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# PROBLEMS AND SOLUTIONS

## PROBLEMS

### 03.6.1. The Central Limit Theorem for Student's Distribution

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Let $x_1, \dots, x_n$ be a random sample from Student's $\mathrm{t}(\nu)$ distribution, where $\nu \in \mathbb{R}_+$. Investigate whether $z_n := \sum_{i=1}^n x_i/\lambda_n$ is asymptotically N(0,1) for a suitable choice of $\lambda_n$.

### 03.6.2. Unbiasedness of the OLS Estimator with Random Regressors

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Consider the linear regression model

$$y = X\beta + u,$$

where $X$ is an $n \times k$ matrix of random regressors, $u$ is an $n$-vector of error terms, and $\beta$ is a $k$-vector of parameters. Suppose $X$ has full column rank with probability one. It is a standard textbook claim that the ordinary least squares (OLS) estimator $\hat{\beta} = (X'X)^{-1}X'y$ of $\beta$ is unbiased if $E(u|X) \overset{a.s.}{=} 0$, where $\overset{a.s.}{=}$ signifies almost sure equality. Specifically, it is claimed that unbiasedness follows from the law of iterated expectations and the relation $E(\hat{\beta}|X) \overset{a.s.}{=} \beta + (X'X)^{-1}X'E(u|X)$. As it turns out, this argument is flawed.

(a) Show by example that $E(u|X) \overset{a.s.}{=} 0$ does not imply existence of $E(\hat{\beta})$.
(b) Provide stronger conditions under which $E(\hat{\beta})$ exists (and equals $\beta$).

## SOLUTIONS

### 02.6.1. Oblique Projectors[1]—Solution

Götz Trenkler (the poser of the problem)
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It is well known that an oblique projector $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{K} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

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where $r$ denotes the rank of $\mathbf{P}$ and $\mathbf{U}$ is a unitary matrix (cf. Hartwig and Loewy, 1992). Because $\mathbf{K}$ is not Hermitian we have $\mathbf{K} \neq \mathbf{0}$.

Consider the matrix

$$\mathbf{R} = \mathbf{U}\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

where $\mathbf{C} = (\mathbf{I}_r + \mathbf{K}\mathbf{K}^*)^{-1}$ and $\mathbf{D} = \mathbf{K}^*\mathbf{C}$. Then

$$\mathbf{PR} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

$$\mathbf{RP} = \mathbf{U}\begin{pmatrix} \mathbf{C} & \mathbf{CK} \\ \mathbf{K}^*\mathbf{C} & \mathbf{K}^*\mathbf{CK} \end{pmatrix}\mathbf{U}^*$$

are both Hermitian. Furthermore, $\mathbf{RPR} = \mathbf{R}$ and $\mathbf{PRP} = \mathbf{P}$. Thus $\mathbf{R}$ satisfies the four conditions of the Moore–Penrose inverse of $\mathbf{P}$, i.e., $\mathbf{R} = \mathbf{P}^+$.

If $\mathbf{P}^+$ were a projector we would get $\mathbf{C}^2 = \mathbf{C}$, which implies $\mathbf{C} = \mathbf{I}$, and consequently $\mathbf{K} = \mathbf{0}$. This is a contradiction.

### NOTE

1. Two solutions are published, proposed independently by Götz Trinkler (the poser of the problem) and by Hans Joachim Werner. Excellent solutions have been independently proposed by G. Dhaene, L. Lauwers, S. Lawford, and H. Neudecker.

### REFERENCE

Hartwig, R.E. & R. Loewy (1992) Maximal elements under the three partial orders. *Linear Algebra and Its Applications* 175, 39–61.

## 02.6.1. Oblique Projectors—Solution

Hans Joachim Werner
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The claimed result is an immediate consequence of the following more informative characterization.

THEOREM 1. *Let $P$ be an idempotent matrix; i.e., let $P^2 = P$. Then $P^+$ is idempotent if and only if $P = P^*$, with $P^+$ and $P^*$ denoting the Moore-Penrose inverse and the conjugate transpose of $P$, respectively.*

**Proof.** Let $P^2 = P = P^*$. Then $P = P^3 = (P^2)^*$, which in turn directly implies that the Penrose defining equations of $P^+$ are satisfied for $P^+ = P$, and so our proof of sufficiency is complete. For proving necessity, let $P$ be such that $P^2 = P$ and $(P^+)^2 = P^+$. If $P = 0$, then trivially $P^+ = 0 = P^*$. Next, let $P \neq 0$

and let $r$ denote the rank of $P$. According to the well-known singular value decomposition theorem, $P$ can then be written as $P = UD_rV^*$, where $U$ and $V$ are (column-unitary matrices) such that $U^*U = V^*V = I_r$ and where $D_r$ is an $r \times r$ diagonal matrix with the $r$ positive singular values of $P$ along its main diagonal. In view of this decomposition, clearly

$$P^2 = P \Leftrightarrow D_r V^* U = I_r \Leftrightarrow V^* U = D_r^{-1} = U^* V.$$

Because $P^+ = V(D_r)^{-1}U^*$, which again is easily seen by checking the Penrose defining equations of $P^+$, we likewise obtain

$$(P^+)^2 = P^+ \Leftrightarrow (D_r)^{-1}U^*V = I_r \Leftrightarrow U^*V = D_r = V^*U.$$

Combining these observations necessarily results in $D_r = I = U^*V = V^*U$. Consequently, $(V-U)^*(V-U) = 0$, or, equivalently, $V = U$. As claimed, we thus arrive at $P = UU^* = P^*$, and so our proof is complete. ∎

Further equivalent conditions for an idempotent matrix $P$ to be Hermitian are given in the following theorem.

THEOREM 2. *Let P be an idempotent matrix. The following conditions are then equivalent:*

*(a)* $P = P^*$*; i.e., P is Hermitian.*
*(b)* $P = PP^*P$*; i.e., P is a partial isometry.*
*(c)* $\mathcal{R}(P) = \mathcal{R}(P^*)$*; i.e., P is an EP matrix.*
*(d)* $(P^+)^2 = P^+$.
*(e)* $P^+ = P$.
*(f)* $P^+ = P^*$.

That (a) ⇔ (b) ⇔ (c) has been shown recently in Werner (2002). That (a) ⇔ (e) ⇔ (f) can be shown on similar lines and is therefore left to the reader.

*REFERENCE*

Werner, H.J. (2002) Partial isometry and idempotent matrices. Solution 28-7.5 to Problem 28-7 (by Götz Trenkler). *IMAGE: The Bulletin of the International Linear Algebra Society* 29 (October 2002), 31–32.

## 02.6.2. Autoregression and Redundant Instruments—Solution

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The vector of regressors is $x_t = (y_{t-1}\, y_{t-2} \ldots y_{t-k})'$. Let $\gamma_j = \gamma_{-j} = \mathrm{E}[y_t y_{t-j}]$ and $\Gamma_j = (\gamma_j\ \gamma_{j-1} \ldots \gamma_{j-\ell+1})'$. The matrix of cross-covariances of $x_t$ and $z_t$ is

$$
\begin{aligned}
\mathrm{Q}_{xz} &= \begin{pmatrix} \gamma_{p-1} & \gamma_p & \cdots & \gamma_{p+k-2} & \cdots & \gamma_{p+\ell-2} \\ \gamma_{p-2} & \gamma_{p-1} & \cdots & \gamma_{p+k-3} & \cdots & \gamma_{p+\ell-3} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \gamma_{p-k} & \gamma_{p-k+1} & \cdots & \gamma_{p-1} & \cdots & \gamma_{p+\ell-k-1} \end{pmatrix} \\
&= (\mathrm{I}_k\ \mathrm{O}_{k\times(\ell-k)})(\Gamma_{p-1}\ \Gamma_p\ \cdots\ \Gamma_{p+\ell-2}),
\end{aligned}
$$

where $\mathrm{I}_m$ denotes $m \times m$ identity matrix and $\mathrm{O}_{m_1\times m_2}$ – zero $m_1 \times m_2$ matrix. The covariance matrix of $z_t \varepsilon_t$ is

$$
\mathrm{V}_{z\varepsilon} = \sigma^2 \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{\ell-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{\ell-1} & \gamma_{\ell-2} & \cdots & \gamma_0 \end{pmatrix} = \sigma^2(\Gamma_0\ \Gamma_1 \ldots \Gamma_{\ell-1}).
$$

The efficient generalized method of moments estimator based on the instrumental vector $z_t$ effectively uses the instrument optimal in the class of linear transformations of $z_t$. This optimal instrument is $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}z_t$ (see, e.g., West, 2001). We will show that in the matrix $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}$ the right $k \times (\ell - k)$ submatrix is zero, so that the optimal combination of elements of $z_t$ involves only the first $k$ entries.

Recall that the Yule–Walker equations for an AR$(k)$ model contain the following recursion:

$$
\gamma_j = \rho_1\gamma_{j-1} + \rho_2\gamma_{j-2} + \cdots + \rho_k\gamma_{j-k}, \qquad j \geq 1.
$$

This implies that

$$
\begin{aligned}
(\Gamma_{p-1}\ \Gamma_p \ldots \Gamma_{p+\ell-2}) &= \mathrm{P}(\Gamma_{p-2}\ \Gamma_{p-1} \ldots \Gamma_{p+\ell-3}) \\
&= \ldots \\
&= \mathrm{P}^{p-1}(\Gamma_0\ \Gamma_1 \ldots \Gamma_{\ell-1}),
\end{aligned}
$$

where

$$
\mathrm{P} = \begin{pmatrix} \rho' & 0'_{\ell-1-k} & \\ & & 0_\ell \\ & \mathrm{I}_{\ell-1} & \end{pmatrix}
$$

and $0_m$ denotes zero $m \times 1$ vector. Thus

$$
\mathrm{Q}_{xz} = (\mathrm{I}_k\ \mathrm{O}_{k\times(\ell-k)})(\Gamma_{p-1}\ \Gamma_p \ldots \Gamma_{p+\ell-2}) = \sigma^{-2}(\mathrm{I}_k\ \mathrm{O}_{k\times(\ell-k)})\mathrm{P}^{p-1}\mathrm{V}_{z\varepsilon}.
$$

It follows that $\mathrm{Q}_{xz}\mathrm{V}_{z\varepsilon}^{-1}$ equals $\sigma^{-2}(\mathrm{I}_k\ \mathrm{O}_{k\times(\ell-k)})\mathrm{P}^{p-1}$, a matrix whose right $k \times (\ell - k)$ submatrix is indeed zero.

## *REFERENCE*

West, K.D. (2001) On optimal instrumental variables estimation of stationary time series models. *International Economic Review* 42, 1043–1050.