sum_{i,j=0}^{r}A_0(a-i,i)\left[4D_nJ_0D_n'+I_n-n(n+2)\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)\right]A_0(b-j,j).
\end{aligned}$$

That $\sum_{i=1}^{3} W_{\vartheta_2\vartheta_2}^{(i)}(a,b) = -C_{22}(a,b,;\theta_0)$ holds, can now be seen by using the identity

$$\mathbb{E}\left[(\mathrm{vec}(\upsilon_t\upsilon_t'))(\mathrm{vec}(\upsilon_t\upsilon_t'))'\right] = \frac{1}{n(n+2)}\left(I_{n^2} + K_{nn} + \mathrm{vec}(I_n)\,\mathrm{vec}(I_n)'\right) \qquad \text{(S.7)}$$

(see Wong and Wang (1992, p. 274)) and observing that the left hand side equals $\mathbb{E}(\upsilon_t\upsilon_t' \otimes \upsilon_t\upsilon_t')$ and that the impact of the term $\mathrm{vec}(I_n)\mathrm{vec}(I_n)'$ on the right hand side cancels by equality (B.6) (see the definition of $C_{22}(a,b,;\theta_0)$).

*Block* $\mathcal{I}_{\vartheta_1\vartheta_2}(\theta_0)$. First conclude from the expression of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$ (see Appendix A), (A.5), (A.6), and (B.3) that

$$\begin{aligned}
\frac{\partial^2}{\partial\vartheta_1\partial\vartheta_2'}g_t(\theta_0) \;=\;& 2\sum_{a=1}^{r}\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\left(I_n\otimes\Sigma_0^{-1/2}e_t(\theta_0)\right)\left(y_{t+b-a}'\otimes I_n\right)\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20})\\
&-2\sum_{a=1}^{r}\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,h_0(\varepsilon_t'\varepsilon_t)\sum_{i=0}^{r}\left(u_{0,t-a}y_{t+b-i}'\otimes\Sigma_0^{-1}\Pi_{i0}\right)\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20})\\
&-4\sum_{a=1}^{r}\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_1}\pi_a'(\vartheta_{10})\,h_0'(\varepsilon_t'\varepsilon_t)\sum_{i=0}^{r}\left(u_{0,t-a}y_{t+b-i}'\otimes\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{i0}\right)\frac{\partial}{\partial\vartheta_2'}\phi_b(\vartheta_{20}).
\end{aligned}$$

In the first expression on the right hand side,

$$\left(I_n\otimes\Sigma_0^{-1/2}e_t(\theta_0)\right)\left(y_{t+b-a}'\otimes I_n\right) = h_0(\varepsilon_t'\varepsilon_t)\,K_{nn}\left(\Sigma_0^{-1}\epsilon_t y_{t+b-a}'\otimes I_n\right)$$

by the definition of $e_t(\theta_0)$ (see (A.1)) and Result 9.2.2(3) in Lütkepohl (1996). Define

$$W_{\vartheta_1\vartheta_2}^{(1)}(a,b) = 2K_{nn}\mathbb{E}\left[h_0(\varepsilon_t'\varepsilon_t)\left(\Sigma_0^{-1}\epsilon_t y_{t+b-a}'\otimes I_n\right)\right],$$

$$W_{\vartheta_1\vartheta_2}^{(2)}(a,b) = -2\mathbb{E}\left[h_0(\varepsilon_t'\varepsilon_t)\sum_{i=0}^{r}\left(u_{0,t-a}y_{t+b-i}'\otimes\Sigma_0^{-1}\Pi_{i0}\right)\right]$$

$$W_{\vartheta_1\vartheta_2}^{(3)}(a,b) = -4\mathbb{E}\left[\frac{f_0''(\varepsilon_t'\varepsilon_t)}{f_0(\varepsilon_t'\varepsilon_t)}\sum_{i=0}^{r}\left(u_{0,t-a}y_{t+b-i}'\otimes\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{i0}\right)\right]$$

and

$$W_{\vartheta_1\vartheta_2}^{(4)}(a,b) = 4\mathbb{E}\left[\left(h_0(\varepsilon_t'\varepsilon_t)\right)^2\sum_{i=0}^{r}\left(u_{0,t-a}y_{t+b-i}'\otimes\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{i0}\right)\right].$$

We need to show that $\sum_{i=1}^{4} W_{\vartheta_1\vartheta_2}^{(i)}(a,b) = -C_{12}(a,b;\theta_0)$. The employed arguments, based mostly on (3), (5), (B.3), and the fact $\mathbb{C}(\upsilon_t) = n^{-1}I_n$, are similar to those used in the previous cases. First note that

$$\begin{aligned}
W_{\vartheta_1\vartheta_2}^{(1)}(a,b) \;&=\; 2K_{nn}\sum_{c=-\infty}^{\infty}\mathbb{E}\left[h_0(\varepsilon_t'\varepsilon_t)\left(\Sigma_0^{-1}\epsilon_t\epsilon_{t+b-a-c}'\Psi_{c0}'\otimes I_n\right)\right]\\
&=\; \frac{2}{n}\mathbb{E}\left[\rho_t^2 h_0(\rho_t^2)\right]K_{nn}\left(\Psi_{b-a,0}'\otimes I_n\right)\\
&=\; -K_{nn}\left(\Psi_{b-a,0}'\otimes I_n\right),
\end{aligned}$$

where the last equality is due to (B.1). Next,

$$\begin{aligned} W^{(2)}_{\vartheta_1\vartheta_2}(a,b) &= -2\sum_{c=0}^{\infty}\sum_{d=-\infty}^{\infty}\sum_{i=0}^{r}\mathbb{E}\left[h_0\left(\varepsilon_t'\varepsilon_t\right)\left(M_{c0}\epsilon_{t-a-c}\epsilon_{t+b-i-d}'\Psi_{d0}'\otimes\Sigma_0^{-1}\Pi_{i0}\right)\right] \\ &= -\frac{2}{n}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left(h_0\left(\rho_t^2\right)\right)\sum_{c=0}^{\infty}\sum_{i=0}^{r}\left(M_{c0}\Sigma_0\Psi_{c+a+b-i,0}'\otimes\Sigma_0^{-1}\Pi_{i0}\right) \end{aligned}$$

and

$$\begin{aligned} W^{(3)}_{\vartheta_1\vartheta_2}(a,b) &= -4\sum_{c=0}^{\infty}\sum_{d=-\infty}^{\infty}\sum_{i=0}^{r}\mathbb{E}\left[\frac{f_0''\left(\varepsilon_t'\varepsilon_t\right)}{f_0\left(\varepsilon_t'\varepsilon_t\right)}\left(M_{c0}\epsilon_{t-a-c}\epsilon_{t+b-i-d}'\Psi_{d0}'\otimes\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{i0}\right)\right] \\ &= -\frac{4}{n^2}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left(\rho_t^2\frac{f_0''\left(\rho_t^2\right)}{f_0\left(\rho_t^2\right)}\right)\sum_{c=0}^{\infty}\sum_{i=0}^{r}\left(M_{c0}\Sigma_0\Psi_{c+a+b-i,0}'\otimes\Sigma_0^{-1}\Pi_{i0}\right). \end{aligned}$$

From the preceding expressions and (S.5) it is seen that $W^{(2)}_{\vartheta_1\vartheta_2}(a,b)+W^{(3)}_{\vartheta_1\vartheta_2}(a,b)=0$.

Regarding $W^{(4)}_{\vartheta_1\vartheta_2}(a,b)$, we have

$$\begin{aligned} W^{(4)}_{\vartheta_1\vartheta_2}(a,b) &= 4\sum_{c=0}^{\infty}\sum_{d=-\infty}^{\infty}\sum_{i=0}^{r}\mathbb{E}\left[\left(h_0\left(\varepsilon_t'\varepsilon_t\right)\right)^2\left(M_{c0}\epsilon_{t-a-c}\epsilon_{t+b-i-d}'\Psi_{d0}'\otimes\Sigma_0^{-1}\epsilon_t\epsilon_t'\Sigma_0^{-1}\Pi_{i0}\right)\right] \\ &= \frac{4}{n^2}\mathbb{E}\left(\rho_t^2\right)\mathbb{E}\left[\rho_t^2\left(h_0\left(\rho_t^2\right)\right)^2\right]\sum_{c=0}^{\infty}\sum_{i=0}^{r}\left(M_{c0}\Sigma_0\Psi_{c+a+b-i,0}'\otimes\Sigma_0^{-1}\Pi_{i0}\right) \\ &= \boldsymbol{\tau}_0\sum_{c=a}^{\infty}\sum_{i=0}^{r}\left(M_{c-a,0}\Sigma_0\Psi_{c+b-i,0}'\otimes\Sigma_0^{-1}\Pi_{i0}\right), \end{aligned}$$

where the last equality holds by the definitions of $h_0(\cdot)$ and $\boldsymbol{\tau}_0$. Combining the preceding derivations yields $\sum_{i=1}^{4}W^{(i)}_{\vartheta_1\vartheta_2}(a,b)=-C_{12}(a,b;\theta_0)$, as desired.

*Block* $\mathcal{I}_{\sigma\sigma}(\theta_0)$. From the expression of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$ (see Appendix A) and (B.3) we obtain

$$\begin{aligned} \frac{\partial^2}{\partial\sigma\partial\sigma'}g_t(\theta_0) &= 2h_0\left(\varepsilon_t'\varepsilon_t\right)D_n'\left(\Sigma_0^{-1/2}\varepsilon_t\varepsilon_t'\Sigma_0^{-1/2}\otimes\Sigma_0^{-1}\right)D_n+\frac{1}{2}D_n'\left(\Sigma_0^{-1}\otimes\Sigma_0^{-1}\right)D_n \\ &\quad +h_0'\left(\varepsilon_t'\varepsilon_t\right)D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\left(\varepsilon_t\varepsilon_t'\otimes\varepsilon_t\varepsilon_t'\right)(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n. \end{aligned}$$

Using (B.3) and the independence of $\rho_t$ and $\upsilon_t$ the expectation of the first term on the right hand side can be written as

$$2\mathbb{E}\left(\rho_t^2h_0\left(\rho_t^2\right)\right)D_n'\left(\Sigma_0^{-1/2}\mathbb{E}\left(\upsilon_t\upsilon_t'\right)\Sigma^{-1/2}\otimes\Sigma^{-1}\right)D_n=-D_n'\left(\Sigma_0^{-1}\otimes\Sigma_0^{-1}\right)D_n,$$

where the equality is based on (B.1) and the fact $\mathbb{E}\left(\upsilon_t\upsilon_t'\right)=n^{-1}I_n$. Thus, we can conclude that

$$\begin{aligned} \mathbb{E}\left(\frac{\partial^2}{\partial\sigma\partial\sigma'}g_t(\theta_0)\right) &= D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\mathbb{E}\left[h_0'\left(\varepsilon_t'\varepsilon_t\right)\left(\varepsilon_t\varepsilon_t'\otimes\varepsilon_t\varepsilon_t'\right)\right]\left(\Sigma^{-1/2}\otimes\Sigma^{-1/2}\right)D_n \\ &\quad -\frac{1}{2}D_n'\left(\Sigma^{-1}\otimes\Sigma^{-1}\right)D_n. \end{aligned}$$

Using (B.3) and (A.2) one obtains

$$\begin{aligned}\mathbb{E}\left[h_0'\left(\varepsilon_t'\varepsilon_t\right)\left(\varepsilon_t\varepsilon_t'\otimes\varepsilon_t\varepsilon_t'\right)\right] &= \left[\mathbb{E}\left(\rho_t^4\frac{f_0''\left(\rho_t^2\right)}{f_0\left(\rho_t^2\right)}\right)-\mathbb{E}\left(\rho_t^4\left(h_0\left(\rho_t^2\right)\right)^2\right)\right]\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)\\ &= \frac{n\left(n+2\right)}{4}\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)-\boldsymbol{i}_0\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right),\end{aligned}$$

where the latter equality is based on (S.6) and the definition of $\boldsymbol{i}_0$ (see (12)). Thus,

$$\begin{aligned}\mathbb{E}\left(\frac{\partial^2}{\partial\sigma\partial\sigma'}g_t\left(\theta_0\right)\right) &= \frac{1}{4}D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\left[n(n+2)\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)-2I_{n^2}\right](\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n\\ &\quad -\boldsymbol{i}_0D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n.\end{aligned}$$

Because $\mathbb{E}\left(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t'\right)=D_n\mathbb{E}\left((\mathrm{vech}(\upsilon_t\upsilon_t'))(\mathrm{vech}(\upsilon_t\upsilon_t'))D_n'\right.$ the right hand side equals $-\mathcal{I}_{\sigma\sigma}\left(\theta_0\right)$ if the expression in the brackets can be replaced by $\mathrm{vec}(I_n)\mathrm{vec}(I_n)'$. From (S.7) it is seen that this expression can be replaced by $\mathrm{vec}(I_n)\mathrm{vec}(I_n)'+K_{nn}-I_{n^2}$. Thus, the desired result follows because

$$\left(K_{nn}-I_{n^2}\right)(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n=(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\left(K_{nn}-I_{n^2}\right)D_n=0$$

by Results 9.2.2(2)(b) and 9.2.3(2) in Lütkepohl (1996).

*Block* $\mathcal{I}_{\lambda\lambda}\left(\theta_0\right)$. By the definition of $\mathcal{I}_{\lambda\lambda}\left(\theta_0\right)$ and the expression of $\partial^2g_t\left(\theta\right)/\partial\lambda_1\partial\lambda_2'$ (see Appendix A) it suffices to note that

$$\mathbb{E}\left[\frac{1}{f\left(\rho_t^2;\lambda_0\right)}\frac{\partial^2}{\partial\lambda\partial\lambda'}f\left(\rho_t^2;\lambda_0\right)\right]=\frac{\pi^{n/2}}{\Gamma\left(n/2\right)}\int_0^\infty\zeta^{n/2-1}\frac{\partial^2}{\partial\lambda\partial\lambda'}f\left(\zeta;\lambda_0\right)d\zeta=0,$$

where the former equality is based on (10) and the latter on Assumption 6(ii) (cf. the corresponding part of the proof of Proposition 2, Block $\mathcal{I}_{\lambda\lambda}\left(\theta_0\right)$).

*Blocks* $\mathcal{I}_{\vartheta_1\sigma}\left(\theta_0\right)$ and $\mathcal{I}_{\vartheta_1\lambda}\left(\theta_0\right)$. The former is an immediate consequence of the expression of $\partial^2g_t\left(\theta\right)/\partial\vartheta_1\partial\sigma'$ (see Appendix A), the independence of $\epsilon_t$ and $\partial\epsilon_t'\left(\vartheta_0\right)/\partial\vartheta_1$, and the fact $\mathbb{E}\left(\partial\epsilon_t'\left(\vartheta_0\right)/\partial\vartheta_1\right)=0$ (see (A.4)) which imply $\mathbb{E}\left(\partial^2g_t\left(\theta_0\right)/\partial\vartheta_1\partial\sigma'\right)=0$.

As for $\mathcal{I}_{\vartheta_1\lambda}\left(\theta_0\right)$, it is seen from the expression of $\partial^2g_t\left(\theta\right)/\partial\vartheta_1\partial\lambda'$ (see Appendix A), the definition of function $h$ (see above (A.1)), and (A.4) that we need to show that

$$\mathbb{E}\left[\frac{1}{f_0\left(\varepsilon_t'\varepsilon_t\right)}\left(u_{0,t-a}\otimes I_n\right)\Sigma_0^{-1}\epsilon_t\frac{\partial}{\partial\lambda'}f'\left(\varepsilon_t'\varepsilon_t;\lambda_0\right)\right]=0,\quad a=1,...,r,$$

and similarly when $1/f_0\left(\varepsilon_t'\varepsilon_t\right)$ is replaced by $f_0'\left(\varepsilon_t'\varepsilon_t\right)/\left(f_0\left(\varepsilon_t'\varepsilon_t\right)\right)^2$. These facts follow from the independence of $u_{0,t-a}$ and $\epsilon_t$ and the fact $\mathbb{E}\left(u_{0,t-a}\right)=0$.

*Block* $\mathcal{I}_{\vartheta_2\sigma}\left(\theta_0\right)$. From the expression of $\partial^2g_t\left(\theta\right)/\partial\vartheta_2\partial\sigma'$ (see Appendix A) and (A.5) we find that

$$\begin{aligned}&\frac{\partial^2}{\partial\vartheta_2\partial\sigma'}g_t\left(\theta_0\right)\\ &= -2h_0\left(\varepsilon_t'\varepsilon_t\right)\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'\left(\vartheta_{20}\right)\sum_{a=0}^{r}\left(\epsilon_t'\otimes y_{t+b-a}\otimes\Pi_{a0}'\right)\left(\Sigma_0^{-1}\otimes\Sigma_0^{-1}\right)D_n\\ &\quad -2h_0'\left(\varepsilon_t'\varepsilon_t\right)\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'\left(\vartheta_{20}\right)\sum_{a=0}^{r}\left(y_{t+b-a}\otimes\Pi_{a0}'\right)\Sigma_0^{-1}\epsilon_t\left(\epsilon_t'\otimes\epsilon_t'\right)\left(\Sigma^{-1}\otimes\Sigma^{-1}\right)D_n.\end{aligned}$$

By independence of $\epsilon_t$ and equation (5), $y_{t+b-a}$ on the right hand side can be replaced by $\Psi_{b-a,0}\epsilon_t$ when expectation is taken. Thus, using the definition of $e_{t0}$ (see (A.1)) and straightforward calculation the expectation of the first term on the right hand side becomes

$$
\begin{aligned}
& -2\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}\mathbb{E}\left[e_{0t}'\otimes\Psi_{b-a,0}\epsilon_t\otimes\Pi_{a0}'\Sigma_0^{-1/2}\right](\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n \\
= & -2\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}A_0(b-a,i)\,\mathbb{E}\left[(e_{0t}'\otimes\varepsilon_t\otimes I_n)\right](\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n \\
= & \sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}A_0(b-a,i)\,(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n,
\end{aligned}
$$

where, again, $A_0(b-a,i)=\Psi_{b-a,0}\Sigma_0^{1/2}\otimes\Pi_{a0}'\Sigma_0^{-1/2}$ and the latter equality is due to $\mathbb{E}(e_{0t}'\otimes\varepsilon_t\otimes I_n)=\mathbb{E}(\varepsilon_t e_{0t}'\otimes I_n)=-2^{-1}I_{n^2}$ (see (B.5)).

The expectation of the second term in the preceding expression of $\partial^2 g_t(\theta_0)/\partial\vartheta_2\partial\sigma'$ can similarly be written as

$$
-2\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\,\mathbb{E}\left[h_0'(\varepsilon_t'\varepsilon_t)\sum_{a=0}^{r}(\Psi_{b-a,0}\epsilon_t\otimes\Pi_{a0}')\,\Sigma_0^{-1}\epsilon_t\,(\varepsilon_t'\otimes\varepsilon_t')\right](\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n,
$$

where, by (B.3) and (A.2), the expectation equals

$$
\begin{aligned}
& \left\{\mathbb{E}\left[\rho_t^4\frac{f_0''(\rho_t^2)}{f_0(\rho_t^2)}\right]-\mathbb{E}\left[\rho_t^4\left(h_0(\rho_t^2)\right)^2\right]\right\}\sum_{a=0}^{r}A_0(b-a,i)\,\mathbb{E}(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t') \\
= & \left(\frac{n(n+2)}{4}-\boldsymbol{i}_0\right)\sum_{a=0}^{r}A_0(b-a,i)\,\mathbb{E}(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t').
\end{aligned}
$$

Here we have used (S.6), the definition of $\boldsymbol{i}_0$ (see (12)), and straightforward calculation. Combining the preceding derivations shows that

$$
\begin{aligned}
\mathbb{E}\left(\frac{\partial^2}{\partial\vartheta_2\partial\sigma'}g_t(\theta_0)\right) = & \ 2\left(\boldsymbol{i}_0-\frac{n(n+2)}{4}\right)\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}A_0(b-a,i)\,\mathbb{E}(\upsilon_t\upsilon_t'\otimes\upsilon_t\upsilon_t') \\
& \times(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n \\
& +\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}A_0(b-a,i)\,(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n \\
= & \ 2\sum_{b=1}^{s}\frac{\partial}{\partial\vartheta_2}\phi_b'(\vartheta_{20})\sum_{a=0}^{r}A_0(b-a,i)\,D_nJ_0D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})D_n,
\end{aligned}
$$

where the last expression equals $-\mathcal{I}_{\vartheta_2\sigma}(\theta_0)$ and the latter equality can be justified by using the definition of $J_0$, the identity (S.7), and arguments similar to those already used in the case of block $\mathcal{I}_{\sigma\sigma}(\theta_0)$ (see the end of that proof).

*Block* $\mathcal{I}_{\vartheta_2\lambda}(\theta_0)$. From the expression of $\partial^2 g_t(\theta)/\partial\vartheta_2\partial\lambda'$ (see Appendix A) and (A.5) it is seen that we need to show that

$$\sum_{i=0}^{r}\mathbb{E}\left[\frac{1}{f_0(\varepsilon_t'\varepsilon_t)}\left(y_{t+a-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1}\epsilon_t\frac{\partial}{\partial\lambda'}f'(\varepsilon_t'\varepsilon_t;\lambda_0)\right]=0,\quad a=1,...,r,$$

and

$$\sum_{i=0}^{r}\mathbb{E}\left[\frac{f_0'(\varepsilon_t'\varepsilon_t)}{(f_0(\varepsilon_t'\varepsilon_t))^2}\left(y_{t+a-i}\otimes\Pi_{i0}'\right)\Sigma_0^{-1}\epsilon_t\frac{\partial}{\partial\lambda'}f(\varepsilon_t'\varepsilon_t;\lambda_0)\right]=0,\quad a=1,...,r.$$

The argument is similar in both cases and also similar to that used in the proof of Proposition 2 (see Block $\mathcal{I}_{\vartheta_2\lambda}(\theta_0)$). For example, consider the former and use (5) and independence of $\epsilon_t$ to write the left hand side of the equality as

$$\begin{aligned}&\sum_{i=0}^{r}\mathbb{E}\left[\frac{1}{f_0(\varepsilon_t'\varepsilon_t)}\left(\Psi_{a-i,0}\epsilon_t\otimes\Pi_{i0}'\right)\Sigma_0^{-1}\epsilon_t\frac{\partial}{\partial\lambda'}f'(\varepsilon_t'\varepsilon_t;\lambda_0)\right]\\ =\;&\sum_{i=0}^{r}A_0(a-i,i)\,\mathbb{E}(\upsilon_t\otimes\upsilon_t)\,\mathbb{E}\left[\frac{\rho_t^2}{f_0(\rho_t^2)}\frac{\partial}{\partial\lambda'}f'(\rho_t^2;\lambda_0)\right],\end{aligned}$$

where that equality is due to (B.3). Because $\mathbb{E}(\upsilon_t\otimes\upsilon_t)=\mathrm{vec}(\mathbb{E}(\upsilon_t\upsilon_t'))=n^{-1}\mathrm{vec}(I_n)$ the last expression is zero by (B.6). A similar proof applies to the other expectation.

*Block* $\mathcal{I}_{\sigma\lambda}(\theta_0)$. One obtains from the expression of $\partial^2 g_t(\theta)/\partial\sigma\partial\lambda'$ (see Appendix A) that $\mathbb{E}(\partial^2 g_t(\theta_0)/\partial\sigma\partial\lambda)$ is a sum of two terms. One is

$$\begin{aligned}-D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\mathbb{E}\left[\frac{1}{f_0(\varepsilon_t'\varepsilon_t)}(\varepsilon_t\otimes\varepsilon_t)\frac{\partial}{\partial\lambda'}f'(\varepsilon_t'\varepsilon_t;\lambda_0)\right]&=-D_n'(\Sigma_0^{-1/2}\otimes\Sigma_0^{-1/2})\mathbb{E}(\upsilon_t\otimes\upsilon_t)\\ &\quad\times\mathbb{E}\left[\frac{\rho_t^2}{f_0(\rho_t^2)}\frac{\partial}{\partial\lambda'}f'(\rho_t^2;\lambda_0)\right],\end{aligned}$$

where the equality is based on (B.3) and, using (10), the last expectation can be written as

$$\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty\zeta^{n/2}\left.\frac{\partial}{\partial\lambda'}f'(\zeta;\lambda)\right|_{\lambda=\lambda_0}d\zeta=\frac{\pi^{n/2}}{\Gamma(n/2)}\frac{\partial}{\partial\lambda'}\int_0^\infty\zeta^{n/2}f'(\zeta;\lambda)\,d\zeta|_{\lambda=\lambda_0}=0.$$

Here the former equality is justified by Assumption 6(ii) and the latter by (B.1). By similar arguments it is seen that the second term of $\mathbb{E}(\partial^2 g_t(\theta_0)/\partial\sigma\partial\lambda)$ becomes $-\mathcal{I}_{\sigma\lambda}(\theta_0)$. □

**Remaining parts of the proof of Theorem 1**. We demonstrate (B.11) for some typical components of $\partial^2 g_t(\theta)/\partial\theta\partial\theta'$ and note that the remaining components can be handled along similar lines. Of $\partial^2 g_t(\theta)/\partial\vartheta_i\partial\vartheta_j'$ $i,j\in\{1,2\}$ we only consider $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$. In what follows, $c_1, c_2, ...$ will denote positive constants. From the expression of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$ (see Appendix A), Assumption 3, and the definitions of the quantities involved (see (A.1), (A.6), (A.5), (18)) it can be seen that

$$\begin{aligned}
\mathbb{E}_{\theta_0}\left(\sup_{\theta\in\Theta_0}\left\|\frac{\partial^2}{\partial\vartheta_1\partial\vartheta_2'}g_t(\theta)\right\|\right) &\leq c_1\mathbb{E}_{\theta_0}\left(\sup_{\theta\in\Theta_0}\|e_t(\theta)\|\sum_{i=1}^{r}\left\|\frac{\partial}{\partial\vartheta_2}u_{t-i}'(\vartheta_2)\right\|\right)\\
&\quad +c_2\mathbb{E}_{\theta_0}\left(\sup_{\theta\in\Theta_0}\sum_{i=1}^{r}\|u_{t-i}(\vartheta_2)\|\left\|\frac{\partial}{\partial\vartheta_2}e_t'(\theta)\right\|\right)\\
&\leq c_3\mathbb{E}_{\theta_0}\left(\left(\sum_{j=-r}^{s}\|y_{t+j}\|\right)^2\sup_{\theta\in\Theta_0}\left|h\left(\epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta);\lambda\right)\right|\right)\\
&\quad +c_4\mathbb{E}_{\theta_0}\left(\left(\sum_{j=-r}^{s}\|y_{t+j}\|\right)^4\sup_{\theta\in\Theta_0}\left|h'\left(\epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta);\lambda\right)\right|\right).
\end{aligned}$$

Finiteness of the last two expectations can be established similarly, so we only show the latter. First conclude from (A.2) and Assumption 7 that, with $\Theta_0$ small enough,

$$\begin{aligned}
\sup_{\theta\in\Theta_0}\left|h'\left(\epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta);\lambda\right)\right| &\leq 2a_1+2a_2\left(\sup_{\theta\in\Theta_0}\epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta)\right)^{a_3}\\
&\leq c_5\left(1+\sup_{\theta\in\Theta_0}\|\epsilon_t(\vartheta)\|^{2a_3}\right)\\
&\leq c_6\left(1+\left(\sum_{j=-r}^{s}\|y_{t+j}\|\right)^{2a_3}\right),
\end{aligned}$$

where the last equality is obtained from the definition of $\epsilon_t(\vartheta)$ (see (18)). Thus, it follows that we need to show finiteness of $\mathbb{E}_{\theta_0}\left(\left(\sum_{j=-r}^{s}\|y_{t+j}\|\right)^{4+2a_3}\right)$ or, by Minkowski's inequality and (5), finiteness of

$$\mathbb{E}_{\theta_0}\left(\|\epsilon_t\|^{4+2a_3}\right)\leq c_7\mathbb{E}_{\lambda_0}\left(\rho_t^{4+2a_3}\right)=\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty\zeta^{n/2+1+a_3}f(\zeta;\lambda_0)\,d\zeta<\infty,$$

where the former inequality is justified by (B.3) and the latter by Assumption 7.

From (18) and the expression of $\partial^2 g_t(\theta)/\partial\sigma\partial\sigma'$ (see Appendix A) it can be seen that the treatment of $\partial^2 g_t(\theta)/\partial\sigma\partial\sigma'$ is very similar to that of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$ and the same is true for $\partial^2 g_t(\theta)/\partial\vartheta_i\partial\sigma'$ $(i=1,2)$. Next consider $\partial^2 g_t(\theta)/\partial\lambda\partial\lambda'$. The dominance assumptions imposed on the third and fifth functions in Assumption 7 together with the triangular inequality and the Cauchy-Schwarz inequality imply that, with $\Theta_0$ small enough,

$$\mathbb{E}_{\theta_0}\left(\sup_{\theta\in\Theta_0}\left\|\frac{\partial^2}{\partial\lambda\partial\lambda'}g_t(\theta)\right\|\right)\leq 2a_1+2a_2\mathbb{E}_{\theta_0}\left(\left(\sup_{\theta\in\Theta_0}\epsilon_t(\vartheta)'\Sigma^{-1}\epsilon_t(\vartheta)\right)^{a_3}\right),$$

where finiteness of the right hand side was established in the case of $\partial^2 g_t(\theta)/\partial\vartheta_1\partial\vartheta_2'$. The treatment of the remaining components, $\partial^2 g_t(\theta)/\partial\vartheta_i\partial\lambda'$ and $\partial^2 g_t(\theta)/\partial\sigma\partial\lambda'$, involve no new features, so details are omitted.

Finally, because

$$-\left(T-s-r\right)^{-1}\partial^2 l_T(\hat{\theta})/\partial\theta\partial\theta' = -\left(T-s-r\right)^{-1}\sum_{t=r+1}^{T-s}\partial^2 g_t(\hat{\theta})/\partial\theta\partial\theta',$$

the consistency claim is a straightforward consequence of the fact that $\partial^2 g_t(\theta)/\partial\theta\partial\theta'$ obeys a uniform law of large numbers. $\square$