

## BOOK REVIEWS

### BANACH ALGEBRAS AND AUTOMATIC CONTINUITY (London Mathematical Society Monographs: New Series 24)

By H. GARTH DALES: 907 pp., £110.00 (LMS MEMBERS' PRICE £77.00),  
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Various called 'metric rings' or 'normed rings', Banach algebras had been under investigation for several years when, in 1941, they burst onto the broader mathematical scene with their use by Gelfand [4] in his elementary proof of Wiener's theorem on absolutely convergent Fourier series. (The term *Banach algebra*, attributed by Rickart [7] to Ambrose [1], is well established nowadays, though other choices are extant; see [2].) Since then, the subject has developed into a major theory in its own right, as well becoming a basic tool in areas such as harmonic analysis and operator theory, to mention just two.

Banach algebras have both algebraic and topological structure, with the basic requirements that the norm topology be complete and multiplication be separately (and hence jointly) continuous. The interplay between the two structures is at the heart of the subject, and it is moot as to which of algebra and analysis should have primacy. Rickart [7] approaches the subject with 'feet in analysis and head in algebra'. The present author explicitly reverses these roles, and Rickart also avers that 'the essence of the subject as an independent discipline is to be found in its algebraic development'.

How far does the algebraic structure determine, or indeed constrain, the topology? For which Banach algebras (as domain or codomain) are algebraic homomorphisms automatically continuous; when are derivations into Banach bimodules automatically continuous; which algebras can be given a Banach algebra topology? Such questions are the substance of the second part of the title. They have been looked at since the beginnings of the subject [3], and they have been considered to some extent in other monographs about Banach algebras—[7, 2, 5, 6] for example—and in the lecture notes [8]. In the book under review here, they are the major goal in the presentation of the whole subject.

In a volume of this size, one may be surprised by there being only five chapters. However, the divisions are clear and to the point. The first chapter deals with the algebraic background, some of which is standard, but much of which is not. Indeed, material on ordered fields, valuation algebras and such like may appear technical obscurities, but the reader can be assured that everything is there for a purpose, and will be needed for later work. On the other hand, it is possibly more digestible to read this chapter by dipping into it when necessary.

Banach algebras *per se* make their appearance in Chapter 2, where the basic concepts are developed, with a wide range of examples. Material from Chapter 1 is revisited, now with topological overtones. The classical development is there, but there is much more—including detailed consideration of bounded approximate identities and factorization questions, bidual algebras, radical algebras, intertwining maps and cohomology, much of which was previously available only in the journal

literature. Of particular value here is the skilful integration of the old with the new, seamlessly moving from early work to recent results.

Chapter 3 deals with Banach algebras with an involution, and is brief compared with the others. The major examples,  $C^*$ -algebras and group algebras, have an enormous literature of their own, and only results germane to the general thrust of the book have been included. In particular, the coverage of the treatise on  $*$ -algebras [5, 6] (reviewed on pp. 504–507) is to a large extent disjoint from the present volume.

Commutative Banach algebras are discussed in Chapter 4. Here we have the standard examples—function algebras, group algebras, convolution algebras—with classical results mixed in with much recent material. Radical algebras begin to make their mark, both as naturally arising examples (weighted convolution algebras), and as important universal objects. The Esterle classification scheme for such algebras is discussed in detail. Results associated with prime ideals also appear as a preparation for Chapter 5. The chapter ends with a brief discussion of more general commutative topological algebras, showing the limits of many Banach algebra results, and explaining the Michael problem.

Chapter 5 is devoted to automatic continuity theory itself, and here all the previous threads come together, including the ‘technical’ results of Section 1.7. At just under 200 pages, the chapter is substantial, as is its content. The basic measure of the discontinuity of a linear operator  $T : X \rightarrow Y$  between Banach spaces is its separating space  $\mathfrak{S}(T) = \{y \in Y : \text{there is } x_n \rightarrow 0 \text{ with } Tx_n \rightarrow y\}$ . The properties of  $\mathfrak{S}(T)$ , together with those of its annihilators, particularly for operators  $T$  which are homomorphisms or derivations, are what drives many of the arguments. A complete list of the major results covered is too long to give here, but a list of sections, indicatively annotated, will show the material covered:

- 5.1 ‘Uniqueness of norm’ (Johnson’s result for semisimple algebras, including the Aupetit/Ransford proof, the case of finite-dimensional radicals);
- 5.2 ‘The separating space and the stability lemma’ (gliding hump theorem, stability properties of separating spaces, derivations on commutative algebras, Thomas’s proof of the commutative Singer–Wermer conjecture);
- 5.3 ‘The continuity ideal and the singularity set’ (continuity of intertwining maps, results involving prime ideals (Cusack, Bade and Curtis), finiteness results for singularity sets);
- 5.4 ‘The main boundedness theorem’ (of Bade and Curtis, application to homomorphisms from regular function algebras, structure of a general homomorphism from a strong Ditkin algebra, prime ideals again, the analogues for  $C^*$ -algebras, homomorphisms from algebras of differentiable functions);
- 5.5 ‘Linear functionals’ (continuity of positive linear functionals and higher point derivations);
- 5.6 ‘Continuous and discontinuous derivations’ (amenability and weak amenability, Read’s commutative radical Banach algebra, the case of  $L^1(G)$ , algebras of compact operators, construction of discontinuous derivations);
- 5.7 ‘Embedding algebras into Banach algebras’ (Allan’s result on power series algebras, the results of Esterle and the author on homomorphisms from  $C(X)$ , discussion of the role of  $\text{CH}$ ).

Finally, there is a series of appendices that deal (very expeditiously) with specific areas of analysis that provide necessary tools, and an extensive 670-entry bibliography, with each item cross-referenced to where it arises in the text. Readers are encouraged to check the author’s website for any corrections or addenda, and to

submit any comments and updates for possible addition to the site. On a stylistic note, definitions tend to be given in their most general setting, and the reviewer found that persistence was sometimes needed to track them down in the index—for example, there is no entry there under ‘semisimple’, nor as a qualifier under ‘Banach algebra’; rather, it is under ‘algebra’.

Of necessity, there are topics that are not covered, or are relegated to a brief mention in the notes at the end of each section (these latter give historical comments, together with indications of further results). In many ways, the selection of topics is determined by the author’s interests and the overarching goal of the automatic continuity results of the final chapter. As implicitly indicated above, it is happily the case that many of the ‘omitted’ topics that come to mind are in fact included in other modern monographs. The result is a volume which is encyclopædic in what it covers, yet to a large extent complementary to others in the general area.

This is a masterful exposition by one of the leading contributors to the field. For the worker in the area, the volume will be indispensable. For someone wanting to learn about the methods and major results of automatic continuity, it is the definitive, if somewhat daunting, exposition of the state of the art as at the close of the 20th century.

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### BANACH ALGEBRAS AND THE GENERAL THEORY OF \*-ALGEBRAS: VOL. II, \*-ALGEBRAS

(Encyclopedia of Mathematics and its Applications 79)

By THEODORE W. PALMER: 823 pp., £75.00, ISBN 0-521-36638-0  
(Cambridge University Press, 2001).

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A  $*$ -algebra is a complex algebra,  $\mathcal{A}$ , with a fixed conjugate linear involution  $a \mapsto a^*$ , where  $a \in \mathcal{A}$ . Algebras having an involution arise principally as convolution algebras on groups and as algebras of operators on Hilbert spaces. These particular algebras are studied in abstract harmonic analysis and  $C^*$ -algebra theory, and are

the subjects of more specialised books such as [9], [14] (see the review on pp. 508–509 of this issue), [4] and [10]. The present volume supplements those books by placing convolution and operator algebras in the broader context of  $*$ -algebras.

Group algebras and operator algebras come additionally equipped with a Banach space norm such that the algebra product is jointly continuous, and are thus examples of Banach  $*$ -algebras. In these examples the involution is continuous as well, but that is not assumed in general Banach  $*$ -algebras.

The book under review is the second of two volumes by Theodore Palmer. These volumes develop the theories of Banach algebras and  $*$ -algebras independently, and then combine the results to describe the theory of Banach  $*$ -algebras. The eight chapters of Volume I [13] dealt with Banach algebras, and Volume II deals with  $*$ -algebras in its first two chapters and Banach  $*$ -algebras and the convolution algebras of locally compact groups in its last two chapters. Volume I appeared in 1994 as number 49 in the series 'Encyclopedia of Mathematics and its Applications' and was reviewed in [2]. That review compares Palmer's volumes with earlier texts on Banach algebras and Banach  $*$ -algebras, including [15], [5], [11], [1] and [8]. Almost simultaneously with this second volume, the earlier reviewer's own book on Banach algebras [3] has appeared (see the review on pp. 502–504 of this issue). The focus of [3] is on automatic continuity, however, and so its content is quite different from that of Palmer's.

All basic results about  $*$ -algebras, Banach  $*$ -algebras and  $C^*$ -algebras are to be found in Palmer's Volume II. It benefits richly from the author's own investigations of  $*$ -algebras, and many discoveries and perspectives are published here for the first time. Here is a more detailed description of the contents.

Chapter 9, the first in Volume II, deals with  $*$ -algebras under no additional hypotheses. Properties of  $*$ -algebras such as being hermitian (every  $h$  with  $h = h^*$  has real spectrum) or symmetric (the spectrum of  $a^*a$  is always non-negative) depend on the involution only, as does the notion of a completely positive map. These topics thus clearly fall under the heading of  $*$ -algebras, and have sections devoted to them. However, results about  $C^*$ -algebras, such as the Gel'fand–Naimark, Kaplansky density and Kadison's theorems, are also treated at this level of generality as statements about  $*$ -representations of  $*$ -algebras on Hilbert space. The rationale is that  $C^*$ -algebras form a purely algebraic category, and all geometric and norm information is encoded in the  $*$ -algebraic structure. This is shown most clearly by the author's geometric characterisation of  $C^*$ -algebras (the Vidav–Palmer theorem).

The main theme of the chapter is how well a given  $*$ -algebra,  $\mathcal{A}$ , may be  $*$ -represented on Hilbert space. Two notions are introduced: the *reducing ideal* in  $\mathcal{A}$ , which is the intersection of the kernels of all  $*$ -representations; and the  *$*$ -representation topology* on  $\mathcal{A}$ , which is the coarsest topology that makes all  $*$ -representations continuous. A  $*$ -algebra is said to be *reduced* if its reducing ideal is equal to zero or, equivalently, if the  $*$ -representation topology is Hausdorff. Many examples are given in illustration, and it is seen that the involutions that are generally studied are far from generic. The examples include a complete list of  $*$ -algebras with dimension at most 3. There are twenty-eight of them; five of these are semisimple algebras, but only three are reduced. The five 4-dimensional semisimple  $*$ -algebras are also listed, and only two of them are reduced.

Chapter 10 imposes additional hypotheses on  $*$ -algebras. These properties are still weaker than being a Banach  $*$ -algebra, but isolate various significant features of Banach  $*$ -algebras. For example, every pre- $*$ -representation,  $T$ , of a Banach

$*$ -algebra,  $\mathcal{A}$ , on a pre-Hilbert space is normed (which means that  $T_a$  is a bounded operator for each  $a$  in  $\mathcal{A}$ ) and therefore extends by continuity to the completion of the pre-Hilbert space. Palmer calls  $*$ -algebras having this property *BG $*$ -algebras*. The category of BG $*$ -algebras is much better behaved than the category of Banach  $*$ -algebras. It possesses the important properties, which the Banach  $*$ -algebras do not, of being closed under passage to  $*$ -ideals and quotients, and that the reducing ideal defines a radical subcategory in the sense of [6, Chapter 9]. Furthermore, every BG $*$ -algebra has a universal  $*$ -representation. Palmer calls  $*$ -algebras having this last property *G $*$ -algebras*. Numerous characterisations of G $*$ -algebras are given, and the proof that every BG $*$ -algebra is a G $*$ -algebra goes via one of these characterisations. That is the author's approach throughout. Several other classes of  $*$ -algebras are introduced as steps in the proof that every Banach  $*$ -algebra is a BG $*$ -algebra. Proofs are thus simplified and clarified, and the essential ideas are highlighted. In the same vein, an algebraic class of  $*$ -algebras (which includes hermitian Banach  $*$ -algebras) is introduced and a generalised Shirali–Ford theorem (every  $*$ -algebra in the class is completely symmetric) is proved.

One of the alternative characterisations of G $*$ -algebras is that they are those  $*$ -algebras which have an enveloping C $*$ -algebra. Despite the delayed appearance of these ideas (the author did much of the work in the 1970's) they remain topical—for instance, the question of when so-called ‘Hecke algebras’ have enveloping C $*$ -algebras is the subject of the thesis [7]. When a  $*$ -algebra,  $\mathcal{A}$ , has an enveloping C $*$ -algebra, its ideal structure yields information about the  $*$ -ideal structure of  $\mathcal{A}$  because there is a continuous surjection from the primitive ideal space of the enveloping C $*$ -algebra to the  $*$ -primitive ideal space of  $\mathcal{A}$ . Algebras for which this map is a homeomorphism are called  *$*$ -regular* by Palmer.

The work of the earlier chapters feeds into Chapter 11, which is about Banach  $*$ -algebras. Fundamental results about Banach  $*$ -algebras have already been established for G $*$ -, BG $*$ - and other larger classes of algebras. These are reviewed in sections on  $*$ -representations, unitary structure, hermitian algebras, ideals and algebras with minimal ideals. In addition, automatic continuity results for positive functionals are established. The chapter has a long concluding section on Hilbert algebras and Tomita algebras, which leads to an introduction to Tomita–Takesaki theory.

A recurring theme throughout the chapters on algebras is the idea of a radical subcategory, following the approach of [6]. In all, thirteen different radical subcategories are identified in the various categories of algebras discussed in the two volumes. Some, such as the Jacobson radical of a Banach algebra, are familiar, but others, such as the symmetric or Leptin radical of a  $*$ -algebra, are less so.

Chapter 12 is about locally compact groups and their convolution algebras. It surveys the structure theory of locally compact groups. The theory of approximation of connected groups by Lie groups and the basic machinery of Lie groups are treated in sufficient depth for the reader to be able to begin to use these tools. There is a section that gives the first unified account of, and adds to, recent progress on the structure of totally disconnected groups. Another section gives an introduction to the unitary representation theory of locally compact groups, drawing on results in the earlier chapters on  $*$ -representations of  $*$ -algebras.

The chapter concludes by updating the author's survey [12] of classes of locally compact groups. These classes are defined in terms of either topological structure (for example, groups with pre-compact conjugacy classes), representations (for example,

Type I groups) or properties of the group convolution algebras (for example, *Boidol* groups are those for which  $L^1(G)$  is  $*$ -regular). Over a long period, many papers have been published exploring the interrelationships between these various classes. Palmer gives us access to this literature through his comprehensive survey. He knows the literature thoroughly and presents the relationships between classes, describes examples of groups distinguishing the classes, and exposes open questions very clearly by means of diagrams and tables. This excellent survey will be an essential reference for researchers in the field for a long time to come.

The proposed Chapters 13 and 14 on cohomology of Banach algebras and  $K$ -theory are not included, but the references intended for these chapters have been retained. Since the bibliography states where each reference is (or would have been) cited, the reader can still benefit from the author's preparation for these topics.

Although parts of it would serve as good references for courses on  $C^*$ -algebras or abstract harmonic analysis, this is not a textbook. It is more a guidebook to  $*$ -algebras and convolution algebras, with almost every part benefiting from the first-hand experience of the author's own investigations. He expertly maps out for us the concepts and results forming the theory of  $*$ -algebras. Introductions to chapters and sections inform us of where we are going, and how we are going to get there. Sometimes we are shown alternative routes. Along the way, we see a multitude of examples, many of them not previously worked out. Tables and diagrams are used successfully to chart all this information.

Historical remarks appear throughout the text, telling us the sources of ideas, attributing results, and alerting us to changes of terminology. The author seems to have digested every item in the ninety-nine page bibliography, and shares his detailed knowledge of the literature through these valuable and interesting comments.

With this volume, Theodore Palmer gives a uniquely complete account of  $*$ -algebras and convolution algebras of locally compact groups, and also makes significant contributions of his own. It will become a standard reference on Banach  $*$ -algebras and abstract harmonic analysis.

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CLASSICAL HARMONIC ANALYSIS AND LOCALLY COMPACT GROUPS  
(London Mathematical Society Monographs: New Series 22)

By HANS REITER and JAN D. STEGEMAN: 327 pp., £60.00 (LMS members' price £42.00), ISBN 0-19-851189-2 (Clarendon Press, Oxford, 2000).

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It is unusual for mathematical monographs to be circulated in illicit photocopied versions, particularly when the subject is as rarefied as 'Classical harmonic analysis and locally compact groups'. But the late Hans Reiter's 1968 monograph, which was out of print within a decade, belongs to this elite class. So the case for a second edition was clear, and who better to prepare that edition than Jan Stegeman, who was significantly involved as Professor Reiter's student in the writing of the original version? What makes the textbook still relevant today is its emphasis on Banach algebra questions related to harmonic analysis. The focus is on  $L^1(G, \mu)$  where  $\mu$  is a (left) Haar measure on a locally compact group  $G$ , but the various themes are extended to cover both Segal algebras (Banach algebras which are dense subalgebras of  $L^1(G, \mu)$ ) and Beurling algebras (weighted versions of  $L^1(G, \mu)$ ). When  $G$  is abelian, we can regard all these algebras as Banach function algebras on  $\widehat{G}$ , the dual group of  $G$ .

The book devotes the first two chapters to the study of  $L^1(\mathbb{R}^n)$ , and associated Banach algebras of functions on the locally compact group  $\mathbb{R}^n$ . The prime example is the algebra of Fourier transforms of  $L^1(\mathbb{R}^n)$ . A particular theme is the structure of closed ideals and the question of whether a function can be approximated in norm by functions vanishing on a neighbourhood of its zero set (the problem of *spectral synthesis*).

With the third chapter, we reach the level of generality of the rest of the book. This chapter presents the basic theory of locally compact groups and Haar measure. A feature of the treatment is to investigate what happens to phenomena when passing to  $G/H$ , where  $H$  is a normal subgroup. Here, the following formula due to André Weil plays a crucial role: for any continuous function of compact support  $f$ , we have

$$\int_{G/H} \left( \int_H f(x\xi) d\xi \right) d\dot{x} = \int_G f(x) dx$$

where  $dx$ ,  $d\xi$  and  $d\dot{x}$  are the (appropriately normalized) Haar measures on  $G$ ,  $H$  and  $G/H$ , respectively. This is applied to analysing the various Banach algebras  $L^1(G)$  (with convolution multiplication),  $L^\infty(G)$  and Beurling algebras on  $G$ .

The next chapters specialize to the abelian case. Chapter 4 outlines the structure theory and duality theory, and summarizes the principal results of the harmonic analysis (Riemann–Lebesgue lemma, uniqueness theorem, inversion theorem, Plancherel's theorem and Bochner's theorem) without proof. The final section proposes a *principle of relativization* as the study of what happens to functions or formulas (or theorems) when one passes from a group to a subgroup. Where the subgroup is closed and normal, this is equivalent to passing from a group to a quotient group in the dual setting.

Chapter 5 constructs functions (including the Fourier transforms of 'trapezium'

functions) which are building blocks for the study of  $L^1(G)$ : the results are relativized and the chapter concludes with a proof of Poisson's formula in the abstract setting.

Chapter 6 extends to the general case the theorem of Wiener–Lévy, that analytic functions operate on the Fourier transforms of  $L^1$  functions. It then covers a number of results of spectral synthesis, including the counter-example of Malliavin. Wiener's theorem (that if an ideal  $I$  in  $L^1(G)$  satisfies the condition that for each  $x \in \widehat{G}$  there is a function in  $I$  whose Fourier transform does not vanish at  $x$ , then  $I = L^1(G)$ ) holds for Segal algebras and certain Beurling algebras, in particular the weighted algebras  $L^1(\mathbb{R}^n, w_x)$ , where  $w_x(x) = (1 + |x|^2)$ , and their analogues for the  $l$ -adic group  $\mathbb{Q}_l$ .

In Chapter 7, the book deals with the *spectrum* of a function (the support of its Fourier transform). Again, spectral synthesis and Beurling algebras make an appearance. The chapter closes with an account of Ditkin sets.

There follows a useful and, in its time, influential chapter on non-abelian groups. (One word of warning: Reiter's notation is highly personal, and not all of it was accepted by workers in the field. In particular, he did not use the term *amenable* to signify a group that carries an invariant mean. Readers should consult the appendix, where this and other connections between Reiter's and subsequent nomenclature are made.)

The new material, accounting for about a third of the book, is partly incorporated (seamlessly) in the text: the sections on measure theory and Segal algebras are greatly expanded; the main result on Ditkin sets is extended to certain Beurling algebras. For the rest, Appendix A considers four topics (Domar's theorem, Malliavin's theorem, Segal algebras and the difference spectrum) in some detail. Appendix B briefly summarizes the additions to the book, and gives references to more recent results.

Professor Reiter's style (and his quirks) have been largely respected. An aspect that gives this reviewer particular satisfaction is the loving care for the correctness of detail, and the absence of sloppy English. Only occasionally can one detect that the book is not written in the first language of either of the authors. I suspect that this edition will be the last, but I can recommend it both to those who want a careful account of the foundations of the subject, and to those who want to dip into it to find examples of what can go wrong and right in classical harmonic analysis.

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## SIMPLICIAL AND OPERAD METHODS IN ALGEBRAIC TOPOLOGY

(Translations of Mathematical Monographs 198)

By V. A. SMIRNOV (translated by G. L. RYBNIKOV) 235 pp., US\$89.00,  
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An operad is a family of topological spaces  $\{\mathcal{E}(j), j \geq 0\}$  whose elements are to be thought of as abstract  $j$ -ary operations similar to the  $j$ -fold product in a group. There are many pairings between the various spaces  $\mathcal{E}(j)$  which are to be envisioned as keeping track of the commutativity and associativity relations between multiplications, and then relations between the relations, and so on ad infinitum. For

example, in the operad of little  $n$ -cubes  $\mathcal{C}(j)$  is the space of  $j$ -tuples of small, disjoint  $n$ -dimensional cubes within (and with edges parallel to those of) the standard unit cube in Euclidean  $n$ -space. By collapsing the complement of the union of the little cube interiors to the base point, each such  $j$ -tuple may be thought of as a map from the  $n$ -sphere to itself which is the sum, in the homotopical sense, of  $j$  self-maps of the  $n$ -sphere.

Operads were first studied by category theorists such as Jon Beck, who noticed a connection with the spaces of iterated multiplication operations in iterated loop spaces. Ioan James, at Oxford, had been the first to manufacture combinatorial models of loop spaces, showing that the free group on a space  $X$  with base-point was a model for the loop space on the suspension of  $X$ ,  $\Omega\Sigma X$ . Around 1963, James Stasheff, in his Oxford thesis, used similar structures to characterise when an H-space was actually a loop space. In the late 1960's Michael Boardman, at Warwick by then, had devised the little cubes operad, and had used it to characterise  $n$ -fold loop spaces for  $1 \leq n \leq \infty$ , and to give combinatorial models for the  $n$ -fold loop space on the  $n$ th suspension of  $X$ . Boardman's ideas concerning the action by the little  $n$ -cube operad on  $n$ -fold loop spaces eventually appeared in [1]. In the meantime these ideas were popularised, researched and further developed by Peter May and his students in Chicago in an extensive series of papers, starting with [3] and including [4] and [2], for example.

In 1981 the author of this volume realised that the concept of an operad, which had become so central to the study of iterated loop spaces, could be equally useful in the investigation of algebraic structures. Recently, this point of view has been amplified by a series of operadic oeuvres in topical areas of physics such as string theory.

The author extended the notion of an operad to the category of chain complexes, and this volume is an account of his work. For example, it is shown that the singular chain complex  $C_*(X)$  admits an action of the operad  $E_\infty$  which, when  $X$  is simply connected, determines the weak homotopy type of  $X$ . The notions of algebras and coalgebras over operads are developed. This means, inter alia, constructing their bar and cobar resolutions, and obtaining spectral sequences from these structures. Several applications are explained: for example, by means of a generalisation of Poisson algebras over operads, the author demonstrates his method by calculating such cohomology algebras as  $H^*(\Omega^n(\mathbb{R}P^\infty/\mathbb{R}P^n); \mathbb{Z}/2)$  when  $n \geq m$ . There are also chapters dealing with operad methods applied to cobordism theories, to the cohomology of groups and Banach algebras and to the calculation of differentials in the Adams spectral sequence.

All in all, I found this book to be a good read that gives the impression of an important mathematical method that still has a lot of mileage in it.

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MODULES OVER NON-NOETHERIAN DOMAINS  
(Mathematical Surveys and Monographs 84)

By LÁSZLÓ FUCHS and LUIGI SALCE: 613 pp., US\$109.00, ISBN 0-8218-1963-1  
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Even if one starts out with the conviction that commutative algebra is the study of commutative Noetherian rings and modules over them (a commutative ring is said to be *Noetherian* if all its ideals are finitely generated), when one progresses down recommended paths of study, one encounters results and examples which shake that conviction: for example, the integral closure of a commutative Noetherian integral domain  $R$  in its field of fractions turns out to be another commutative ring  $R'$  that contains  $R$  as a subring, but  $R'$  need not be Noetherian; and the perfect closure of a reduced commutative Noetherian ring of prime characteristic is a commutative ring which is in general not Noetherian. Thus the restriction of attention to commutative rings and integral domains that are Noetherian is neither natural nor wise.

However, the theory of modules over commutative Noetherian rings is well and attractively developed, because the Noetherian condition, satisfied by many rings that arise in algebraic geometry, appears to give just enough ‘finiteness’ to make many highly satisfactory results possible.

The book under review (to quote its authors) ‘deals with the theory of modules over commutative integral domains, paying only scant attention to the structure of the underlying domains and practically ignoring the Noetherian case’. The authors consider that the module theories of valuation domains, which are perhaps some of the simplest non-Noetherian domains, and of Prüfer domains, have played a key rôle in remarkable developments that took place in the final two decades of the last century.

After the scene-setting Chapter I, the authors devote the next three chapters to development of properties of various classes of non-Noetherian integral domains that will feature prominently later in the book. Chapter II is concerned with valuation domains (that is, integral domains in which the ideals are totally ordered by inclusion), with emphasis on the maximal, the almost maximal, the Henselian and the strongly discrete valuation domains. Prüfer domains, the global versions of valuation domains, can be viewed as providing a non-Noetherian generalization of the concept of a Dedekind domain, and they form the subject matter of Chapter III. Krull domains, which are quite close to being Noetherian and inherit numerous properties from Noetherian domains, are among the further examples of non-Noetherian domains discussed in Chapter IV.

Most of the remaining twelve chapters are devoted to the study of particular classes of modules over non-Noetherian domains. Thus finitely generated modules are studied in Chapter V, projective modules and related modules in Chapter VI, divisible modules in Chapter VII, injective modules in Chapter IX, uniserial modules in Chapter X, polyserial modules in Chapter XII, RD-injective modules and pure-injective modules in Chapter XIII, torsion modules in Chapter XIV, and torsion-free modules in Chapters XV and XVI. Ideas of a topological nature are explored in

Chapter VIII, while ideas originating in the theory of Abelian groups are to the forefront in Chapter XI, entitled 'Heights, invariants and basic submodules'.

The fund of examples of non-Noetherian domains provided by the first four chapters serves the authors well in the subsequent module-theoretic parts of the book. There are situations where the study of some aspect of a general module-theoretic concept seems hopeless over an arbitrary non-Noetherian domain, and sometimes, in such situations, the authors specialize the underlying domain to a case where some theory can be developed. For example, in Chapter IX, the authors sidestep the 'seemingly hopeless task of obtaining a full structural characterization of injective modules over a non-Noetherian domain' and examine the subclass of codivisorial injective modules over Krull domains; and most of Chapter XIV is devoted to torsion modules over valuation domains, because there appears to be only a little that can be said about torsion modules over general domains.

Although the concepts of 'injective module' and 'projective module' are well used in the book, the authors make clear at the outset that no sophisticated result from homological algebra is required. The focus of the authors' presentation is very personal, and reflects their own interests and research; they have not tried to be exhaustive. There are exercises, ranging from routine problems to new material going slightly beyond what is covered in the text, at the ends of the sections. The notes at the ends of the chapters include historical comments. The authors have deliberately omitted standard material covered in textbooks and monographs, and have skipped details of proofs of results that can be found in several textbooks on algebra. Thus the reader should not be surprised to be sent to other sources for proofs of some of the results presented in the book.

Even so, the result is big: it measures  $260\text{mm} \times 182\text{mm} \times 35\text{mm}$  and weighs nearly 1300gm. There is no doubt that the authors have performed a significant service of considerable scholarship in collecting together in one place so much information about modules over non-Noetherian domains, and that this volume will be a valuable source of information to workers in that field.

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R. Y. SHARP

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