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PROBLEMS AND SOLUTIONS

PROBLEMS

03.5.1. A Concise Derivation of the Wallace and Hussain Fixed Effects Transformation

Badi H. Baltagi
Texas A&M University

Consider the panel regression model with two-way error component disturbances defined by

$$u_{it} = \mu_i + \lambda_t + \nu_{it} \qquad i = 1,\ldots,N \qquad t = 1,\ldots,T.$$

Wallace and Hussain (1969) derive the fixed effects transformation for this model by "trial error and generalization." Use the Frisch–Waugh–Lovell theorem described in Davidson and MacKinnon (1993, p. 19) to obtain a concise derivation of this Within transformation.

REFERENCES

Wallace, T.D. & A. Hussain (1969) The use of error components models in combining cross-section and time-series data. *Econometrica* 37, 55–72.
Davidson, R. & J.G. MacKinnon (1993) *Estimation and Inference in Econometrics*. New York: Oxford University Press.

03.5.2. Consistent Standard Errors for Target Variance Approach to GARCH Estimation

Dennis Kristensen and Oliver Linton
London School of Economics

In a recent paper Engle and Sheppard (2001) have used a "target variance" approach to estimate a class of multivariate generalized autoregressive conditional heteroskedasticity (GARCH) models. The question we pose here is how to derive the asymptotic distribution of the estimators and find the correct standard errors for a univariate version of this. Suppose that

$$y_t = \varepsilon_t \sigma_t, \qquad \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \gamma y_{t-1}^2,$$

where ε_t is independent and identically distributed mean zero and variance one. The parameters are assumed to be positive, $\omega, \beta, \gamma > 0$, and satisfy

$$\beta + \gamma < 1. \qquad \textbf{(1)}$$

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Under (1), the unconditional variance, σ^2, is well defined and is given by

$$\sigma^2 = E(\sigma_t^2) = \frac{\omega}{1-\gamma-\beta}.$$

We can reparameterize the model in terms of $(\sigma^2, \beta, \gamma)$:

$$\sigma_t^2 = \sigma^2(1-\gamma-\beta) + \beta\sigma_{t-1}^2 + \gamma y_{t-1}^2.$$

The target variance approach is to choose the estimator of $\theta = (\beta, \gamma)$ as $\hat{\theta} = \arg\min_{\theta\in\Theta} \ell_T(\theta, \hat{\sigma}^2)$ where

$$\ell_T(\theta, \hat{\sigma}^2) = -\frac{1}{2T}\sum_{t=1}^{T} \log \sigma_t^2(\theta, \hat{\sigma}^2) - \frac{1}{2T}\sum_{t=1}^{T} \frac{y_t^2}{\sigma_t^2(\theta, \hat{\sigma}^2)}$$

and

$$\sigma_t^2(\theta, \hat{\sigma}^2) = \hat{\sigma}^2(1-\gamma-\beta) + \beta\sigma_{t-1}^2 + \gamma y_{t-1}^2,$$

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} y_t^2.$$

This method has a nice computational advantage over the usual GARCH estimation because the optimization is only with respect to the two parameters β, γ, and avoids the usual problem where there is a high covariance between the maximum likelihood estimates of ω and β.

Derive the asymptotic variance of $\sqrt{T}(\hat{\theta} - \theta)$ and suggest consistent standard errors.

REFERENCE

Engle, R.F. & Sheppard, K. (2001) Theoretical and Empirical Properties of Dynamic Conditional Correlation Multivariate GARCH. NBER Working paper w8554.

SOLUTIONS

02.5.1. A Mixingale Inequality Using an Exponential Moment

Robert M. de Jong
Michigan State University

First note that for all $K > 0$,

$$\begin{aligned}
&\|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p \\
&\quad \le \|E(X_t I(|X_t| \le K)|\mathcal{G}_{-\infty}^{t-m}) - EX_t I(|X_t| \le K)\|_p \\
&\qquad + \|E(X_t I(|X_t| > K)|\mathcal{G}_{-\infty}^{t-m}) - EX_t I(|X_t| > K)\|_p \\
&\quad \le 6\alpha(m)^{1/p} K + 2(E|X_t|^p I(|X_t| > K))^{1/p},
\end{aligned}$$

where the last equation follows from the inequality

$$\|E(Y_t|\mathcal{G}_{-\infty}^{t-m}) - EY_t\|_p \leq 6\alpha(m)^{1/p-1/r}\|Y_t\|_r$$

with $r = \infty$ and $Y_t = X_t I(|X_t| \leq K)$ and the inequality

$$\|E(X_t I(|X_t| > K)|\mathcal{G}_{-\infty}^{t-m}) - EX_t I(|X_t| > K)\|_p$$
$$\leq 2\|X_t I(|X_t| > K)\|_p = 2(E|X_t|^p I(|X_t| > K))^{1/p}.$$

Next, note that for all $K \geq 1$,

$$\begin{aligned} E|X_t|^p I(|X_t| > K) &= \int_{K^p}^{\infty} P(|X_t| > s^{1/p})\, ds \\ &= \int_{K^p}^{\infty} P(\exp(|X_t|) > \exp(s^{1/p}))\, ds \\ &\leq E\exp(|X_t|)\int_{K^p}^{\infty} \exp(-s^{1/p})\, ds \\ &\leq E\exp(|X_t|)C_{p1}K^{p-1}\exp(-K) \end{aligned}$$

for some constant C_{p1}, where the first inequality is the Markov inequality, implying that for all $K \geq 1$,

$$\begin{aligned} \|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p &\leq 6K\alpha(m)^{1/p} + (2C_{p1}E\exp(|X_t|)K^{p-1}\exp(-K))^{1/p} \\ &\leq 6K\alpha(m)^{1/p} + 2C_{p1}^{1/p}(E\exp(|X_t|))^{1/p}K^{1-1/p}\exp(-K/p). \end{aligned}$$

Now setting $K = 2pE\exp(|X_t|)\log(1 + \alpha(m)^{-1})$ (and noting that $K > 1$) will turn the preceding inequality into

$$\begin{aligned} &\|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p \\ &\quad \leq 12pE\exp(|X_t|)\log(1 + \alpha(m)^{-1})\alpha(m)^{1/p} \\ &\qquad + 2C_{p1}^{1/p}(E\exp(|X_t|))^{1/p}(2pE\exp(|X_t|) \\ &\qquad \times \log(1 + \alpha(m)^{-1}))^{1-1/p}(1 + \alpha(m)^{-1})^{-2E\exp(|X_t|)}. \\ &\quad = a + b, \quad \text{say}. \end{aligned}$$

Now note that $E\exp(|X_t|) > 1$, implying that the last expression can be bounded by

$$\begin{aligned} &a + C_{p2}E\exp(|X_t|)(1 + \alpha(m)^{-1})^{-1}(\log(1 + \alpha(m)^{-1}))^{1-1/p} \\ &\quad \leq a + C_{p3}E\exp(|X_t|)\alpha(m)^{1/p}\log(1 + \alpha(m)^{-1}) \end{aligned}$$

for constants C_{p2} and C_{p3}. Setting $C_p = 12p + C_{p3}$ now gives

$$\|E(X_t|\mathcal{G}_{-\infty}^{t-m}) - EX_t\|_p \leq C_p\alpha(m)^{1/p}\log(1 + \alpha(m)^{-1})E\exp(|X_t|),$$

which is the stated result.

02.5.2. Durbin–Watson Statistic and Random Individual Effects

Stanislav Anatolyev
New Economic School, Moscow

In all regressions, the residuals consistently estimate corresponding regression errors. Therefore, to find a probability limit of the Durbin–Watson statistic, it suffices to compute the variance and first-order autocovariance of the errors across the stacked equations

$$\operatorname*{plim}_{n\to\infty} DW = 2\left(1 - \frac{\varrho_1}{\varrho_0}\right),$$

where

$$\varrho_0 = \operatorname*{plim}_{n\to\infty} \frac{1}{nT}\sum_{t=1}^{T}\sum_{i=1}^{n} u_{it}^2, \qquad \varrho_1 = \operatorname*{plim}_{n\to\infty} \frac{1}{nT}\sum_{t=2}^{T}\sum_{i=1}^{n} u_{it}u_{i,t-1},$$

and u_{it}'s denote regression errors. Note that the errors are uncorrelated where the index i switches between individuals, hence summation from $t = 2$ in ϱ_1.

Consider the original regression (1) where $u_{it} = \mu_i + v_{it}$. Then $\varrho_0 = \sigma_v^2 + \sigma_\mu^2$ and

$$\varrho_1 = \frac{1}{T}\sum_{t=2}^{T} \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} (\mu_i + v_{it})(\mu_i + v_{i,t-1}) = \frac{T-1}{T}\sigma_\mu^2.$$

Thus

$$\operatorname*{plim}_{n\to\infty} DW_{OLS} = 2\left(1 - \frac{T-1}{T}\frac{\sigma_\mu^2}{\sigma_v^2+\sigma_\mu^2}\right) = 2\,\frac{T\sigma_v^2 + \sigma_\mu^2}{T(\sigma_v^2+\sigma_\mu^2)}.$$

Consider the Within regression (2) where $u_{it} = v_{it} - \bar{v}_{i\cdot}$. Then

$$\varrho_0 = \frac{1}{T}\sum_{t=1}^{T} \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T-1}{T}v_{it} - \frac{1}{T}\sum_{\tau\neq t} v_{i\tau}\right)^2 = \frac{T-1}{T}\sigma_v^2$$

and

$$\varrho_1 = \frac{1}{T}\sum_{t=2}^{T} \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T-1}{T}v_{it} - \frac{1}{T}v_{i,t-1} - \frac{1}{T}\sum_{\substack{\tau\neq t\\ \tau\neq t-1}} v_{i\tau}\right)$$
$$\times\left(\frac{T-1}{T}v_{i,t-1} - \frac{1}{T}v_{it} - \frac{1}{T}\sum_{\substack{\tau\neq t\\ \tau\neq t-1}} v_{i\tau}\right)$$
$$= -\frac{T-1}{T^2}\sigma_v^2.$$

Thus

$$\text{p}\lim_{n\to\infty} DW_{Within} = 2\,\frac{T+1}{T}.$$

Consider the Between regression (3) where $u_{it} = \mu_i + \bar{v}_{i\cdot}$. Then

$$\varrho_0 = \frac{1}{T}\sum_{t=1}^{T} \text{p}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\mu_i + \bar{v}_{i\cdot})^2 = \sigma_\mu^2 + \frac{1}{T}\,\sigma_v^2$$

and

$$\varrho_1 = \frac{1}{T}\sum_{t=2}^{T} \text{p}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\mu_i + \bar{v}_{i\cdot})^2 = \frac{T-1}{T}\left(\sigma_\mu^2 + \frac{1}{T}\,\sigma_v^2\right).$$

Thus

$$\text{p}\lim_{n\to\infty} DW_{Between} = \frac{2}{T}.$$

The generalized least squares transformation orthogonalizes the errors; therefore,

$$\text{p}\lim_{n\to\infty} DW_{GLS} = 2.$$

Because all computed probability limits except that for DW_{OLS} do not depend on the variance components, the only way to construct an asymptotic test of $H_0 : \sigma_\mu^2 = 0$ vs. $H_A : \sigma_\mu^2 > 0$ is by using DW_{OLS}. Under H_0, $\sqrt{nT}\,(DW_{OLS} - 2) \xrightarrow{d} N(0,4)$ as $n \to \infty$ (estimation of β does not affect the limiting distribution). Under H_A, $\text{p}\lim_{n\to\infty} DW_{OLS} < 2$. Hence a one-sided asymptotic test for $\sigma_\mu^2 = 0$ for a given level α is

$$\text{Reject if } DW_{OLS} < 2\left(1 + \frac{z_\alpha}{\sqrt{nT}}\right),$$

where z_α is the α-quantile of the standard normal distribution.