BOOK REVIEWS

ANALYTIC NUMBER THEORY
(American Mathematical Society Colloquium Publications 53)

By Henryk Iwaniec and Emmanuel Kowalski:
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What constitutes 'Analytic number theory' as a subject, and sets it apart from other areas of number theory, is not entirely clear-cut. There is a grey area between elementary and analytic number theory, and any field of number theory is likely to interact with every other part. In its broadest definition, analytic number theory is the application of analytic methods (real, complex or harmonic) to problems involving integers or integer-like sets (the ring of integers in an algebraic number field, for example). In particular, the distribution of primes (or prime-like elements) plays a central rôle in the subject. The book under review has an enormous scope, covering large areas of modern-day analytic number theory. The classical application of real, complex and Fourier analysis is present, but also the modern use of ideas from modular forms and spectral analysis is described. In addition, interactions with algebraic geometry are given, and applications are made to algebraic number-fields.

From one perspective, analytic number theory, as conceived in this work, is largely concerned with the study of

$$\text{the sum } \sum_{n\leqslant N} a_n f(n), \qquad \text{or the series } \sum_{n=1}^{\infty} a_n f(n). \tag{1}$$

Here, a_n picks out an arithmetic property of n. For example, a_n might be the characteristic function of the set of primes, or (what amounts to essentially the same thing in applications) the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{otherwise.} \end{cases}$$

The function $f(n)$ detects some other property of n. For example, Dirichlet characters $\chi(n)$ detect the condition $n \equiv a \pmod q$, or the functions n^{-s} $(s \in \mathbb{C})$ can be used to study integers in short intervals. The sum (1) can thus cover the classical problems of finding primes in arithmetic progressions and primes in short intervals, as well as the more modern examples developed throughout this text. Along with this unifying theme, the reader will notice the frequent references to modular forms. This should come as no surprise, since the writers are experts in the introduction of such notions into different areas of analytic number theory.

Among the fundamental subjects covered in some way in this book, we mention: the prime number theorem, Dirichlet's theorem on primes in arithmetic progressions, characters (Dirichlet and Hecke), summation formulae, functional equations, zeros of L-functions and zero-free regions (including automorphic L-functions), sieve methods, the use of bilinear forms, exponential sums, Dirichlet polynomials,

mollifiers, sums over finite fields (including the Weil bound by Stepanov's elementary method), sums over primes, Kloosterman sums, the Bombieri–Vinogradov theorem, Linnik's theorem on the least prime in an arithmetic progression, additive problems, equidistribution, imaginary quadratic fields, zeros of $\zeta(s)$ on $\operatorname{Re} s = \frac{1}{2}$, and connections with random matrix theory. It would have been impossible for the authors to have covered all these subjects in depth. They have therefore opted to tackle some aspects in detail, while giving sketches and surveys for others (with references to more specialised papers or books). The advantage of such a wide coverage is that interactions can be seen between apparently different fields within the subject. Also, the earlier, 'classical' results can often be viewed as specific examples of a modern, more general, approach.

The exposition is generally of high quality, although of necessity some sections consist only of brief sketches. The authors emphasise the ideas behind the proofs, and take the view that the mathematics should be 'beautiful'. One drawback of their approach, however, is that many mathematicians will need to read through this book several times if they are to understand all the comments of the authors. The subject cannot be viewed as a straight-line development, and so there are many references forward in the text, which may be comprehensible only on a second or third reading. This is but a small price to pay, given the encyclopaedic nature of this volume.

Who should buy this book? Certainly, every researcher in analytic number theory should have this tome on their shelf, for no other work covers such a wide range of subjects. Nor do many other authors have the depth of insight of the present writers. They are experts in this area, and have produced an important reference work that should also be of value to researchers in other branches of number theory. The authors state in their introduction: "Formal prerequisites for much of the book are rather slight." So, does this work supersede other books that one might recommend to research students starting out (for example, [**1**-**3**]? In my opinion, a student would benefit most from this book if they already had a background in number theory gleaned from [**2**] or some similar work. For many students, a mastery of the classical theory contained in [**1**] and the early chapters of [**3**] (where some of the technical details that are omitted from the book under review will be found), would also be most helpful. This book by Iwaniec and Kowalski would then be an excellent introduction to 21st-century analytic number theory.

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INTRODUCTION TO FOLIATIONS AND LIE GROUPOIDS
(Cambridge Studies in Advanced Mathematics 91)

By I. MOERDIJK and J. MRČUN:
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There are several approaches to the concept of a *foliation* on a manifold M. The more intuitive is to envisage a decomposition of M into immersed submanifolds, where the latter comprise the leaves of a foliation $(M, \mathcal{F})$. Alternatively, M may admit an integrable distribution, a group action, a certain cocycle, or an equivalence relation, all of which could foliate M in either a non-singular or a singular way. Such constructions reflect upon the classical origins of the subject: the solution flow of differential equations, their phase portraits, and the orbits and distribution of vector fields. For these differing (but often equivalent) approaches, a unifying principle is the idea of a *foliation atlas* into which several of the topological and geometric characteristics of $(M, \mathcal{F})$ are structured. There is some scope in constructing foliations, such as those generated by flows, by certain group actions, the foliation by the fibres of a fibre bundle, the suspension of a diffeomorphism, and by various other geometric techniques.

Generalizing Poincaré's earlier idea of the 'first return map' is the notion of *holonomy*. Intuitively, this describes how the leaves of $(M, \mathcal{F})$ may veer off asymptotically, but return eventually within a kind of phase shift. It is an important geometric concept, since knowing the holonomy group (often via the germs of locally defined diffeomorphisms) means having significant information about both $(M, \mathcal{F})$ and its transverse structure. The holonomy of $(M, \mathcal{F})$ is a subject that was studied in much depth by Haefliger, and it plays an important role in the acclaimed stability theorems of Ehresmann, Reeb and Thurston, a series of results linking the holonomy concept to the topological type of the leaves. One background feature concerns the possibility of a topological space admitting an orbifold structure, thus generalizing the usual atlas/chart constructions for a manifold. For instance, an orbifold structure on the space of leaves $M/\mathcal{F}$ is realized in the case where $(M, \mathcal{F})$ has compact leaves with finite holonomy, as in the case of certain compact Lie group actions, or when a discrete group acts properly on M.

As can be seen in most textbooks on the subject, emphasis is laid on the geometric-topological properties of foliations admitting compact leaves – a rich and deep topological problem, which was studied by Thurston, Haefliger, Novikov, and others in the 1960s and 1970s. If $(M, \mathcal{F})$ admits a compact leaf, then there are restrictions on the possible nature of the foliation, and the type of holonomy group that it can admit, besides the structure of the cohomology in leaf-wise directions. Much of this work was directed at codimension-one foliations on 3-manifolds, where the underlying procedures ought not to be overlooked by a serious student of the subject. An instrumental result is the renown theorem of Novikov, which states that a (smooth) tranversely orientable codimension one-foliation on a compact 3-manifold with finite fundamental group has to admit a compact leaf.

Further on the geometric side, one discovers a close interplay between Morse theory, holonomy, and the noteworthy Godbillon–Vey class $\mathrm{gv}(\mathcal{F})$, a concept that

figures in several aspects of the theory and its relationship with dynamical systems; see the two-volume work by Candel and Conlon [**2**, **3**]. Many differential-geometric concepts, such as connections, curvature and holonomy, play an important role in the theory of foliations and foliated principal bundles. The theory of Riemannian foliations involves a variety of differential-geometric techniques, and has developed somewhat independently. Such foliations often admit holonomy-invariant metrics and (transversely) parallelizable properties in relationship to the normal bundle of the foliation. An important class of such foliations arise from the kernel of a Maurer–Cartan form with values in a Lie algebra; these are usually referred to as *Lie foliations*. Homogeneous and Lie foliations are instrumental in the various structure theorems of Molino [**9**], which relate the properties of transversely parallelizable foliations to the leaf closures of $(M, \mathcal{F})$, as well as to the Lie algebra of transverse vector fields. Other aspects of the theory of Riemannian foliations are developed in the books by Reinhart [**10**] and Tondeur [**11**].

Since foliations on $(M, \mathcal{F})$ can be defined in terms of equivalence relations, it seems natural that foliations can be used in the testing-ground for the theory of groupoids and their relationship to monodromy/holonomy questions. An essential construction concerns how the holonomy coverings of $(M, \mathcal{F})$ can be pieced together to form the Lie groupoid $\mathrm{Hol}(M, \mathcal{F})$, the holonomy groupoid of the foliation [**6**]. It is a fundamental object, used for studying the invariants of $(M, \mathcal{F})$ and, on an analytic slant, leads to the foliation (reduced) C*-algebra $C_r^*(M, \mathcal{F})$, which is central to the noncommutative theory as developed by Connes [**4**]. At the same time, the algebraic and geometric properties of Lie groupoids provide a rich enough subject in their own right, and one that is driven mainly by the quest to generalize much of the traditional theory of Lie groups to this context. The construction of the Lie groupoid $\mathrm{Hol}(M, \mathcal{F})$ can be seen as the product of a certain globalisation procedure that is applicable to more general classes, such as locally topological groupoids (see, for instance, [**1**]).

Whereas Lie groups go hand-in-hand with Lie algebras, the study of Lie groupoids in relationship to Lie algebroids involves a number of peculiarities: there can be Lie algebroids that are not the Lie algebroids of an associated Lie groupoid. In a geometric sense, a Lie algebroid can be considered as generalizing the concept of a tangent bundle. Specifically, if $\mathfrak{X}(M)$ denotes the set of vector fields on M, then a Lie algebroid over M consists of a vector bundle $E \longrightarrow M$, equipped with a Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(E)$, and a bundle map $\Upsilon : E \longrightarrow TM$, called *the anchor*. Further, there is an induced map $\widetilde{\Upsilon} : \Gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras such that, given sections in $\Gamma(E)$ together with a differentiable function, a version of the Leibniz rule is satisfied. The problem of the integrability of Lie algebroids can be traced back to Lie's work on those classes of algebras and groups that were to bear his name. When M is just a point, we recover the definition of a Lie algebra and Lie's third theorem says that this Lie algebra can be integrated to a Lie group. But general integrability questions for Lie algebroids are subtly different from those of Lie algebras, and different classes of obstructions can arise [**5**]. Thus a main part of the subject's development has revolved around the prospect of extending Lie's various structure theorems. Further developments concern other geometric properties of Lie groupoids (algebroids) and their relevance to quantization principles. (For related work on this subject, see [**7**, **8**].)

The book by Moerdijk and Mrčun serves as a concise and user-friendly introduction to the foundations of foliation theory and Lie groupoids, delving into just about all of the above-mentioned facets of the subject, and more. Given the range of topics covered, all of them neatly housed within a manageable hundred and seventy-three pages of text, the book reflects a great deal of care and thought on the part of the authors. The text is somewhat panoramic in nature, and leans more towards the differential-geometric topics at about the half-way stage. Although some passages are rather short, there is no apparent neglect of essential details, and the clarity of exposition is maintained right to the end. Those interested in the subject should find the book a delight to work through, and should thereby find a pathway into the more specialized topics mentioned earlier. It would benefit the reader to have an acquaintance with some basic differential geometry and algebraic topology for the sake of reasonable and fair prerequisites. Otherwise, the treatment of each topic is as admirably self-contained as one might hope for. The instructional value of the book seems quite high, and it could certainly be recommended as text on which to base a graduate course.

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