BOOK REVIEWS

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S0024609301218530

COMPOSITIONS OF QUADRATIC FORMS
(de Gruyter Expositions in Mathematics 33)

By Daniel B. Shapiro: 417 pp., DM248.00, isbn 3-11-012629-X (Walter de Gruyter, Berlin, 2000).

The identities

$$(X_1^2+X_2^2)(Y_1^2+Y_2^2)=(X_1Y_1+X_2Y_2)^2+(X_1Y_2-X_2Y_1)^2$$

and

$$(X_1^2+X_2^2)(Y_1^2+Y_2^2)=(X_1Y_1-X_2Y_2)^2+(X_1Y_2+X_2Y_1)^2$$

were explicitly mentioned in Fibonacci's *Liber Quadratorum* of 1225, but they must have been known much earlier, and Weil in his book [**5**, p. 11] observes that Diophantus must already have been familiar with these identities.

A similar 4-square identity was found by Euler in 1748, and an 8-square identity by Degen in 1818. All of these identities may be explained by considering the norm maps in complex numbers, quaternions and Cayley numbers.

In 1898 Hurwitz (in [**1**], see also [**2**]) posed the general problem: for which positive integers r, s and n does there exist a formula

$$(X_1^2+X_2^2+\cdots+X_r^2)(Y_1^2+Y_2^2+\cdots+Y_s^2)=Z_1^2+\cdots+Z_n^2,$$

where $X=(X_1,\ldots,X_r)$ and $Y=(Y_1,\ldots,Y_s)$ are a system of indeterminates, and each $Z_k=Z_k(X,Y)$ is a bilinear form in X and Y? Hurwitz showed that this problem is equivalent to the existence of a solution of a system of matrix equations depending on r, s and n. In the case when $r=s=n$, Hurwitz showed that n can only be $1, 2, 4$ or 8.

Amazingly enough, during the mid-1960s Pfister discovered that formulas do exist for the case where $r=s=n$ equal to any power of 2, provided that one allows denominators to enter into the expression of $Z_k=Z_k(X,Y)$; see, for example, [**4**]. During this time, Pfister introduced other quadratic forms of dimension 2^n which generalize the sums of squares and have some remarkable properties. These forms are called *Pfister forms*, and they play a crucial role in the algebraic theory of quadratic forms.

Shapiro's book consists of two parts. In Part I, general quadratic forms are considered, instead of just the sums of squares. As an example, one may consider the problem of finding formulas $\sigma(X)q(Y)=q(Z)$ where (V,q) and (S,σ) are quadratic spaces over some field F of characteristic not equal to 2, and each Z_k is a bilinear form in the systems of indeterminates X, Y with coefficients in F. In this case, we are saying that q and σ admit a composition.

The key notions in Part I are the sets $\mathrm{Sim}(V,q)$ consisting of all similarities of a given quadratic space (V,q) over F and their subspaces, which carry quadratic forms in a natural way. One quickly realizes that this is a convenient way of generalizing Hurwitz's matrix equations mentioned above. Part I contains many interesting results on the spaces of similarities, and their relationship with Clifford algebras and modules over Clifford algebras, involutions on algebras, central simple algebras and Pfister forms.

Chapter 9, in particular, contains an attractive exploration of the 'Pfister factor conjecture'. For any Pfister form φ of dimension 2^n, there is an explicit construction showing that φ admits a composition with some form σ having a maximal dimension

$$\rho(2^m) = \begin{cases} 2m+1 & \text{if } m \equiv 0 \pmod 4, \\ 2m & \text{if } m \equiv 1 \pmod 4, \\ 2m & \text{if } m \equiv 2 \pmod 4, \\ 2m+2 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

It is not known whether a quadratic form q of dimension 2^m which admits a composition with some form of the maximal dimension $\rho(2^m)$ must necessarily be a scalar multiple of a Pfister form. A positive answer to this question is known as the 'Pfister factor conjecture'.

Part II of this book concentrates upon the determination of the sizes of r, s and n in the composition formula mentioned above. Additionally, Part II utilizes algebraic methods, but it also makes use of algebraic topology, K-theory, and differential geometry. In particular, in Chapter 15 the application of Hopf's map on spheres

$$H_f : S^{r+s-1} \to S^n$$

associated with any normed bilinear map

$$f : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$$

is considered.

K. Y. Lam used the geometry of this Hopf map to uncover certain 'hidden' nonsingular pairings associated with the original map f. Lam used these ideas to show that there can be no normed bilinear maps of sizes [10, 11, 17] and [16, 16, 23]. (For the significance of these results, see [**3**] and the reviewed book, page 229.)

Shapiro's book is a beautiful volume, containing many fascinating results. It is a work of love, which evolved from Shapiro's lectures given at the Universität Regensburg (Germany) in 1977, and which were tested, improved and changed over many years. Shapiro himself contributed significantly to this topic, and in particular in the 1970s he proved the Hurwitz–Radon theorem for arbitrary regular quadratic forms over any field of characteristic not equal to 2, and he investigated the quadratic forms that admit compositions.

This book is written very carefully, the proofs are elegant, and at the end of each chapter there are interesting comments concerning the history of the subject, several thought-provoking exercises, references and acknowledgements. Furthermore, this book contains a number of valuable items that have not been previously published.

Some major problems in the algebraic theory of quadratic forms have recently been solved. They include the famous Milnor conjecture, which relates the Milnor K-theory of fields modulo 2, graded Witt rings of quadratic forms, and Galois cohomology with $\mathbb{F}_2$-coefficients. Additionally, some basic questions concerning the maximal dimensions of anisotropic quadratic forms were solved. New and powerful techniques from K-theory, homotopy theory and algebraic geometry are now available. Perhaps some of these new techniques can be applied to attack the open questions presented in Shapiro's book.

Shapiro presents us in this book with some very beautiful work, and with some new challenges and open problems. This is a suitable book for young mathematicians interested in the interplay between quadratic form theory, algebraic topology, K-theory and combinatorics, as well as an invaluable reference source that is appropriate for specialists and for the novice alike.

References

1. A. Hurwitz, 'Über die Komposition der quadratischen Formen von beliebig vielen Variabeln', *Nachr. Ges. Wiss. Göttingen* (*Math.-Phys. Kl.*) (1898) 309–316; reprinted in *Math. Werke*, vol. 2 (Birkhäuser, Basel, 1963) 565–571.
2. A. Hurwitz, 'Über die Komposition der quadratischen Formen', *Math. Ann.* 88 (1923) 1–25; reprinted in *Math. Werke*, vol. 2 (Birkhäuser, Basel, 1963) 641–666.
3. K. Y. Lam, 'Some new results on composition of quadratic forms', *Invent. Math.* 79 (1985) 467–474.
4. A. Pfister, 'Multiplikative quadratische Formen', *Arch. Math.* 16 (1965) 363–370.
5. A. Weil, *Number theory, an approach through history from Hammurapi to Legendre* (Birkhäuser, Boston, 1984).

The University of Western Ontario — Ján Mináč

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S0024609301228537

ARITHMETICITY IN THE THEORY OF AUTOMORPHIC FORMS
(Mathematical Surveys and Monographs 82)

By Goro Shimura: 302 pp., US$69.00, isbn 0-8218-2671-9
(American Mathematical Society, Providence, RI, 2000).

The study of the Riemann zeta function $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ has spawned a huge amount of mathematics. Much is known about this function, but there are still many unsolved questions—for example, the Riemann hypothesis is still open, nearly 150 years after it was postulated. Another unsolved problem is whether $\zeta(2n+1)$ is irrational for all integers $n \geqslant 1$ (although Ball and Rivoal have recently announced a proof that $\zeta(2n+1)$ is irrational for infinitely many n). On the other hand, the values $\zeta(2n)$ for $n \geqslant 1$ are well-known—they are rational multiples of π^{2n}, the rational fudge-factors being related to Bernoulli numbers. This last result is an example of the phenomenon of so-called *special values* of the zeta function. The philosophy is that if $L(s)$ is a complex function built from an arithmetic object, then for certain explicit $\sigma \in \mathbb{C}$ (where typically σ will be an integer or half an integer), the value of $L(\sigma)$ should be the product of a 'transcendental' factor, which will typically be a period value, or the result of a certain integral, and an algebraic factor, which will typically be a subtle rational number related to the arithmetic of the object that $L(s)$ is encoding. In the case of $\zeta(s)$, one should perhaps think of ζ as encoding facts about the integers. (In many cases it is unknown whether the 'transcendental' factor is in fact transcendental, so perhaps this term should be used with caution.)

As sometimes happens in mathematics, one good idea can be generalised and generalised until it becomes a really powerful force. Dirichlet constructed functions generalising $\zeta(s)$, now known as *Dirichlet L-functions*, and used them to prove that there are infinitely many primes in an arithmetic progression $a, a+d, a+2d, \ldots$ if a and d are coprime. The algebraic factors of certain special values of Dirichlet L-functions were shown to be related to class groups of number fields. Later on, people defined L-functions associated to elliptic curves, modular forms, Galois representations, motives, ..., and results about special values were proved and conjectured. In fact, there are currently far more unproven conjectures on special values of these functions than there are proofs. A key example of the conjectures in this area is the Birch–Swinnerton-Dyer conjecture, relating the special value of the L-function of an elliptic curve over $\mathbb{Q}$ at the point $s=1$ to the rational points on the curve.

The book under review is a very thorough study of the special values of L-functions associated to automorphic forms for a very wide class of reductive groups. Shimura has been studying these L-functions (or at least special cases of them) for over 30 years, as one can instantly see by glancing at the references—of the 52 books or articles mentioned, 39 of them are by Shimura himself. Shimura formulates the notion of $\overline{\mathbb{Q}}$-rationality of an automorphic form, in great generality, and considers the L-function associated to certain forms of this type. One of his main results is that for certain explicit $\sigma \in \mathbb{C}$ the L-function $L(\sigma)$ can be written as a product of an algebraic number and an explicit transcendental factor, hence verifying another instance of the philosophy outlined above. Special cases of this result were known before, in many cases due to Shimura himself.

The proof of the result is of a very technical, yet classical, nature. It is well known that the theory of Eisenstein series is essential in proving results of this nature. Shimura studies holomorphic and non-holomorphic Eisenstein series, and proves enough arithmeticity results about them to deduce his results on L-functions using generalisations of standard tricks. Shimura takes special care in proving his results in the maximal possible generality, dealing with forms of both integral and half-integral weights over very general classes of symplectic and unitary groups, working not just over the rationals, but also over more general totally real and CM fields. As a consequence, the results are comprehensive, but this has the inevitable result that the notation can get heavy at times.

The fact that there is a reasonable notion of the $\overline{\mathbb{Q}}$-rationality of an automorphic form is strongly related to the fact that a wide class of moduli spaces possess canonical models defined over number fields. These moduli spaces are now known as *Shimura varieties*, although they are not explicitly referred to by Shimura under this name. Shimura takes up the study of these canonical models in the early chapters of the book. His approach involves a study of abelian varieties and theta functions, and is very classical, completely avoiding the notions of schemes and Grothendieck's foundations of algebraic geometry. Once one knows that these canonical models exist, it is perhaps not surprising that there is a good notion of the $\overline{\mathbb{Q}}$-rationality of an automorphic form, at least for automorphic forms coming from algebraic geometry. Shimura also introduces the notion of $\overline{\mathbb{Q}}$-rationality of some non-holomorphic forms, and this is a much more delicate point, requiring a study of real differential operators on the moduli spaces in question.

After this study of $\overline{\mathbb{Q}}$-rationality, Shimura goes on to explain the complex functions that play the key role in his results, namely the Eisenstein series and the L-functions (referred to in the book as 'zeta functions') associated to the forms he is concerned with. A large part of the book is devoted to an investigation of various arithmetic properties of the Eisenstein series in question, and from these results the main results concerning the special values of the L-functions are obtained.

Shimura has taken great pains to write a book which will cover arithmeticity results in great generality, and as a result the book will have the status of being basically the only reference when results are needed in this generality. Indeed, the reviewer does not even know of another reference for the special case of arithmeticity results for L-functions associated to Siegel modular forms over totally real fields in the generality treated by this volume.

Imperial College, London KEVIN BUZZARD

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S0024609301238533

SINGULARITIES OF PLANE CURVES
(London Mathematical Society Lecture Note Series 276)

By EDUARDO CASAS-ALVERO: 345 pp., £29.95 (LMS members' price £22.46), ISBN 0-521-78959-1 (Cambridge University Press, 2000).

The study of singularities of plane curves has a long history: the first significant development was Newton's introduction of the 'Newton polygon' as a tool for solving a polynomial equation $f(x, y) = 0$ for y in terms of x. During the 19th century, this method was developed by Puiseux to prove his theorem, and singularities took their place in the general development of the algebraic geometry of plane curves; for example, they appear in Plücker's formulae relating the numerical invariants of a curve.

The study of families of curves in a plane (or a more general surface) leads naturally to counting points with multiplicities (this is clearly needed for the so-called 'theorem of Bézout'). Two smooth curves have intersection number (at least) 2 at a point if and only if they have the same tangent direction at the point. This led M. Noether to invent, for each point and for each direction at that point, an 'infinitely near point'. Noether's work was developed by Enriques [**3**] to obtain an extensive calculus of infinitely near points of various kinds, which may be used to study families of curves, singular points on curves, and ideals in rings of functions. Noether also showed that singularities of curves could be 'resolved' by Cremona transformations of the plane.

The growth of topology in the early 20th century led by the early 1930s to the notion of topological equivalence of curve singularities, and the proof that this is equivalent to the equality of certain combinatorial invariants. Zariski, who was one of the key players at that time, encoded this in his 1965 notion of *equisingularity*. It is now possible to formulate many conditions that are equivalent to equisingularity, and to study singularities of plane curves from many different viewpoints.

Numerous books have been written in which singularities of plane curves have played a significant role. Most books on algebraic curves contain a proof of resolution of singularities (for modern books, this usually means resolution by blowings-up). The topic can also be regarded as a good approach to presenting an introduction to the techniques of complex analytic geometry. There also exist manuscripts (mostly unpublished at present, but the book [**2**] by Eisenbud and Neumann is important) treating the topic from a more topological point of view. Of particular interest is the vast tome of Brieskorn [**1**] on algebraic curves, which begins with a 200-page historical survey, and develops the major themes mentioned above.

The contents of the book under review are as follows. After a chapter of preliminaries, the Newton–Puiseux algorithm is developed, and the result is used to establish basic algebraic properties. The next chapter defines branches, local rings, parametrisations, intersection multiplicities and linear systems. Next, Casas introduces infinitely near points and the notion of proximity, establishes the existence of resolutions, and gives a definition of equisingularity. Chapter 4 deals with the conditions of asking curves to go through infinitely near points with prescribed multiplicities, obtains the proximity inequalities, and describes the unloading algorithm. Then the relation between the Puiseux characteristic exponents and the invariants defined via infinitely near points is obtained, and related also to the semigroup of a branch.

The remaining topics are more specialised. Next, the singularities of the polar curves of a given curve are studied. This is a problem full of pitfalls, which the author has studied in some depth, and he gives a careful presentation, including information about polar invariants and about decomposition theorems. He proceeds to a local study of linear families of plane curves, including a version of Bertini's theorem, and some sufficiency theorems. The final chapter begins by studying valuations of the local ring of the curve, and contains the relation between infinitely near points and Zariski's complete (that is, integrally closed) ideals.

The book by Casas is quite different from all the others mentioned. The style has been polished by repeated presentation in lectures to students, so that the book is very much written to introduce novices to the study of singularities; in particular, there are useful diagrams, and carefully chosen exercises at the end of each chapter. The approach is derived from that of Enriques: the central concept in the book is that of a curve passing through a cluster of infinitely near points with prescribed multiplicities. However, although the viewpoint may be regarded as old-fashioned, the selection of topics includes many questions of current interest. The scope is restricted to algebraic and analytic properties of curves over the complex numbers; their topology is not discussed.

References

1. E. Brieskorn and H. Knörrer, *Plane algebraic curves* (Birkhäuser, Basel, 1986).
2. D. Eisenbud and W. Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Stud. 110 (Princeton Univ. Press, 1985).
3. F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche* (Niccolo Zanichelli, Bologna, 1915/1918/1924).

The University of Liverpool C. T. C. Wall

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S002460930124853X

EQUILIBRIUM STATES IN ERGODIC THEORY
(London Mathematical Society Student Texts 42)

By Gerhard Keller: 178 pp., £16.95 (LMS members' price £12.71),
isbn 0-521-59534-7 (Cambridge University Press, 1998).

This book gives an introduction to the ergodic theory of equilibrium states. The concept of equilibrium states originates from statistical mechanics, where it denotes those states towards which a physical system (not interacting with the outside) is driven by its internal fluctuations. After having entered into mathematical usage, an 'equilibrium state' became a probability measure on a topological space which is characterised by certain variational principles.

The present book considers equilibrium states from two points of view. One is that of statistical mechanics on lattices, where the group $\mathbb{Z}^d$ induces a canonical action. The other is that of time-discrete dynamical systems, which are given by an action of $\mathbb{Z}$ (or of $\mathbb{Z}^+$). These two points of view are interwoven here. The ergodic theory is presented for general $\mathbb{Z}^d$-actions, and the variational principles associated with the characterisation of equilibrium states are exploited to obtain results on invariant measures for dynamical systems.

The book consists of six chapters. The first three chapters provide the basic theory. Chapter 1 has an introductory character. It discusses equilibrium states and pressure on finite spaces, thereby introducing the main examples on an elementary level. Gibbs measures and large deviation estimates for Gibbs measures on finite lattices (with the lattice size tending to infinity) are presented. Chapters 2 and 3 are devoted to abstract ergodic theory, which is developed for general $\mathbb{Z}_+^d$ and $\mathbb{Z}^d$ actions, respectively. In particular, a proof of the Birkhoff ergodic theorem for $\mathbb{Z}_+^d$ actions is provided. Then ergodicity and mixing, the ergodic decomposition, and factors and extension are introduced. In Chapter 3, entropy theory is developed up to the Shannon–McMillan–Breiman theorem. Again, everything is formulated and proved for the general case of $\mathbb{Z}_+^d$ actions.

In Chapter 4, equilibrium states and pressure are introduced for continuous $\mathbb{Z}^d$ actions on a compact metrisable space X. The pressure $p(\psi)$ of an upper semi-continuous function $\psi : X \to \mathbb{R}$ (called a *local energy function*) is defined from the variational point of view by

$$p(\psi) = \sup_{\mu \in \mathscr{M}(\mathscr{T})} \Big(h(\mu) + \int \psi \, d\mu \Big),$$

where $\mathscr{M}(\mathscr{T})$ is the set of all invariant measures for the $\mathbb{Z}^d$-action $\mathscr{T}$, and $h(\mu)$ is the measure-theoretic entropy. Equilibrium states (for ψ) are defined as invariant measures for which this supremum is achieved. Then continuity properties of the entropy function $\mu \mapsto h(\mu)$ and the convex-geometrical characterisation of equilibrium states as derivatives of the pressure function are discussed. A variational principle for the pressure is given, which allows one to obtain $p(\psi)$ by an approximation using *constructible δ-equilibrium states*. This is then used to characterise equilibrium states for expansive actions.

Chapter 5 deals with Gibbs measures (associated with a regular local energy function) for configuration spaces over the d-dimensional integer lattice. A main result of this chapter is the identification of the translation-invariant Gibbs measures with the equilibrium states for a given regular local energy function. Another major result is the derivation of a large deviations principle for Gibbs measures.

Finally, Chapter 6 deals with (discrete-time) dynamical systems, which are assumed to be *continuous fibred* with respect to some reference measure. The classical situation of such a system is a piecewise differentiable map on a subset of $\mathbb{R}^d$, with the reference measure being the Lebesgue measure. After *Sinai–Bowen–Ruelle measures* have been introduced as being those invariant measures which are Birkhoff limits of points from a set of positive measure (with respect to the reference measure), the existence of absolutely continuous equilibrium states is investigated. The book concludes with a discussion of conformal iterated function systems, for which the existence of invariant measures of maximal Hausdorff dimension is shown by identifying them as equilibrium states.

This is a very well-written book. It is well organised, clear, and coherent. While the theory is developed in quite a general setting, there are always examples (often shift systems, tent maps, and other simple systems) demonstrating the notions and the results. For readers with a background in measure theory, it is essentially self-contained. The book can be warmly recommended to graduate students and researchers in ergodic theory, in dynamical systems, in statistical mechanics, and in related fields. It would also be suitable as a basis for a very good graduate course.

Technische Universität Ilmenau HANS CRAUEL

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S0024609301258536

CLASSICAL INVARIANT THEORY
(London Mathematical Society Student Texts 44)

By PETER J. OLVER: 280 pp., £15.95 (LMS members' price £11.96), ISBN 0-521-55821-2 (Cambridge University Press, 1999).

This impressive book concentrates on the classical invariant theory of binary forms in characteristic zero (with a closing chapter on the general case of multivariate forms, again in characteristic zero, which we shall not mention further) so as to bring out a beautifully rich mixture of algebra (invariants of quadratics, cubics, and so on; matrix and Lie groups; Lie algebras; rings of invariants), geometry (group actions on spaces; symmetries, especially of curves), history (references to works of the founding fathers and mother) and applications (mathematical physics; applied mathematics; quantum groups; computer vision). The exposition is clear, well motivated (both mathematically and historically) and tightly coherent, with the basic cases of quadratic and other forms being reworked from ever more sophisticated points of view in order to illuminate and exemplify the discussion. The reader is required to take an active rôle by working through the many exercises and comments; indeed, there are a number of intriguing remarks about further directions for research. Much of the text could be tackled by a good undergraduate (with singular benefit to his or her education and appreciation of the breadth and interconnectedness of old and new mathematics), but older hands will find much to admire as well.

After a brief history of the subject, the book opens with a prelude on quadratic polynomials and forms, concentrating on the discriminant, its geometrical meaning and its rôle in the matter of canonical forms. These topics are reworked in the next chapter for cubics and quartics, with Hessians and resultants now making an appearance; in turn, these serve to introduce the notions of invariants and covariants, and to motivate a discussion of Hilbert's basis theorem (of invariant theory), and of syzygies.

The next two chapters give a brief but thorough introduction to transformation groups, representation theory and the general notion of invariants; here, the mathematics is perhaps just a bit too concrete—a little abstraction could have added some structural insight. Next comes an innovatory approach to the computation of invariants and covariants, using so-called 'transvections' based on invariant differential operators (the idea goes back to Cayley); the connections with the more usual symbolic (or 'umbral') approach are taken up in the following chapter. The advantage here is that transvectants can be applied equally well to give an invariant theory of smooth or analytic functions (rather than just forms); moreover, the symbolic method can be made unusually precise. The connection between the two is made via a Fourier-like transform which changes differential data into algebraic data; this approach enables the author to give a particularly insightful proof of Hermite's famous reciprocity relation on covariants of binary forms.

Next there is a chapter on Sylvester's 'algebro-chemical theory' (*sic*), which is a graphical method of viewing invariants and covariants. The fundamental syzygies then correspond to certain allowable 'chemical reactions', that is, to certain operations on associated digraphs—as the author wryly notes, a picture is worth a thousand algebraic manipulations! Finally, for binary forms, the author introduces an innovatory method, based on Cartan's normalization procedure via so-called 'moving frames', to construct invariants (both ordinary and differential) for sufficiently regular Lie group actions, and so presents a novel solution to the basic equivalence problem of binary forms; here, the use of the 'signature curve' to discuss symmetries of curves is especially pretty. The discussion of binary forms concludes with a look at infinitesimal methods using matrix Lie algebras, and then Hilbert differential operators (which are averaging operators) are used to derive the Hilbert basis theorem of invariant theory from the ring-theoretic version—a fittingly striking end to a beautiful book.

The University of Edinburgh LIAM O'CARROLL

Bull. London Math. Soc. 33 (2001) DOI: 10.1112/S0024609301268532

MODULAR FORMS AND GALOIS COHOMOLOGY
(Cambridge Studies in Advanced Mathematics 69)

By HARUZO HIDA: 343 pp., £42.50 (US$69.95), ISBN 0-521-77036-X
(Cambridge University Press, 2000).

GEOMETRIC MODULAR FORMS AND ELLIPTIC CURVES

By HARUZO HIDA: 361 pp., £42.00, ISBN 981-02-4337-5
(World Scientific, Singapore, 2000).

The Shimura–Taniyama conjecture, the topic which unifies these two volumes, is one of the cornerstones of modern arithmetic-algebraic geometry. It states that the Hasse–Weil L-function of every elliptic curve, which is defined over the rational numbers, is the Hecke L-function of a modular form which is a Hecke eigencuspform defined over the rationals. The conjecture was first proved for semi-stable elliptic curves by Wiles and Taylor [**2**], improved by Diamond, and finally proved completely by Breuil, Conrad, Diamond and Taylor.

It may not appear so at first sight, but this was fundamentally a conjecture about numbers. In fact, the proof of Fermat's last theorem was the motivation for Wiles' original Shimura–Taniyama result.

As Professor Hida points out, modern algebraic number theory draws upon a very wide spectrum of methodology. In the case of the Shimura–Taniyama conjecture, one needs as background a total familiarity with modular forms (particularly Hida modular forms), as well as the associated Galois representations and their deformations, also schemes, group-schemes and their cohomology, plus elliptic curves and their moduli varieties. In addition, most crucially, one needs access to an expert to explain the synthesis.

These two volumes provide the access to such an expert.

The author hopes that these two volumes will supply postgraduate students and researching mathematicians (already familiar with basic algebraic number theory and class field theory) with an adequate background to enable them to read Wiles' proof of Fermat's last theorem. The first book introduces modular forms, Galois cohomology and representations. It represents the contents of a number of graduate courses given at UCLA during 1995–8. The second book originated as a series of eighteen lectures at a 1992 CIMPA conference in Nice, supplemented by a chapter on Jacobians and a final chapter taking the reader up to Wiles' proof.

In writing these two volumes, Professor Hida has done the mathematical community a considerable service. Reading them (perhaps in conjunction with [**1**]), a strong mathematician, working hard and with considerable tenacity, could bring her/himself up to date with the most important advances in modern number theory. However, such solitary scholarship would be hard going, and in my opinion, sadly, there is at most one UK mathematical department where a PhD student could hope to learn this material from graduate courses.

References

1. G. Cornell, J. H. Silverman and G. Stevens (eds), *Modular forms and Fermat's last theorem* (Springer, New York, 1977).

2. A. Wiles and R. Taylor, 'Ring-theoretic properties of certain Hecke algebras', *Ann. of Math.* 141 (1995) 443–551 and 553–572.

University of Southampton VICTOR SNAITH

INDEX OF BOOK REVIEWS