BOOK REVIEWS

ALGEBRAIC COMBINATORICS ON WORDS
(Encyclopedia of Mathematics and its Applications 90)

By M. LOTHAIRE: 504 pp., £60.00, ISBN 0 521 81220 8
(Cambridge University Press, 2002).

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Consider the following problem about strings of symbols. Call a string a *square* if consists of two adjacent copies of the same string. (An example is the English word hotshots, which is just $(\texttt{hots})^2$.) Must every sufficiently long string contain a square?

If the underlying alphabet is arbitrarily large, the answer is clearly 'no': for example, the string abcdef ... xyz contains no square. On the other hand, if the alphabet consists of two symbols, the answer is 'yes': a simple argument shows that every string of length greater than or equal to 4 contains a square. Over an alphabet of three symbols, however, the answer is 'no': consider the morphism μ such that

$$\mu(\texttt{a}) = \texttt{abc}, \quad \mu(\texttt{b}) = \texttt{ac}, \quad \text{and} \quad \mu(\texttt{c}) = \texttt{b}.$$

If we iterate μ starting with the symbol a, then it can be shown that each of the strings $\mu(\texttt{a}) = \texttt{abc}$, $\mu^2(\texttt{a}) = \mu(\mu(\texttt{a})) = \texttt{abcacb}$, and so on, contains no square.

This beautiful result is due to the Norwegian mathematician Axel Thue in 1906, and is the first significant result in what is now called *combinatorics on words* – 'word' being another term for a string of symbols. In the past thirty years this field has seen explosive growth, particularly in France and Italy.

The book under review is a collection of thirteen chapters, written by different authors, each chapter dealing with some aspect of combinatorics on words. (The name of the author, M. Lothaire, is a collective pseudonym derived from Lotharingia, an ancient name for the area comprising roughly modern-day Lorraine, Alsace, Belgium, Luxembourg, and north-west Germany.) The authors and titles are as follows: Jean Berstel and Dominique Perrin, 'Finite and infinite words'; Jean Berstel and Patrice Séébold, 'Sturmian words'; Julien Cassaigne, 'Unavoidable patterns'; Aldo de Luca and Stefano Varricchio, 'Sesquipowers'; Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon, 'The plactic monoid'; Véronique Bruyère, 'Codes'; Christiane Frougny, 'Numeration systems'; Filippo Mignosi and Antonio Restivo, 'Periodicity'; Christophe Reutenauer, 'Centralizers of noncommutative series and polynomials'; Dominique Foata and Guo-Niu Han, 'Transformations on words and q-calculus'; Jacques Désarménien, 'Statistics on permutations and words'; Volker Diekert, 'Makanin's algorithm'; and Tero Harju, Juhani Karhumäki, and Wojciech Plandowski, 'Independent systems of equations'.

Many of the chapters present material that is difficult to find, or even unavailable, elsewhere. Chapter 2, on Sturmian words, is particularly noteworthy. *Sturmian words* can be defined in many equivalent ways, but the simplest is probably as follows: an infinite word over a two-letter alphabet is Sturmian if it has exactly $n+1$ distinct subwords (factors) of length n, for all $n \geqslant 0$. An example is the infinite

Fibonacci word

$$0100101001001\ldots,$$

which is the fixed point of the morphism sending $0 \to 01$ and $1 \to 0$. Sturmian words have many interesting combinatorial and number-theoretic properties, and have arisen in fields as diverse as computer graphics (where they arise in the display of digitized straight lines), symbolic dynamics, and music. This comprehensive chapter collects many results from the literature on Sturmian words, sometimes with new proofs.

Chapter 7 discusses numeration systems, that is, generalizations of the familiar representation of numbers in base 10. The author covers β-expansions, which are representations in an arbitrary real base $\beta > 1$ and the interesting connections between these expansions and Pisot and Perron numbers. Also discussed is the representation of complex numbers in Gaussian integer bases and the connection to fractals.

Chapter 8 is about periodicity. Here, one of the most famous results is the Fine–Wilf theorem, which states that if two periodic infinite words with periods p and q agree on $p + q - \gcd(p, q)$ consecutive symbols, then they are identical. This theorem has recently been generalized by several different groups of researchers; among other things, this chapter discusses a generalization to three periods.

Although none of the authors is a native English speaker, the exposition is generally quite good. Sometimes a Gallicism creeps in, as on p. 165, where the crucial sentence describing the main topic of the chapter is hard to parse because of poor agreement. I did notice a number of minor errors: on p. 20, for example, a transducer is said to process an input beginning with the most significant digit, but actually begins with the least significant digit. And, as inevitably occurs with chapters written by different authors, there is sometimes a clash of notation: x^{ζ} denotes a set of words on p. 12, and a word on p. 34, but ζ itself denotes a word on p. 115. But overall the material is presented comprehensibly, even pleasingly. Each chapter is accompanied by exercises and detailed notes on the origins of the subject. A bibliography with about 500 references makes it easy to pursue further research.

The volume under review is actually the second by the same pseudonymous author; the first is entitled *Combinatorics on words* [**1**], and a third volume is currently being written. However, this book can be read largely independently of the first volume, and will be of interest to number theorists, combinatorialists, theoretical computer scientists, and algebraists.

Reference

1. M. Lothaire, *Combinatorics on words*, 2nd edn, Encyclopaedia of Math. Appl. 17 (Cambridge University Press, 1997).

University of Waterloo JEFFREY SHALLIT

OPERADS IN ALGEBRA, TOPOLOGY AND PHYSICS
(Mathematical Surveys and Monographs 96)

By Martin Markl, Steve Shnider and Jim Stasheff: 349 pp., US$89.00,
isbn 0-8218-2134-2
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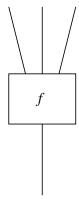
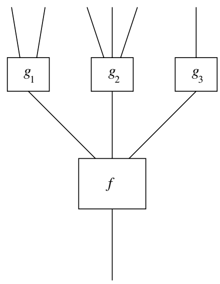
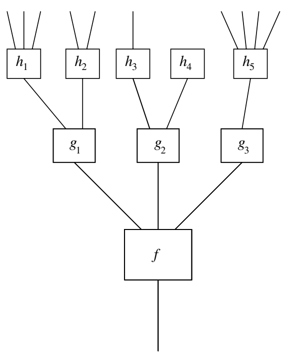
Operads are powerful tools, and this is the book in which to read about them. However, if you're like most mathematicians, your first question will be: '*What is an operad?*' Luckily, the answer is simple. An operad O consists of a set O_n of abstract 'n-ary operations' for each n, together with rules for composing these operations. We can think of an n-ary operation as a little black box with n wires coming in, and one wire coming out.



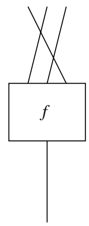
We are allowed to compose these operations as follows, feeding the outputs of the operations $g_1, \ldots, g_n$ into the inputs of the n-ary operation f, and obtaining a new operation, which we call $f \circ (g_1, \ldots, g_n)$.



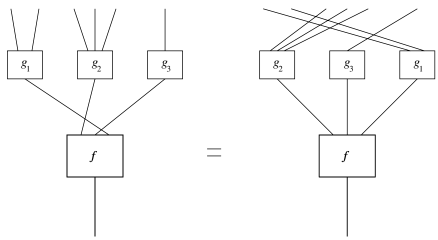
Finally, we demand that there be a unary operation serving as the identity for composition, and impose an 'associative law' that makes a composite of composites like this well-defined.



We can permute the inputs of an n-ary operation f and get a new operation, which we call $f\sigma$ if σ is the permutation of the inputs.



We demand that this give a right action of each permutation group S_n on each set O_n. Finally, we demand that these actions be compatible with composition. For example, the following equality must hold.



That's all! Unfortunately, if one writes these definitions using equations, they look quite formidable. There must be hundreds of mathematicians roaming the earth who think that operads are difficult and abstruse, because they've seen the definitions without any pictures. These people should be grateful that they weren't taught how to tie their shoes in an equally wrong-headed manner.

With this answered, your next question is probably: '*Why should I care about these things?*' There are many reasons, but historically, the first comes from topology. In homotopy theory, the main way to probe a space X is by looking at maps $f : S^k \to X$. We define the 'kth loop space' of X, $\Omega^k X$, to be the space of all such maps sending the north pole to a chosen point $* \in X$. The set of connected components of $\Omega^k X$ is called the 'kth homotopy group' of X; this is a group for $k > 0$, and an abelian group for $k > 1$.

Most homotopy theorists would gladly sell their souls for the ability to compute the homotopy groups of an arbitrary space. However, there is extra information lurking in the space $\Omega^k X$ that gets lost when we consider only its connected components. Starting in the late 1950s, a large number of excellent topologists – including Adams and Mac Lane, Stasheff, Boardman and Vogt, and May – struggled to understand *all* the structure possessed by an k-fold loop space. For example, $\Omega^1 X$ is something like a topological group, thanks to our ability to 'compose' loops. However, the usual group laws such as associativity hold only up to homotopy. To make matters even trickier, these homotopies satisfy certain laws of their own, but only up to homotopy – and so on *ad infinitum*. Similarly, $\Omega^k X$ is something like an abelian topological group for $k > 1$, but again only up to homotopies that themselves satisfy certain laws up to homotopy, and so on – and in a manner that gets ever more complicated for higher k.

After more than decade of hard work, it became clear that operads are the easiest way to organize all these higher homotopies. Just as a group can act on a set, so can an operad O, each abstract operation $f \in O_n$ being realized as actual n-ary operation on the set in a manner that preserves composition, the identity, and the permutation group actions. A set equipped with an action of the operad O is usually called an 'algebra over O'. It turns out that the structure of a k-fold loop space is completely captured by saying that it is an algebra over a certain operad! Even better, if we choose this operad O to be 'cofibrant', any space equipped with a homotopy equivalence to a k-fold loop space will also become an algebra over O. This is the simplest example of how operads are used to describe 'homotopy-invariant algebraic structures', in which all laws hold up to an infinite sequence of higher homotopies.

For an operad to do this job, it must really have a *topological space* of operations O_n for each n, since the fact that various laws hold up to homotopy is expressed by the existence of certain continuous paths in these spaces. Similarly, composition and the permutation group actions should be *continuous maps*. Finally, we should only consider algebras that are topological spaces on which the operad acts continuously.

In short, topology really requires operads and their algebras in the category of topological spaces rather than sets. The ability to transplant the theory of operads to various different categories is an important aspect of their power. After a long pedagogical warmup, the authors of this book wisely treat operads in an arbitrary symmetric monoidal category. They also prove the worth of this level of generality by discussing many examples in detail. For example, they describe how operads in the category of chain complexes have been used to study deformation quantization – and also string theory, where the operations of gluing together Riemann surfaces are important. Indeed, these physics applications have led to a kind of renaissance in the theory of operads!

The plan of the book is as follows. Part I gives a nice introductory tour of operads and their applications. Part II starts by describing operads in a symmetric monoidal category, focusing on their relation to trees and introducing the associahedron operad as a key example. The authors then consider applications to homotopy theory, leading up to a general theory of homotopy-invariant algebraic structures. The next section treats a variety of algebraic topics including the theory of 'quadratic operads', which are operads in the category of vector spaces generated by binary operations satisfying ternary relations. Three classic examples are the operads whose algebras are associative algebras, commutative algebras and Lie algebras. The authors describe a (co)homology theory for the algebras of any quadratic operad, which generalizes Hochschild (co)homology for associative algebras, Harrison (co)homology for commutative algebras, and the usual (co)homology of Lie algebras. Then the authors turn to operads coming from geometry, particularly certain moduli spaces and configuration spaces. Finally, they discuss some generalizations including cyclic operads, which show up in cyclic cohomology, and modular operads, which show up in closed string field theory. The book does not cover the applications of operads to the theory of n-categories; luckily, Tom Leinster is writing a book on that subject. For all other aspects of operad theory, this book by Markl, Shnider and Stasheff is the place to start.

University of California, Riverside JOHN BAEZ

INTRODUCTION TO THE BAUM–CONNES CONJECTURE
(Lectures in Mathematics)

By Alain Valette: 104 pp., CHF33.00 (20.56 euro), ISBN 3-7643-6706-7
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The origins of the Baum–Connes conjecture can be traced back to index theorems of Atiyah and Singer and the early work on analytic K-homology. The conjecture asserts that for any locally compact, second countable, Hausdorff group G, the assembly map

$$\mu : \mathrm{K}_*(\underline{E}G) \to \mathrm{K}_*(\mathrm{C}_r^*(G))$$

is an isomorphism. The ingredients of the conjecture are as follows. On the right-hand side we find the K-theory of the reduced C^*-algebra $\mathrm{C}_r^*(G)$. On the left-hand side we have the equivariant K-homology of the universal example $\underline{E}G$ for proper G-actions. These are linked by an index-type assembly map μ. It is thus clear that, even to state the conjecture, one needs to acquire some knowledge of C^*-algebras, K-theory, K-homology (unified in Kasparov's KK-theory), and index theory.

Valette's book, remarkably, manages to do that and much more in some 100 pages. He restricts his treatment to discrete groups, which is a vast universe with many wild creatures. The material in the book is arranged so as to provide answers to the following questions:

1. Why should we care about the Baum–Connes conjecture?
2. How do we state it?
3. What are the interesting examples?
4. How do we prove it?

There are many possible answers to the first question, and Valette chooses the path leading up from the conjectures that the group ring $\mathbb{C}G$ of a discrete torsion-free group contains no non-trivial idempotents, zero divisors or invertible elements. The conjecture of Kadison and Kaplansky asserts that the reduced C^*-algebra of a discrete torsion-free group contains no non-trivial idempotents. These conjectures are implied by the Baum–Connes conjecture (as is the Novikov conjecture). An example is provided by a theorem of Higson and Kasparov, who proved the conjecture for groups with the Haagerup property; that is, groups that act metrically properly and isometrically on an affine Hilbert space.

The first five chapters of the book introduce the tools required to state the Baum–Connes conjecture. Although the pace is brisk, all the necessary theorems are stated, and the most important ones are proved. The chapter on K-homology contains a number of very carefully worked out examples of representatives of K-homology classes. Kasparov's KK-theory is introduced and immediately used in Chapter 6, in one of the two constructions of the Baum–Connes assembly map that are presented in the book. The topological meaning of the conjecture is illustrated with the help of a Chern character defined by Baum and Connes, which, after tensoring with $\mathbb{C}$, identifies the left-hand side of the conjecture with cohomology of G with coefficients in the vector space generated by elements of finite order. This theme is developed in a chapter devoted to specific examples of the assembly map and its relation to the Chern character.

The last part of the book deals with questions three and four on our list. Groups with the 'rapid decay' property (known as the (RD) groups) are singled out as a source of interesting examples. Then two strategies for proving the Baum–Connes conjecture are introduced: the so called 'Dirac-dual Dirac' method, and the approach due to Vincent Lafforgue, which uses his Banach KK-theory. This theory was a key tool in his proof of the Baum–Connes conjecture in a number of cases, for example, for co-compact lattices in a rank one simple Lie group, or in $SL_3(\mathbb{R})$ or $SL_3(\mathbb{C})$. (Lafforgue's result also covers many groups that are not discrete.)

The appendix by Guido Mislin brings in a detailed discussion of the classifying space for proper actions, explaining the relation between the analytic and topological description.

This book will provide a very useful guide to any reader of at least advanced postgraduate level who is interested in this, one of the most beautiful applications of modern functional analysis. The exposition is clear and precise, and all the main points are explained in sufficient detail. The many explicit examples are among the strongest points of the text.

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