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Appendix A from Teh, "Galileo's Gauge: Understanding the Empirical Significance of Gauge Symmetry" (PHOS, vol. 83, no. 1, p. 93)

The 't Hooft-Polyakov Monopole

It is a familiar theme in the foundations of physics that there is a strong analogy between gauge theory and general relativity (GR). However, while geometrically nontrivial solutions (i.e., black holes) of GR were discovered very soon after the theory's incipiency, the historical discovery of the gauge theory analog (i.e., topological solitons) took a rather more complex path. Roughly speaking, topological solitons are nonlinear, finite-energy, particle-like solutions of field theory: their stable particle-like nature is due to a topologically nontrivial 'twisting' of the field configuration, which can be quantified in terms of a conserved 'topological charge'.

The key insight into the existence of soliton solutions was provided by 't Hooft and Polyakov, who managed to guess a soliton *ansatz* for $SU(2)$ YMH theory, now fittingly called the 't Hooft-Polyakov (tHP) monopole—'monopole' because of its resemblance to the Dirac monopole in electrodynamics.

In order for a YMH solution to have a finite energy, the field ϕ must tend asymptotically to the *vacuum manifold* $\mathcal{V} \coloneqq \{\phi | V(\phi) = 0\}$, on pain of the energy integral of the theory blowing up. Such asymptotic statements are best modeled in terms of the behavior of field $\phi(x)$ as $|x|$ tends to infinity. On the one hand, we know that spatial infinity can be modeled as a very large sphere S^2_∞ in $\mathbb{R}^3$; and on the other hand, we know that the $\mathcal{V}$ has the topology of a sphere S^2. Thus, the asymptotic data for ϕ as $|x| \to \infty$ is given by the map

$$\phi^\infty : S^2_\infty \to \mathcal{V} \equiv S^2. \tag{A1}$$

This qualitative analysis suffices to reveal the sense in which a field configuration can be characterized in terms of a topological property: intuitively, the topological character of a configuration is determined by the number of times ϕ^∞ wraps S^2 around S^2. In more technical language, we say that the different types of solutions are characterized by the homotopy class of ϕ^∞. In fact these classes are in 1:1 correspondence with the second homotopy group $\pi_2(S^2) = \mathbb{Z}$, and we say that a solution in the homotopy class $N \in \mathbb{Z}$ has topological charge N.

By combining an asymptotic analysis of the fields (based on the finite energy requirement) with the requirement of spherical symmetry, 't Hooft and Polyakov were able to write down the explicit form that (A, ϕ) should take if it is to be a topological soliton solution for the YMH theory. We will not need the details of the solution in what follows; indeed it will suffice to quote the asymptotic form of ϕ:

$$\phi^\infty = v\hat{x}, \tag{A2}$$

where v is a real number and $\hat{x}$ is a unit 3 vector. The group $\mathcal{G}_I$ of gauge transformations that leave this boundary condition invariant is then easily determined to be

$$\{g : \mathbb{R}^3 \to G \mid g(x) \to \exp(i\alpha t \cdot x), \quad |x| \to \infty\}. \tag{A3}$$

We thus see that the DES symmetry group of the theory can be computed by means of the quotient group operation in (13). The result is given by a straightforward but tedious exercise in algebraic topology, and it depends on the topological charge N of the solution. To wit, when $N = 0$, $\mathcal{G}_{\mathrm{DES}} = \mathbb{Z} \times U(1)$, and when $N \geq 1$, $\mathcal{G}_{\mathrm{DES}} = \mathbb{Z}_{|N|} \times \mathbb{R}$.[36]

Let us take stock. Earlier in section 2, we considered Greaves and Wallace's claim of GlobalInLocal, which they take to be illustrated by the example of KGM gauge theory (the end of sec. 3.2), whose DES symmetry is a globalization of its local symmetry. However, we also saw in section 4.2 that the general definition of a theory's DES symmetry turns on its boundary conditions and the asymptotic behavior of its fields: thus, the KGM example shows that only in one such

36. See Giulini (1995) for this result and the details of the computation. For an alternative—and prior—derivation of the dyonic spectrum (the 'Witten effect'), see the introduction of a topological term in Witten (1979).

case, DES symmetry happens to coincide with a globalization of gauge symmetry. But, our consideration of the tHP monopole furnishes us with an example in which the two concepts come apart.

A second problem noted earlier with the claim GlobalInLocal is that the concept of a global transformation does not make sense in the context of a general gauge theory. This fact is also illustrated in the case of monopoles, albeit ones whose long range field is non-Abelian (e.g., where the gauge symmetry is 'broken down' to a $U(2)$ subgroup), unlike that of the tHP monopole (which has a $U(1)$ long-range magnetic field). For instance, Preskill (1984, sec. 4.5) shows that global transformations cannot be defined in the vicinity (i.e., at the subsystem boundary) of a monopole with a long-range non-Abelian magnetic field. What this shows is that there are physical subsystems for which the asymptotic global transformations that Greaves and Wallace describe in the KGM case cannot even be defined—and where this is due not to topologically nontrivial space-times but instead to non-Abelian gauge groups and a topologically nontrivial subsystem boundary. This again shows that it is in general a mistake to think of global symmetry as somehow contained within gauge symmetry.

ADDITIONAL REFERENCES LISTED IN THE APPENDIX

Giulini, D. 1995. "Asymptotic Symmetry Groups of Long-Ranged Gauge Configurations." *Modern Physics Letters* A 10:2059–70.
Preskill, J. 1984. "Magnetic Monopoles." *Annual Review of Nuclear and Particle Science* 34:461–530.