Appendix to 'Efficiency and future generations' by John Broome

The arguments in this paper are based on a standard finite general equilibrium model containing public goods (e.g. Foley 1970).

The model

There are T public goods and N private goods. A typical vector of goods is $(e, g) = (e_1, \ldots e_T, g_1, \ldots g_N)$, where the public goods are e and the private goods g. Aggregate production is represented by a vector y of goods in which positive components represent outputs and negative components inputs. The production set Y is the set of aggregate productions that are technically possible. There are I people. A typical consumption vector of person i is $x^i = (e, g^i) = (e_1, \ldots e_T, g_1^i, \ldots g_N^i)$. Each person has a consumption set X^i of consumptions that are possible for her, and a preference relation $\succeq^i$ on X^i. She has an initial endowment of private goods $w^i = (w_1^i, \ldots w_N^i)$.

An *allocation* is a vector $x = (x^1, \ldots x^I) = (e, g^1, \ldots e, g^I)$, where each person has the same consumption of public goods e. An allocation is *feasible* if and only if $(x^i) \in X^i$ for all i and $(e, \sum_i (g^i - w^i)) \in Y$. (In the text of the article, I use 'possible' for feasible.) Let Z be the set of feasible allocations.

For each i, the indifference relation $\sim^i$ and strict preference relation $\succ^i$ are defined in the standard way.

An allocation $x = (x^1, \ldots x^I)$ is *Pareto indifferent* to an allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I)$ if and only if $x^i \sim^i \hat{x}^i$ for all *I*. x *Pareto dominates* $\hat{x}$ if and only if $x^i \succeq^i \hat{x}^i$ for all i and x is not Pareto indifferent to $\hat{x}$. A feasible allocation is *Pareto efficient* if and only if no feasible allocation Pareto dominates it. Otherwise it is *Pareto inefficient*.

Basic assumptions:

Y is closed and each X^i is closed.

Each $\succeq^i$ is complete, transitive and reflexive.

Each $\succeq^i$ is continuous. That is: for all $x^i \in X^i$ the sets $\{\hat{x}^i | \hat{x}^i \succeq^i x^i\}$ and $\{\hat{x}^i | x^i \succeq^i \hat{x}^i\}$ are closed.

Each $\succeq^i$ is locally nonsatiated. That is: for all $x^i \in X^i$, every neighbourhood of x^i contains an $\hat{x}^i$ such that $\hat{x}^i \succ^i x^i$.

The set Z of feasible allocations is bounded.

Interpretation. The model has its standard intertemporal interpretation. Each good is dated. In the application to climate change, e_t is minus the emission of greenhouse gas at the tth date. In the first part of this appendix, I assume there is a fixed population of people. Each person has fixed dates of birth and death, but these dates differ between people. Unless people have altruistic preferences about goods consumed after they die, the private goods in each person's consumption set are all dated to dates when she is alive.

A general equilibrium model like this models the economy by means of a single intertemporal equilibrium, which ascribes prices to all goods at all times. This is not realistic. There are in practice few markets at the present time for future goods. Financial markets dealing in debt are a sort of proxy, but only a rough one. The most serious problem is that the people who are supposed to trade in these markets do not all exist at the same time. Future prices appear in practice in the form of present people's expectations of future prices, which depend on their expectations of future people's preferences. These expectations may be mistaken.

First lesson

No theorem 1. No general theorem supports the first lesson. Several theorems state sufficient conditions for a state to be efficient, but none that I know states sufficient conditions for a state to be inefficient. Even if it contains externalities, a state might be Pareto efficient because of special features of the economy. For example, a negative externality might come in discrete lumps. The marginal lump emitted may be a genuine externality, in that removing it would have a positive external benefit. But the private cost of removing this lump may exceed the external benefit of doing so. If so, the state might be Pareto efficient despite the externality. For this reason, I expressed the first lesson with the qualification ‘except in some unusual circumstances’.

However, I can informally describe one necessary condition for a state with externalities to be inefficient. If a state containing an externality is Pareto inefficient, that means a Pareto improvement is possible. To make the Pareto improvement will normally require some agent to change its production or consumption in a way that is inherently costly, and to be compensated for the cost by a transfer of goods from other agents. So compensating transfers must be possible if Pareto inefficiency is to result from an externality.

In the case of climate change, present people can change their consumption and production methods for the sake of future people, but (unless they have altruistic preferences for future goods) this will be a Pareto improvement only if they can be compensated for doing so by a transfer from future people. They will have to receive present goods, which they can consume and benefit from; a transfer of future goods in the form of debts would not be good enough unless they can be traded for present goods. Future people can compensate present people with present goods, through the mechanism of overlapping lives described in section 2.1.

An intergenerational externality such as climate change creates Pareto inefficiency between generations only in so far as intergenerational transfers are possible through overlapping lives. If there could be no such transfers, intergenerational externalities would not lead to Pareto inefficiency. This constraint is implicit in the structure of the model.

Second lesson

Theorem 2. Given the basic assumptions, if a feasible allocation is Pareto inefficient, there is a feasible Pareto efficient allocation that Pareto dominates it.

Proof of theorem 2. The set Z of feasible allocations is closed because Y is closed and each X^i is closed. It is bounded by assumption.

Take a feasible allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I)$ that is Pareto inefficient. Take the set of allocations $\{(x^1, \ldots x^I) | (\forall i)(x^i \succeq^i \hat{x}^i)\}$ that Pareto dominate or are Pareto indifferent to $\hat{x}$. This is a closed set because the individual sets $\{(x^i) | x^i \succeq^i \hat{x}^i\}$ are closed. Let Z' be the intersection of Z with this set. Z' is closed because it is the intersection of closed sets. It is bounded because Z is bounded.

By Debreu’s (1954) representation theorem, each person i’s preferences can be represented by a continuous real-valued utility function u^i. Each allocation x in Z' has a total of utilities $U(x) = \sum_i u^i(x^i)$. U is a continuous function on the closed and bounded set Z'. It therefore attains its maximum on this set.

Take an allocation x' in Z' that has maximum total of utility. x' is feasible because it is in Z. Any feasible allocation that Pareto dominates x' Pareto dominates $\hat{x}$ and is therefore in Z'. But no allocation in Z' Pareto dominates x' because any Pareto dominating allocation would have a greater total of utility. x' is therefore Pareto efficient. x' is consequently not Pareto indifferent to $\hat{x}$ because if either of a pair of Pareto indifferent allocations is efficient, so is the other, and $\hat{x}$ is not efficient. Since x' is in the set of allocations that either Pareto dominate or are Pareto indifferent to $\hat{x}$, x' Pareto dominates $\hat{x}$.

In sum, x' is a feasible Pareto efficient allocation that Pareto dominates $\hat{x}$, as the theorem requires. END OF PROOF

Third lesson

Theorem 3. Suppose a feasible allocation $x = (x^1, \ldots x^I) = (e, g^1, \ldots e, g^I)$ is such that there is a price vector $r = (r_1, \ldots r_N)$ for private goods, and for each i there is a price vector $q^i = (q_1{}^i, \ldots q_T{}^i)$ for public goods with the following properties:

$p.(e, \sum_i (g^i - w^i)) \geq p.y$ for any vector $y \in Y$, where $p = (\sum_i q^i, r)$. (So the production sector is profit-maximizing at p.)

For all i, $x^i \succeq^i \hat{x}^i$ for any $\hat{x}^i$ such that $p^i.\hat{x}^i \leq p^i.x^i$, where $p^i = (q^i, r)$. (So x^i is preference maximizing at p^i, given a budget constraint.)

Then, given the basic assumptions, x is Pareto efficient.

Interpretation. This is the so-called first theorem of welfare economics applied to an economy with public goods. Note that in theorem 3 the price of each private good is the same for each person and for producers, but each public good has different prices for different people. The price to producers of a public good is the sum of its prices for each person.

In this theorem people are not to be interpreted as actually buying public goods. Their preference-maximizing over public goods is purely notional. The preference-maximizing condition specifies that each person would be willing to buy just the prevailing amount of public goods were she required to pay for them at her own prices $(q_1{}^i, \ldots q_T{}^i)$. These prices indicate her willingness to pay for public goods at the margin. For greenhouse gas, these prices measure the marginal external costs imposed on the person by emissions.

On the other hand, the production system in the theorem, which produces both public and private goods, is supposed to be genuinely profit-maximizing at the given prices. Producers are supposed genuinely to pay the prices of their inputs. For public goods, these prices are the sum of the private willingnesses to pay. In practice they would need to be imposed by the government.

The conditions of the theorem express the condition that producers' external costs are internalized. It is Paul Samuelson's (1954) condition for Pareto efficiency with public goods.

Proof of theorem 3. Suppose the allocation x is not efficient. Then there is a feasible allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I) = (\hat{e}, \hat{g}^1, \ldots \hat{e}, \hat{g}^I)$ that Pareto dominates x.

Because $\hat{x}$ Pareto dominates x, $\hat{x}^i \succ^i x^i$ for some i. Since x^i is preference maximizing at p^i, it follows that $p^i.\hat{x}^i > p^i.x^i$ for this i.

Because $\hat{x}$ Pareto dominates x, $\hat{x}^i \succeq^i x^i$ for all i. Suppose $p^i.\hat{x}^i < p^i.x^i$. Then there is a neighbourhood of $\hat{x}^i$ such that for every point $\vec{x}^i$ in this neighbourhood $p^i.\vec{x}^i < p^i.x^i$. By local nonsatiation, one of these points is strictly preferred to $\hat{x}^i$, which means it is strictly preferred to x^i. This contradicts that x^i is preference maximizing at p^i. So $p^i.\hat{x}^i \geq p^i.x^i$ for all i.

Because $p^i.\hat{x}^i \geq p^i.x^i$ for all i and $p^i.\hat{x}^i > p^i.x^i$ for some i, $\sum_i p^i.\hat{x}^i > \sum_i p^i.x^i$. That is, $\sum_i (q^i.\hat{e} + r.\hat{g}^i) > \sum_i (q^i.e + r.g^i)$. That is, $\sum_i q^i.\hat{e} + r.\sum_i \hat{g}^i > \sum_i q^i.e + r.\sum_i g^i$.

However, because x is profit-maximizing at p, $p.(e, \sum_i (g^i - w^i)) \geq p.(\hat{e}, \sum_i (\hat{g}^i - w^i))$, so $p.(e, \sum_i g^i) \geq p.(\hat{e}, \sum_i \hat{g}^i)$. That is $\sum_i q^i.e + r.\sum_i g^i \geq \sum_i q^i.\hat{e} + r.\sum_i \hat{g}^i$.

The supposition that x is not efficient therefore implies a contradiction. So x is efficient. END OF PROOF

Fourth lesson

Theorem 4. Suppose a feasible allocation $x = (x^1, \ldots x^I) = (e, g^1, \ldots e, g^I)$ is Pareto efficient. Assume also that the production set Y is convex, and that for each i the set $\{\hat{x}^i | \hat{x}^i \succeq^i x^i\}$ is convex. Then, under the basic assumptions, there are initial endowments of private goods $w =$

$(w^1, \ldots w^J)$, a price vector $r = (r_1, \ldots r_N)$ for private goods, and for each i a price vector $q^i = (q_1^i, \ldots q_T^i)$ for public goods with the following properties:

$p.(e, \sum_i (g^i - w^i)) \geq p.y$ for any vector $y \in Y$, where $p = (\sum_i q^i, r)$. (So the production sector is profit-maximizing at p.)

For all i, $x^i \succeq^i \hat{x}^i$ for any $\hat{x}^i$ such that $p^i.\hat{x}^i \leq p^i.x^i$, where $p^i = (q^i, r)$. (So x^i is preference-maximizing at p^i given a budget constraint.)

Proof of theorem 4. A proof appears in Foley (1979: 68–9). Some slight modifications are required since Foley uses slightly different assumptions.

Interpretation. This is a version of the second theorem of welfare economics for an economy with public goods. It is the converse of the first theorem. It shows that, provided the production set and upper-contour sets for preferences are convex, a Pareto efficient allocation can be supported by a price system in which the price of a public good is the sum of people's willingnesses to pay for it. If the Pareto efficient allocation is to be a market equilibrium the price of public goods will have to be established by government taxes, and initial endowments $(w^1, \ldots w^J)$ will need to be distributed appropriately by lump-sum taxes and subsidies.

Constrained efficiency

I now drop the assumption that the population of people is fixed. As before, each possible state contains a number of people living for various periods that overlap. There is one group of people, the present people, who are the same in every state, but there are also future people who exist in some states and not in others. Both the identities and numbers of future people may vary between states.

There is a set (which need not be finite) of *possible people*. For convenience, I shall assume that each possible person has fixed preferences. (If we wanted to allow for varying preferences, we could treat a possible person as a possible person–preference pair.) Each possible person i has a consumption set X^i, a preference relation $\succeq^i$ on X^i, and an initial endowment w^i. There is an aggregate production set Y.

The *present population* is a finite nonempty subset Π of the set of possible people. It has J members indexed by $i = 1, \ldots J$. A *population* is a union $\Pi \cup \Phi$ of Π with some other, disjoint, finite subset Φ of possible people, who are the *future population*. $\Pi \cup \Phi$ has I members indexed by $I = 1, \ldots I$.

Take a particular population $\Pi \cup \Phi$. An *allocation* for this population is a vector $x = (x^1, \ldots x^I) = (e, g^1, \ldots e, g^I)$ so that each person's consumption of public goods is the same. An allocation is *quasi-feasible* if and only if $x^i \in X^i$ for all $i \in \Pi \cup \Phi$, and $(e, \sum_{i \in \Pi \cup \Phi} (g^i - w^i)) \in Y$. Owing to the nonidentity effect, not all quasi-feasible allocations are causally compatible with the population's being $\Pi \cup \Phi$. An allocation is *feasible* if and only if it is quasi-feasible and causally compatible with its population.

A quasi-feasible allocation for $\Pi \cup \Phi$ is *Pareto quasi-efficient* if and only if no quasi-efficient allocation for $\Pi \cup \Phi$ Pareto dominates it.

A *present allocation* for the present population Π is a vector $x^\Pi = (x^1, \ldots x^J) = (e, g^1, \ldots e, g^J)$. A present allocation is *feasible* if and only if $x^i \in X^i$ for all $i \in \Pi$, and $(e, \sum_{i \in \Pi} (g^i - w^i)) \in Y$. The *future production* of a feasible present allocation x^Π together with a production vector $(e, g) \in Y$ is the vector $(e, g - \sum_{i \in \Pi} (g^i - w^i))$. The *constraint set* of a feasible present allocation x^Π together with a production vector (e, g) is a set of T+N dimensional vectors y such that $y \geq (e, g - \sum_{i \in \Pi} (g^i - w^i))$. (Notation for vectors: $y \geq z$ means $y_s \geq z_s$ for each component s.) A quasi-feasible allocation x for a population $\Pi \cup \Phi$ includes a feasible present allocation for its present population Π consisting of its first J components, and it implies a production vector $(e, \sum_{i \in \Pi \cup \Phi} (g^i - w^i))$. So future production and a constraint set can be ascribed to any quasi-feasible allocation x. I use the notation $f(x)$ for the future production of an allocation x.

The constraint set of an allocation x is *crude* if and only if it is the set $\{y \mid y \geq f(x)\}$.

Given a population $\Pi \cup \Phi$, the constraint set of an allocation $x = (x^1, \ldots x^I) = (e, g^1, \ldots e, g^I)$ is *preference-based* if and only if it is the set $\{(\hat{e}, \bar{g}) \mid (\exists(\hat{g}^{J+1}, \ldots \hat{g}^I))(\bar{g} = \sum_{i \in \Phi} (\hat{g}^i - w^i)$ & $(\forall i \in \Phi)((\hat{e}, \hat{g}^i) \in X^i$ & $(\hat{e}, \hat{g}^i) \succeq^i x^i))\}$. This is the Scitovszky set of future productions that are sufficient to give every future person in Φ a consumption that she does not disprefer to her consumption in x.

A feasible allocation $x = (x^1, \ldots x^J)$ for $\Pi \cup \Phi$ is *constrained efficient* if and only if there is no feasible present allocation $\hat{x}^\Pi = (\hat{x}^1, \ldots \hat{x}^J)$ and production $(\hat{e}, \hat{g}) \in Y$ such that, for all $i \in \Pi$, $\hat{x}^i \succeq^i x^i$, for some $i \in \Pi$, $\hat{x}^i \succ^i x^i$ and future production is a member of the constraint set of x.

We might adopt:

> *Existence assumption*. For any given feasible present allocation $x^\Pi = (x^1, \ldots x^J)$ and production $(e, g) \in Y$, there is a feasible allocation x that has this present allocation and production.

Owing to the nonidentity effect, this is not implied by the rest of the model, but it seems plausible. Under the existence assumption, efficiency can be defined more neatly as follows:

> An allocation $\hat{x}$ *constrained dominates* an allocation x if and only if for all $i \in \Pi$, $\hat{x}^i \succeq^i x^i$, for some $i \in \Pi$, $\hat{x}^i \succ^i x^i$, and future production $f(\hat{x})$ is a member of the constraint set of x.
>
> A feasible allocation is *constrained efficient* if and only if there is no feasible allocation that constrained dominates it.

I use the existence assumption only in generalizing theorem 2 below.

For two of the theorems below, I need to assume that the production set allows present desirable consumption to be substituted for future desirable consumption as an output. To save space, I shall do this in an ad hoc manner, as follows:

> *Substitution assumption*. If $x = (x^1, \ldots x^I)$ and $\bar{x}$ are two quasi-feasible allocations for the population $\Pi \cup \Phi$ and $\bar{x}$ Pareto dominates x, there is a quasi-feasible allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I)$ that Pareto dominates x and such that $\hat{x}^I \succ^i x^i$ for some $i \in \Pi$.

Constrained efficiency: theorems

Theorem 2′. Let constraint sets be crude. Under the basic assumptions and the existence assumption, if a feasible allocation is constrained inefficient, there is a feasible allocation that constrained dominates it and is constrained efficient.

Proof. Let $\hat{x}$ be a feasible allocation that is constrained inefficient with future production $f(\hat{x})$. Let $\hat{x}^\Pi$ be the present allocation consisting of the first J components of $\hat{x}$. Let Z^Π be the set of feasible present allocations that can have future production at least as great as $f(\hat{x})$. That is, $Z^\Pi = \{x \mid (\forall i \in \Pi)(x^i \in X^i)$ & $(\exists(e, g) \in Y)((e, g - \sum_{i \in \Pi} (g^i - w^i)) \geq f(\hat{x}))\}$. Z^Π is closed because Y is closed and each X^i is closed. It is bounded because the set of feasible allocations is bounded. Let $Z^{\Pi\prime}$ be the intersection of Z^Π and the set $\{x^\Pi \mid (\forall i \in \Pi)(x^i \succeq^i \hat{x}^i)\}$ of present allocations that are not dispreferred to $\hat{x}^\Pi$ by any present person. $Z^{\Pi\prime}$ is bounded and closed, being the intersection of closed sets.

The proof then proceeds like the proof of theorem 2 by maximizing the total utility of present people on the closed and bounded set $Z^{\Pi\prime}$, while keeping future production constant. END OF PROOF.

I cannot prove a version of theorem 2 for constraint sets other than crude ones. As in the proof above, there is a bounded set on which the total utility of present people might be maximized, but this set may not be closed.

Theorem 5. Under the basic assumptions and the substitution assumption, given a population $\Pi \cup \Phi$, a quasi-feasible allocation is Pareto quasi-efficient if and only if it is constrained efficient with a preference-based constraint set.

Proof. First, take a quasi-feasible allocation $x = (x^1, \ldots x^I)$ that is not constrained efficient with a preference-based constraint set. I prove that it is not Pareto quasi-efficient. There is a feasible present allocation $\hat{x}^{\Pi} = (\hat{x}^1, \ldots \hat{x}^J)$ and production $(\hat{e}, \hat{g}) \in Y$ such that, (i) for all $i \in \Pi$, $\hat{x}^i \succeq^i x^i$, (ii) for some $i \in \Pi$, $\hat{x}^i \succ^i x^i$ and (iii) future production $(\hat{e}, \bar{g})$ is a member of the preference-based constraint set of x. (iii) implies that $(\hat{e}, \bar{g})$ is sufficient to give every future person in Φ a consumption that she does not disprefer to her consumption in x. That is: there is a vector of private goods $(\hat{g}^{J+1}, \ldots \hat{g}^I)$ such that $\sum_{i \in \Phi} (\hat{g}^i - w^i) = \bar{g} = \hat{g} - \sum_{i \in \Pi} (\hat{g}^i - w^i)$ and, for all $i \in \Phi$, $(\hat{e}, \hat{g}^i) \in X^i$ and $(\hat{e}, \hat{g}^i) \succeq^i x^i$. Take the allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I)$ made up of the present allocation $\hat{x}^{\Pi}$ joined to the vector $(\hat{x}^{J+1}, \ldots \hat{x}^I) = (\hat{e}, \hat{g}^{J+1}, \ldots \hat{e}, \hat{g}^I)$. Then for all $i \in \Pi \cup \Phi$, $\hat{x}^i \succeq x^i$ because of (i) above and because, for all $i \in \Phi$, $(\hat{e}, \hat{g}^i) \succeq^i x^i$. For some $i \in \Pi \cup \Phi$, $\hat{x}^i \succ x^i$ because of (ii) above. Finally, $\hat{x}$ is quasi-feasible because, for all $i \in \Pi \cup \Phi$, $\hat{x}^i \in X^i$ and because total consumption $(\hat{e}, \sum_{i \in \Pi \cup \Phi} (\hat{g}^i - w^i)) = (\hat{e}, \sum_{i \in \Pi} (\hat{g}^i - w^i) + \sum_{i \in \Phi} (\hat{g}^i - w^i)) = (\hat{e}, \hat{g})$, which is a member of Y. In sum, $\hat{x}$ is quasi-feasible and Pareto dominates x. x is therefore not Pareto quasi-efficient.

Next take a quasi-feasible allocation $x = (x^1, \ldots x^I)$ that is not Pareto quasi-efficient. I prove it is not constrained efficient with a preference-based constraint set. There is a quasi-feasible allocation $\bar{x}$ that Pareto dominates x. By the substitution assumption, there is a quasi-feasible allocation $\hat{x} = (\hat{x}^1, \ldots \hat{x}^I)$ that Pareto dominates x and such that $\hat{x}^i \succ^i x^i$ for some $i \in \Pi$. Also, $\hat{x}^i \succeq^i x^i$ for all $i \in \Pi$. The allocation $\hat{x}$ implies a feasible present allocation $\hat{x}^{\Pi} = (\hat{x}^1, \ldots \hat{x}^J)$ and a future production $(\hat{e}, \sum_{i \in \Phi} (\hat{g}^i - w^i))$. We already know that, for some $i \in \Pi$, $\hat{x}^i \succ^i x^i$ and that, for all $i \in \Pi$, $\hat{x}^i \succeq^i x^i$ because $\hat{x}$ Pareto dominates x. The vector $(\hat{g}^{J+1}, \ldots \hat{g}^I)$ satisfies the conditions that ensure future production $(\hat{e}, \sum_{i \in \Phi} (\hat{g}^i - w^i))$ is a member of the preference-based constraint set of x, because for all $i \in \Phi$, $(\hat{e}, \hat{g}^i) \in X^i$ & $(\hat{e}, \hat{g}^i) \succeq^i x^i$. In sum, $\hat{x}^{\Pi}$ and production $(\hat{e}, \hat{g}) \in Y$ are such that, for all $i \in \Pi$, $\hat{x}^i \succeq^i x^i$, for some $i \in \Pi$, $\hat{x}^i \succ^i x^i$ and future production is a member of the constraint set of x. It follows that x is not constrained efficient with a preference-based constraint set. END OF PROOF.

Theorems 3 and 4 are true if ‘Pareto quasi-efficient’ is substituted for ‘Pareto efficient’ in the statements of the theorems. The proofs are exactly the same except that in the proofs (but not the statements of the theorems) ‘quasi-feasible’ must be substituted for ‘feasible’.

Therefore, in view of theorem 5, theorems 3 and 4 are true if ‘constrained efficient’ is substituted for ‘Pareto efficient’, so long as the constraint set is preference-based. This specifies how far the third and fourth lessons extend to the case of variable population.

References

Debreu, G. 1954. Representation of a preference ordering by a numerical function. In *Decision Processes*, ed. R. M. Thrall, C. H. Coombs and R. L. Davis, 159–65. Wiley.

Foley, D. 1970. ~~‘~~Lindahl’s solution and the core of an economy with public goods~~’,~~ *Econometrica*~~,~~ 38~~, pp.~~ 66–72.

Samuelson, P. 1954. The pure theory of public expenditure. *Review of Economics and Statistics* 36: 387–9.