

BOOK REVIEWS

TORSORS AND RATIONAL POINTS (Cambridge Tracts in Mathematics 144)

By ALEXEI SKOROBOGATOV: 187 pp., £35.00, ISBN 0-521-80237-7
(Cambridge University Press, 2001).

Bull. London Math. Soc. 35 (2003) © 2003 London Mathematical Society
DOI: 10.1112/S0024609302211911

A large number of Diophantine problems, many going back to antiquity, can be interpreted (in slightly more modern language) as being to determine all rational, or integral, solutions to a set of simultaneous polynomial equations in several variables. In still more modern language, these equations determine an algebraic variety, which (for simplicity, sufficient for most classical problems, at least) is defined over the field \mathbb{Q} of rational numbers. The task is now twofold: to determine whether or not the variety has any rational points, and if so, to find them explicitly.

It is a measure of our ignorance that the types of variety for which these questions have satisfactory solutions is still extremely small. To illustrate this, one needs only to look at the case of a plane curve defined by a single cubic polynomial $f(x, y)$ with rational coefficients. There is no known algorithm, guaranteed to terminate in a finite number of steps, that will answer in all cases the question: does the curve $\mathcal{C} : f(x, y) = 0$ have any rational points? Although there are partial algorithms, based on a considerable body of theory, that can in many cases either find a rational point or prove that there are none, there are serious theoretical obstacles that prevent these algorithms from handling all cases. For curves of higher degree, and higher-degree varieties (surfaces, three-folds, and so on), the situation is just as bad, if not worse.

Some cases can be easily dealt with: if a plane conic (or curve of degree 2) has no rational points, then this can always be proved by a simple congruence argument. In fancier language, conics satisfy the *Hasse principle*: they have rational points if and only if they have p -adic points for all primes p (and also have real points, though for plane conics the latter condition follows from p -adic solubility). For example, the equation $x^2 + y^2 = 3$ has no rational solutions, as can readily be proved by considering the homogenized form of the equation $X^2 + Y^2 = 3Z^2$ modulo 4. (A simpler form of the same mod 4 argument shows that $X^2 = 2Y^2$ has no integer solutions except $(0, 0)$, and hence that $\sqrt{2}$ is irrational, in a single step which is simpler than the traditional proof which only considers the equations modulo 2.)

The only general-purpose method for showing that a more general variety has no rational points is to show that there are no p -adic points for some prime p . (Here we include the ‘infinite prime’ $p = \infty$, and take the p -adic field \mathbb{Q}_p to be $\mathbb{Q}_\infty = \mathbb{R}$ in that case.) For certain varieties, the existence of such ‘local’ points for all p is sufficient to imply the existence of rational, or ‘global’ points: then we say that this class of varieties satisfies the Hasse principle, also known as the ‘local-global principle’. The example of plane conics (curves of genus 0) is the simplest case where the Hasse principle applies. It does not apply to plane cubics (which, when non-singular, have genus 1), or to curves of any higher genus. This explains the difficulty, theoretical

as well as practical, in deciding the solubility of a plane cubic. Selmer's famous example $3x^3 + 4y^3 = 5$ is a cubic curve that does not satisfy the Hasse principle: it has no rational solutions, despite having p -adic solutions for all primes p .

For curves of genus 1, the study of their failure to satisfy the Hasse principle leads into deep questions concerning the arithmetic of elliptic curves. The failure of the Hasse principle is measured by a group, the Tate–Shafarevich group of the curve, which remains quite mysterious: it is conjectured to be finite, but this was not proved in any single case until work of Rubin and Kolyvagin in the 1980s. This unproved finiteness conjecture is a stumbling block for any algorithm that seeks to determine whether a curve of genus 1 has a rational point.

During the twentieth century, mathematicians have tried to find other classes of variety for which the Hasse principle holds, or to find alternative criteria when it does not. The main contribution to the latter task was provided by Manin, who discovered a general obstruction to the Hasse principle. The technical definition of the Manin obstruction involves Galois cohomology and Brauer groups. Each local point (defined over \mathbb{Q}_p) on a variety defined over \mathbb{Q} maps to a point in \mathbb{Q}/\mathbb{Z} , and for a global point, the sum of its images is 0 in \mathbb{Q}/\mathbb{Z} . Loosely, there is a kind of ‘coherence’ linking the p -adic embeddings of a rational point. Hence it is at least conceivable that one could show, for a variety that is everywhere locally soluble, that no collection of local points (one for each prime) could come from a global, rational point. The full formulation of this idea leads to what is known as ‘the Manin obstruction to the Hasse principle’.

Now the main task may be refined as follows: find classes of varieties for which the Manin obstruction to the Hasse principle is the only obstruction. This is the main subject of the book under review.

There is not room here to explain the general concept of a torsor, except to say that it generalizes the earlier notion of principal homogeneous spaces, which were used (before being so named) in classical descent methods for solving Diophantine problems. The modern notion of a torsor was first systematically developed by Colliot-Thélène and Sansuc, since around 1980. It took until 1999 before the first unconditional example was constructed (by Skorobogatov) of a variety (a bielliptic surface) with no rational points, which the Manin obstruction is insufficient to deal with. This example is given in Chapter 8. This reviewer notes with some satisfaction that a crucial part of the proof of the properties of the constructed surface relies on a very explicit 4-descent on a specific elliptic curve defined over \mathbb{Q} , which was carried out by S. Siksek. (This computation, which is the only explicitly numerical example in the book, appears in an appendix to Chapter 8.)

Until now there has been no source from which to learn the general theory of torsors, or its application to generalized descents. Hence the appearance of the book under review is welcome: from the dust-jacket, one learns that ‘This book represents the first detailed exposition of (1) the general theory of torsors with key examples; (2) the relation of descent to the Manin obstruction, and (3) applications of descent ...’. However, the prospective reader might be misled by this statement; while the first half of the book consists of a section entitled ‘Torsors’, the first chapter starts by stating its aim as ‘to review the general theory of torsors’ and then ‘recalling the definition of torsors’. The reader is referred to other standard texts, such as Milne's *Étale cohomology*, for background and further details. It seems clear, therefore, that this book, which uses all the machinery of modern arithmetic algebraic geometry from the start, is not intended for beginners. On the other hand,

the exposition does include some fairly explicit material linking the general theory to more classical approaches, for example to 2- and 4-descent on elliptic curves in the appendix to Chapter 3.

The second part of the book applies the theory of torsors to the problem of rational points. Torsors may be used to carry out a generalized descent, which in some cases may deliver more information than the Manin obstruction. The Manin obstruction itself is explained in Chapter 5 and its relation with ‘abelian descent’ in Chapter 6. Two extended examples follow: descent on conic bundle surfaces (Chapter 7) and on bielliptic surfaces (Chapter 8). The latter example is handled using torsors under *non-abelian* finite groups, showing that this ‘non-abelian descent’ can provide a stronger obstruction to the existence of rational points than the Manin obstruction. The final chapter also uses non-abelian cohomology in a crucial way, proving Borovoi’s result that the Manin obstruction is the only one for homogeneous spaces of simply connected semisimple groups with connected stabilizer. This involves the use of second non-abelian cohomology, using the language of liens and gerbs.

For a fuller synopsis of the book’s contents, it is worth reading the five-page introductory chapter, though the reader is hereby warned that this is written in much more elementary and accessible language than most of the book. Another positive feature is the collection of ‘Comments’ sections at the end of each chapter, which go some way towards showing how the modern theory relates to more classical methods.

It is clear that the methods of which this book is one of the first expositions outside journal articles have great power, both in unifying several earlier approaches to the problem of rational points, and in solving new classes of problems. These methods are not easy to understand or apply, and it is quite a challenge to make them accessible. A more self-contained treatment would have been a huge undertaking, and the subject is perhaps not yet mature enough for this. Meanwhile, this book will be useful to specialists in arithmetic algebraic geometry in giving in one volume an account of some very recent progress on some of the oldest problems in mathematics.

The University of Nottingham

JOHN E. CREMONA

FLOER HOMOLOGY GROUPS IN YANG–MILLS THEORY (Cambridge Tracts in Mathematics 147)

By S. K. DONALDSON: 236 pp., £50.00 (US\$75.00), ISBN 0 521 80803 0
(Cambridge University Press, 2002).

Bull. London Math. Soc. 35 (2003) © 2003 London Mathematical Society
DOI: 10.1112/S0024609302221918

If M is a three-dimensional manifold, the fundamental group $\pi_1(M)$ is a rich invariant, but one that is hard to use directly. One can examine it indirectly, as with any group, by looking for homomorphisms from $\pi_1(M)$ to some auxiliary group G . For example, the Alexander polynomial of a knot K in three-space is an invariant which encodes information about the homomorphisms $\rho : \pi_1(M) \rightarrow G$, where M is the complement of the knot and G is a two-step solvable group. In simple cases, and with a little luck, knots with the same Alexander polynomial can be distinguished

by looking for homomorphisms to other, perhaps finite, groups G . The number of such homomorphisms to a given G , or the number of equivalence classes under the action of G by conjugation, is an invariant of K .

If G is instead a Lie group, then the set of homomorphisms $\rho : \pi_1(M) \longrightarrow G$ is a topological space. Its quotient by the action of G is the G -representation variety, $R(M, G)$. In the special case $G = \mathrm{SL}(2, \mathbb{C})$, the G -representation variety of a knot complement is a valuable tool in some sophisticated applications to topology: it is an ingredient, for example, in the proof by Gordon and Luecke [1, 6] that a knot is determined up to isotopy by the topology of its complement.

In 1986, Floer defined a new invariant of three-manifolds, starting with the representation variety $R(M, G)$ for the group $G = \mathrm{SO}(3)$. It was constructed originally for closed, oriented manifolds M with trivial first homology [5], but has since been extended in various directions. The invariant takes the form of a finitely-generated abelian group $\mathrm{HF}(M)$, the *Floer homology* of M . In good cases, the representation variety is a finite set, and $\mathrm{HF}(M)$ is the homology of a chain complex whose chain groups have one generator for each representation. These chain groups are readily calculated, but the differential in the chain complex is more elusive: its matrix entries are defined by counting solutions to the anti-self-dual Yang–Mills equations on the cylinder $\mathbb{R} \times M$.

The same Yang–Mills equations that appear in Floer's construction were the tool that the author had used to define new invariants of four-manifolds [2], invariants which opened up several of the previously impenetrable problems in four-dimensional differentiable topology. It was realized early on, by the author and perhaps by Floer himself, that the Floer homology groups fitted into the four-dimensional framework very well. For example, a four-dimensional cobordism between two three-manifolds M_1 and M_2 would give rise (under some mild topological restrictions) to a homomorphism from $\mathrm{HF}(M_1)$ to $\mathrm{HF}(M_2)$; this is something that we can use, for example, to distinguish 'exotic' cobordisms between given three-manifolds, or to establish the non-existence of cobordisms of a specified homotopy type. It is in this four-dimensional setting that the Floer homology groups have made their mark.

Before the publication of the book under review, there had been no easily approachable account of Floer homology since Floer's original paper, and no detailed account at all of its relationship to the four-dimensional invariants. The reason lies in the difficulties of the subject. There are, first, the technical difficulties associated with the Yang–Mills equation, and second, difficulties with the more formal aspects concerning, for example, how to deal with important cases excluded by the 'mild topological restrictions' referred to above. At a certain point, these two sets of difficulties meet, and here the theory is still incomplete and in some ways unsatisfactory. At just the same point, special functions make an unexpected appearance, such as the Weierstrass σ -function that arises in the 'blow-up' formula of Fintushel and Stern [3].

This beautifully written book starts with an account of Floer's original construction, describes the relationship with four-manifold invariants, and highlights some of the remaining problems. Without getting lost in technical detail or excess notation, the author presents a readable account, full of motivation, proceeding one step at a time from the simplest cases. Often, he presents the proof of a proposition only in a slightly simplified form, allowing the reader to see the essential point, while trusting the reader to fill out the extra details of the general case. The first chapter

ends with some appendices, one explaining the connection with Floer's earlier work on the Arnol'd conjecture [4], another discussing how Floer's construction can be motivated by arguments in quantum field theory, a discussion which points to another of Floer's original sources of inspiration: the ideas of Witten contained in [7]. The final chapter examines the blow-up formula, special functions, and the breakdown of the existing theory.

Some familiarity with Yang–Mills theory is assumed, and the book will be an essential reference for students of the subject. But a wider audience will find much to read. The book gives a presentation of basic analytic results and constructions that have more general applicability; and above all, it is full of observations and insight into a difficult and fascinating area that has been the focus of intense research for many years.

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Harvard University

PETER KRONHEIMER

NUMBER THEORY IN FUNCTION FIELDS (Graduate Texts in Mathematics 210)

By MICHAEL ROSEN: 358 pp., £38.50 (US\$49.95), ISBN 0-387-95335-3
(Springer, New York, 2002).

Bull. London Math. Soc. 35 (2003) © 2003 London Mathematical Society
DOI: 10.1112/S0024609302231914

When asked what he had learned from his studies about nature's Creator, the British biologist J. B. S. Haldane is reported to have said, 'An inordinate fondness for beetles'. When faced with a similar question about the Creator of mathematics, a reasonable reply by a mathematician might well be, 'An inordinate fondness for analogy'. Indeed, reasoning by analogy is one of the most powerful tools that we possess. One has only to look at Iwasawa theory, or the impact of arithmetic cohomology theories, à la Grothendieck, on abstract algebraic geometry over an arbitrary base field, to appreciate how powerful 'analogy' truly is.

One of the most fertile areas for finding, and using, analogies is the theory of algebraic curves (projective and smooth) over a finite field, or, equivalently, finite-dimensional field extensions of $\mathbb{F}_p(x)$, where \mathbb{F}_p is the finite field of p elements, p prime, and x is an indeterminate. Such fields are called 'global function fields'. Indeed, it was here that A. Weil established the results that motivated him, in

analogy with singular cohomology of smooth, complex, projective algebraic curves (or Riemann surfaces), to conjecture the existence of the cohomology theories that Grothendieck was later to produce.

Remarkably, global function fields – in their incarnation as algebraic curves – are not just analogous to Riemann surfaces; indeed, they are also analogous to finite-dimensional field extensions of the rational numbers \mathbb{Q} (called ‘global number fields’ or just ‘number fields’). This last analogy also runs very deep, and has stimulated a huge amount of research spanning many decades. It is also the underlying theme in the valuable book being reviewed.

A first instance of how close number fields and global function fields are is provided by the ‘product formula’. Let F be a global function field, and let X be the associated smooth complete curve. The closed points of X are in one-to-one correspondence with the places of F (equal to the equivalence classes of absolute values on F). Let f be a non-zero element of F . As is well known, the number of zeroes and poles of f , counted with the correct multiplicities, are the same; thus the divisor of f always has degree 0. Exponentiating this and using normalized valuations, one finds equivalently that the product over all the normalized absolute values of f is 1. For number fields, if the Archimedean absolute values (coming from embeddings into the complex numbers) are thrown in, a similar product formula is readily established, and is one of the foundational statements of the theory.

Another fundamental analogy between number fields and global function fields lies in the associated ζ -functions and L -series. Indeed, the theory of zeta functions goes all the way back to Riemann’s original paper on the distribution of prime numbers. An analogous complex-valued construction for function fields was originally given in the thesis of E. Artin. Whereas the many properties of zeta functions of number fields, such as the Riemann hypothesis and its generalizations, remain *highly* mysterious, the analogous properties of complex-valued zeta functions of function fields have been known for decades. One high point of Rosen’s book is an appendix giving a self-contained proof of the Riemann hypothesis for function fields.

The list of other problems that have formulations in both number fields and function fields is quite long, and very many of them are covered in *Number theory in function fields*. For instance, there are function-field versions of Artin’s primitive root conjecture and the ABC conjecture. Both of these topics are treated in Rosen’s book. In order to cover the analog of Artin’s primitive root conjecture, Rosen brings the 1937 *Mathematische Annalen* paper of H. Hasse’s student H. Bilharz into the modern era. Elsewhere, the author presents some of the function field ideas that paved the way for Iwasawa’s theory of cyclotomic fields and \mathbb{Z}_p -extensions, and so on.

All of the material in the book is clearly presented at the appropriate level for a graduate course. Numerous and illustrative problems are presented, with the goal of introducing both new techniques and new material.

As our theme here is the many analogies between number fields and function fields, it is natural to ask whether function fields contain any analogs of the integers \mathbb{Z} . For example, the function field $\mathbb{F}_p(x)$ contains the subring $\mathbb{F}_p[x]$ which, like \mathbb{Z} , is a Euclidean domain. In general, one proceeds as follows. Let F be a fixed global function field, and let ∞ be a fixed place. One sets A equal to the set of all functions $f \in F$ with no poles away from ∞ . One readily sees that A is a Dedekind domain with a finite class group and unit group, and is a very general analog of \mathbb{Z} .

Let ∞ be the place of \mathbb{Q} given by the usual absolute value so that the real numbers \mathbb{R} equal \mathbb{Q}_∞ . Of course, one knows that $\mathbb{Z} \subset \mathbb{Q}_\infty = \mathbb{R}$ discretely, and \mathbb{R}/\mathbb{Z} is discrete. Similarly, one has $A \subset F_\infty$ discretely, and F_∞/A can also be seen to be compact (through the use of the Riemann–Roch theorem for instance). Remarkably, as one uses \mathbb{R} and its algebraic closure \mathbb{C} to construct the exponential function, elliptic curves, L -functions, and so forth, so too do F_∞ and its algebraic closure \bar{F}_∞ possess a very rich collection of analytic functions and associated objects.

More specifically, in analogy with \mathbb{Z} -lattices in \mathbb{C} , an ‘ A -lattice in \bar{F}_∞ ’ is a finitely generated discrete A -submodule of \bar{F}_∞ . To the rank 1 lattice $2\pi i\mathbb{Z}$ one attaches the usual exponential function $\exp(x)$. To an A -lattice M of *arbitrary* rank, one attaches its exponential function $\exp_M(x)$, defined by $\exp_M(x) := x \prod_{0 \neq m \in M} (1 - x/m)$. This function is easily seen to be entire and *additive*, thereby giving an analytic isomorphism of $\mathcal{F} := \bar{F}_\infty/M$ with \bar{F}_∞ . The space \mathcal{F} has an obvious A -module structure on it, which can be carried over to \bar{F}_∞ via $\exp_M(x)$. The resulting A -module structure on \bar{F}_∞ is called a ‘Drinfeld module’. Drinfeld modules are remarkably rich analogs of elliptic curves, and are the subject of much current research. They have moduli spaces with associated modular forms. They can also be given Γ -functions and L -series in analogy with elliptic curves; as the L -series of an elliptic curve is derived from the associated \mathbb{Z} -action, so too are the L -series of Drinfeld modules derived from their associated A -action. In particular, such L -series will naturally have values in \bar{F}_∞ . This allows one, for instance, to establish an analog of the famous formula of Euler on $\zeta(2n)$, where $\zeta(s)$ is the Riemann zeta function and n is a positive integer. While much is known about such L -series and modular forms, they remain highly mysterious. Rosen’s book contains an introduction to this chain of ideas which will leave the reader in a good position to appreciate current work.

The interplay over the years between global function fields and number fields has been intense and extremely fruitful. *Number theory in function fields* does an excellent job of introducing these ideas. It will be a welcome resource for any number theorist, and it should become a standard text for graduate students in the area. There is a great pedagogical advantage in viewing difficult classical problems first in the function field arena, where they often have very clear, precise, and suggestive solutions.

The Ohio State University

DAVID GOSS

CELLULAR AUTOMATA: A DISCRETE UNIVERSE

By ANDREW ILACHINSKI: 808 pp., £76.00, ISBN 981-02-4623-4
(World Scientific, Singapore, 2001).

Bull. London Math. Soc. 35 (2003) © 2003 London Mathematical Society
DOI: 10.1112/S0024609302241910

Cellular automata (‘CAs’) are discrete spatially extended dynamical systems, capable of a vast variety of behaviors. Some people study them for their own sake; some use them to model real phenomena; and some speculate that they underlie fundamental physics. The present volume is the most comprehensive single-author book on CAs to date, and provides a useful unified reference to many ideas scattered through the literature. While aimed at an audience of physicists, it should be useful and

comprehensible to mathematicians and computer scientists. While no one book could exhaust such a wide subject, there are several places where this one falls short, and others where it is too generous to ideas that, while popular ten years ago in the complex systems community, have not borne fruit.

After an introduction and a lengthy chapter on formalism (mostly discrete mathematics), the author begins with a phenomenological exploration of basic CA rules. He discusses periodic domains and particles, temporal and spatial correlations, mean-field theory, and Wolfram's grouping of CAs into four somewhat ill-defined classes. He then discusses Langton's λ parameter and the 'edge of chaos' idea. Unfortunately, he repeats early claims that CAs must evolve towards the 'edge of chaos' in order to perform computational tasks, even though this was thoroughly demolished by Mitchell, Hraber and Crutchfield in 1993.

After a nice discussion of Conway's game of life and a sketch of the proof that it can perform universal computation, in Chapter 4 the author gives an introduction to the theory of continuous dynamical systems, and how notions like invariant measure carry over into CAs. Chapter 5 mainly enumerates periodic orbits and bounds transient times.

Chapter 6 gives an introduction to the theory of languages and automata, including non-regular languages. It focuses, however, on the dynamics of the regular languages generated by CAs. That CAs can give rise to context-free and context-sensitive languages is reduced to a brief mention of the work of Hurd. Also, the author misses all of the recent work of Machta *et al.* on the computational complexity of simulating CAs and physical systems, and their relationship to parallel circuit complexity.

Chapter 7 discusses probabilistic CAs and gives an introduction to scaling, phase transitions, and the Ising model of magnetism. It also explains why naive CA simulations often produce unphysical results, and how Creutz and others have designed CA rules that avoid this. It does not explain why most computational physicists still prefer traditional Monte Carlo simulations to CAs.

Chapter 8 has some excellent material on reversible CAs, and on work by Margolus, Takesue, Pomeau, Goles and Vichniac on building thermodynamics from microscopically reversible dynamics. After discussing coupled map lattices and spatio-temporal intermittency, the chapter concludes with a smorgasbord of popular complex systems, including Kauffman's Boolean Nk networks, random maps, and sandpiles. There is a brief section on reaction-diffusion systems, which is the only place in the book where CAs appear as models of pattern formation. (The section on quantum CAs, unfortunately, ignores work by Meyer, Watrous and others.) The chapter presents a CA model of AIDS and immune response, but is far too uncritical about whether such crude models tell us anything important about AIDS.

Chapter 9 gives a good introduction to lattice gases, discussing why CAs can, sometimes, efficiently simulate hydrodynamics and its generalizations. In general, fluid motion in a lattice gas inherits the unphysical anisotropy of the lattice. If the lattice has the right symmetry properties, however, isotropy reappears on large scales. The author's orientation towards physics comes through clearly here, since he does not explain hydrodynamic notation (for example, $\nabla \cdot \vec{v}$) at all.

Chapter 10, on neural networks, has nothing to do with the rest of the book.

Chapter 11 focuses on 'artificial life,' including agent-based models, von Neumann's and Langton's self-reproducing automata, and genetic algorithms. It

presents the author's detailed model of land combat. This certainly has interesting dynamics, but we are left to wonder whether it is at all realistic.

Chapter 12 asks 'Is Nature, underneath it all, a CA?' Many have speculated that the world's apparent continuity masks a fundamentally discrete physics, among them Richard Feynman, John Wheeler and T. D. Lee. (In a sense, students of topological quantum field theories are pursuing this idea.) CAs are possible candidates for such a physics, and Fredkin, Toffoli, Margolus and Wolfram have advocated this vigorously. Many subtle issues are involved: for instance, to capture general relativity's coupling between matter and space-time curvature, the values of the cells must modify the lattice structure. Unfortunately, this chapter is simply a potpourri of speculative theories, without any exploration of whether these have led to any progress in physics. The author is entitled to philosophize, but not to indiscriminately mix field theory, mystical triads, quantum computation and Fritjof Capra. It would have been far better to work through a few models with detail and rigor, and let readers judge their worth.

The book ends with two appendices, one describing currently available CA hardware and software, and the other listing web pages related to CAs. The bibliography is extensive, although far from complete.

Overall, the author's choice of topics is suboptimal: some are out of date, some are extraneous, some deserve a more critical examination, and some are conspicuous by their absence. Nonetheless, there is much useful material here, and we are not aware of anything better with a comparable scope. CA enthusiasts will want copies on their shelves.

University of New Mexico and
the Santa Fe Institute

CRISTOPHER MOORE

University of Michigan and
the Santa Fe Institute

COSMA SHALIZI

DISCREPANCY OF SIGNED MEASURES AND POLYNOMIAL APPROXIMATION

(Springer Monographs in Mathematics)

By VLADIMIR V. ANDRIEVSKII and HANS-PETER BLATT: 438 pp., £59.50
(US\$79.95), ISBN 0-387-98652-9 (Springer, New York, 2002).

Bull. London Math. Soc. 35 (2003) © 2003 London Mathematical Society
DOI: 10.1112/S0024609302251917

This book is devoted to discrepancy estimates for the zeros of polynomials and for signed measures. Most of the topics have emerged in the last two or three decades, and in most cases the book contains far-reaching generalizations of results published earlier.

A somewhat detailed discussion of Chapter 2 will reveal the main subject and the most characteristic features that the book dwells on. It is about zero distribution of monic polynomials. Let E be a compact set on the plane, and let $P_n(z) = z^n + \dots$ be a polynomial of degree n with leading coefficient 1. It is well known that the

supremum norm of P_n satisfies the inequality $\|P_n\|_E \geq (\text{cap}(E))^n$, where $\text{cap}(E)$ denotes logarithmic capacity. Now the chapter roughly deals with the fact that if the norm is close to this theoretical lower bound, then the zeros of P_n are uniformly distributed with respect to the equilibrium measure μ_E of E . There have been several different problems in the mathematical literature leading to this type of question. Jentzsch proved that if $f(z) = \sum_k a_k z^k$ is an analytic function with finite radius of convergence ρ , then every point on the circle $C_\rho = \{z \mid |z| = \rho\}$ is a limit point of the zeros of partial sums of the expansion of f . Szegő complemented this by showing that there is a subsequence $\{s_{n_k}\}$ of the partial sums such that the zeros of s_{n_k} are uniformly distributed in angle (that is, the arguments of the zeros are asymptotically uniformly distributed). In a different direction, Bloch and Pólya investigated the possible number R of zeros of a polynomial $P_n(z) = \sum_{k=0}^n a_k z^k$, provided that $|a_0|, |a_n| \geq \mu'$, $|a_k| \leq \mu$, $k = 1, \dots, n-1$. Their result was extended by Schmidt, and subsequently by Schur, who showed that

$$R^2 \leq 4n \log \frac{|a_0| + \dots + |a_n|}{\sqrt{|a_0||a_n|}}.$$

Erdős and Turán found a common extension of the Jentzsch–Szegő and the Schmidt–Schur inequalities; that is, they showed that if $nv_n(S(\alpha, \beta))$ denotes the number of zeros of P_n in the sector $\alpha < \arg z < \beta$, then

$$\left| v_n(S(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right| \leq 16 \sqrt{\frac{\log P(\rho)}{n}} \quad (1)$$

where

$$P(\rho) = \frac{1}{\sqrt{|a_0||a_n|\rho^n}} \|P_n\|_{C_\rho}.$$

Erdős and Turán also proved the analogue for the line: if all zeros of $P_n(z) = z^n + \dots$ are in the interval $[-1, 1]$, then

$$|v_n([a, b]) - v_{[-1, 1]}(a, b)| \leq \frac{8}{\log 3} \sqrt{\frac{\log \tilde{P}}{n}}, \quad (2)$$

where

$$\tilde{P} = 2^n \|P_n\|_{[-1, 1]}.$$

Here v_n denotes the normalized counting measure on the zeros of P_n and $d\mu_{[-1, 1]}(x) = dx/\sqrt{1-x^2}$ is the equilibrium distribution of the interval $[-1, 1]$. We shall use the terminology that the discrepancy of the arguments of the zeros is the one given on the right of (1), and in a similar manner a short formulation of (2) is that the discrepancy of the zeros is the one given on the right of (2).

Chapter 2 in the book gives considerable generalizations of these results to more general sets. First of all, the Jentzsch–Szegő theorems are extended to compact sets E with connected complement. For example, it is proved that if P_n are monic,

$$\limsup_n \|P_n\|_E^{1/n} \leq \text{cap}(E)$$

and

$$\lim_n v_n(K) = 0$$

for any compact subset of the interior of E , then the distribution of the zeros is

asymptotically the same as the equilibrium distribution μ_E of E ; that is, $\lim_n v_n = \mu_E$. However, the main results of the chapter are the extension the Erdős–Turán theorems to quasiconformal curves and arcs. For example, on quasiconformal curves E , the discrepancy of the zero distribution from the equilibrium measure is given in terms of the quantities

$$\varepsilon_n = \frac{1}{n} \log \|P_n\|_E - \log \operatorname{cap}(E), \quad \delta_n = \frac{1}{n} \log \frac{\|P_n\|_E}{|P_n(z_0)|},$$

where z_0 is any fixed point in the interior of E , and it reads as

$$|(v_n - \mu_E)(A_{\sigma,\tau}(J))| \leq C(\varepsilon_n + \delta_n)^{\alpha/(1+\alpha)},$$

where J is any subarc of E and $A_{\sigma,\tau}(J)$ is a σ, τ -neighbourhood of J defined in terms of inner and outer conformal mappings, $\sigma \geq c(\varepsilon_n + \delta_n)^{\alpha/(1+\alpha)}$ and $\tau \geq c(\varepsilon_n + \delta_n)^{1/(1+\alpha)}$. Here the crucial parameter α depends on the modulus of smoothness of the conformal mappings from the interior and exterior of E onto the unit disk. In general, for piecewise smooth curves, α depends on the angles at the corners. In particular, if E is an interval, then $\alpha = 1$ and we recapture the Erdős–Turán theorem. A similar result is proven for quasiconformal arcs, and for sets bounded by several quasiconformal curves. In the case when the set consists of a finite number of intervals, the constant in the estimate is independent of the set: if $E \subset \mathbb{R}$ consists of a finite number of intervals, then for any interval $I \subset \mathbb{R}$ and for the corresponding strip domain $S(I) = \{z \mid \operatorname{Re} z \in I\}$ the authors prove that

$$|(\mu_E - v_n)(S(I))| \leq 8\sqrt{\varepsilon_n}.$$

If we denote by

$$U^v(z) = \int \log \frac{1}{|z - t|} dv(t)$$

the logarithmic potential of a measure v , then for a monic polynomial $P_n(z)$ and for the normalized zero counting measure v_n , we clearly have $|P_n(z)|^{1/n} = \exp(-U^{v_n}(z))$. Thus, the previous discrepancy questions are special cases of the more general question of giving bounds for the discrepancy

$$D[\sigma] = \sup_{J \subseteq E} |\sigma(J)|$$

of a signed measure $\sigma = \sigma^+ - \sigma^-$ with positive unit Borel measures σ^\pm . The following chapters deal with such discrepancy estimates in terms of different kinds of bounds for the potential U^σ . Chapter 3 utilizes the two-sided bound

$$\varepsilon_\sigma(\delta) = \|U^\sigma\|_{E_\delta},$$

where $E_\delta = \{z \mid |\Phi(z)| = 1 + \delta\}$ is the δ -level line of the conformal map Φ from the outer domain of E onto the exterior of the unit disk. A typical estimate of this sort claims that if E is a K -quasiconformal curve and $\sigma^+(J) \leq M\mu_E(J)^\beta$ for all subarcs $J \subseteq E$, then

$$D[\sigma] \leq C \left(\varepsilon_\sigma(\delta) \log \frac{1}{\delta} + \delta^{1/(2K^2)} + M\delta^{\beta/2} \right).$$

The analogous local result is technically much more difficult, and is treated in the later part of Chapter 3.

Chapter 4 gives discrepancy estimates in terms of one-sided bounds for the potential U^σ , while in Chapter 5 discrepancy theorems are given in terms of the energy

$$I[\sigma] = \int U^\sigma d\sigma = \iint \log \frac{1}{|z - t|} d\mu(t) d\mu(z).$$

For example, if $\sigma^+ = \mu_E$, then the results give the beautiful inequality

$$D[\sigma] \leq C \sqrt{I[\sigma] \log \frac{1}{I[\sigma]}}.$$

The list of applications of the main results is truly impressive. Jentzsch–Szegő-type results are applied to zero distributions of different kinds of sequences of polynomials approximating a given function f on E . In particular, sequences of best approximating polynomials to a non-analytic functions have equilibrium zero distribution through a subsequence (but not necessarily through the whole sequence), while the situation is entirely different for near-best approximants. Zeros of polynomials of maximally convergent sequences, as well as $a(\neq 0)$ -values of orthogonal polynomials with respect to thick area measures, again obey the equilibrium distribution rule, and a similar result is verified for Bieberbach polynomials. In the latter few cases, the limit theorems are accompanied by a discrepancy theorem with discrepancy error equal to a power of $(\log n)/n$.

A second, equally impressive, chapter deals with applications of the main discrepancy estimates to distributions of Fekete points, of extreme and alternation points in best polynomial approximation, and to zero distribution of orthogonal polynomials. In these cases the typical discrepancy error is of the order $(\log n)^2/n$, except for orthogonal polynomials with respect to general Szegő weights, where it is $\sqrt{(\log n)/n}$.

One additional chapter deals with complex approximation of piecewise analytic functions on touching domains.

A remarkable feature of the book is that most of the results in it are shown to be sharp. The authors take the trouble to find counterexamples that show the sharpness of the different estimates, as well as the sharpness of the results regarding conditions on the underlying domains, curves or measures.

The main tools in the book are logarithmic potential theory and conformal and quasiconformal mappings, but the individual proofs use different blends of these with geometric function theory, harmonic analysis, complex analysis, approximation theory and different expansion tools (Faber polynomials, orthogonal and Bieberbach polynomials). The book is pretty much self-contained, for Chapter 1 and several appendices treat those aspects of potential theory and geometric function theory that are used in the proofs.

This work is a valuable monograph on a field that has attracted considerable interest in the recent past, and which has various applications in approximation theory and orthogonal polynomials.

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