

# Electromagnetism of One-Component Plasmas of Massless Fermions.

## Supplemental material

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### 1. Kramers-Kronig relations

By virtue of the Riesz-Herglotz theorem (Tkachenko et al. 2012) and directly from (5.21) in (Rylyuk & Tkachenko), in the upper-half plane  $\text{Im } z > 0$ , the following representation for the electrical susceptibility tensor holds

$$\hat{\kappa}(\mathbf{k}, z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_H(\mathbf{k}, \omega)}{\omega - z} d\omega \quad (1.1)$$

and we arrive at the Kramers-Kronig relations,

$$\begin{aligned} \hat{\kappa}_H(\mathbf{k}, \omega) &= \frac{1}{\pi} \text{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_{AH}(\mathbf{k}, \omega')}{\omega' - \omega} d\omega' , \\ \hat{\kappa}_{AH}(\mathbf{k}, \omega) &= -\frac{1}{\pi} \text{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_H(\mathbf{k}, \omega')}{\omega' - \omega} d\omega' , \end{aligned} \quad (1.2)$$

where the subscript AH denotes the anti-hermitian matrix. The fact that the function  $\hat{\kappa}(\mathbf{k}, z)$  satisfies the Kramers-Kronig relations (1.2) is a direct consequence of the causal response of the plasma to an external perturbation. In the absence of the spatial dispersion ( $\mathbf{k} = \mathbf{0}$ ) the conductivity tensor  $\hat{\sigma}(z) = \lim_{\mathbf{k} \rightarrow \mathbf{0}} \hat{\sigma}(\mathbf{k}, z)$  is also a response function and it satisfies to the Kramers-Kronig relations

$$\begin{aligned} \hat{\sigma}_H(\omega) &= \frac{1}{\pi} \text{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\sigma}_{AH}(\omega')}{\omega' - \omega} d\omega' , \\ \hat{\sigma}_{AH}(\omega) &= -\frac{1}{\pi} \text{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\sigma}_H(\omega')}{\omega' - \omega} d\omega' . \end{aligned} \quad (1.3)$$

In this case for the conductivity tensor  $\hat{\sigma}(z)$  the Kubo formula (5.22) in (Rylyuk & Tkachenko) is valid (Zubarev 1971).

The expressions in (5.21) (at  $\mathbf{k} = \mathbf{0}$ ) and in (5.22) in (Rylyuk & Tkachenko) are the same in appearance, the only difference being in the meaning of the averaging operation  $\langle \dots \rangle$ , that is, in how the Coulomb interaction is taken into account in the Hamiltonian - explicitly, or as a screening field. In the first case, the Hamiltonian contains the direct Coulomb interaction between Dirac fermions. In the second case, the Hamiltonian does

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not contain the Coulomb interaction, but the Coulomb interaction between the charges is taken into account by introducing a self-consistent screening field.

## 2. Some mathematical formulas

The Euler-Maclaurin sum formula is

$$\sum_{a \leq n < b} F(n) \simeq \int_a^b F(x) dx + B_1 F(x)|_a^b + \frac{B_2}{2} \frac{\partial F(x)}{\partial x} \Big|_a^b, \quad (2.1)$$

where  $B_n$  are the Bernoulli numbers, in particular,  $B_1 = -1/2$ ,  $B_2 = 1/6$ .

The Poisson formula reads

$$\sum_{n=n_0}^{\infty} F(n) = \int_a^{\infty} F(x) dx + 2\text{Re} \sum_{n=1}^{\infty} \int_a^{\infty} F(x) \exp(2\pi i n x) dx, \quad n_0 - 1 < a < n_0 \quad (2.2)$$

and

$$J_{0,\pm} = -\hbar \int_x^{\infty} dn \int \frac{\partial f_{\text{FD}}(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=\epsilon_{n,\pm}} dk_z \simeq 4\pi^2 \frac{n_e l^3}{v} K_1(x), \quad (2.3)$$

where  $n_e$  is the electron number density,  $l = \beta \hbar v$ ,  $x = \sqrt{2} \beta \hbar \omega_H$  and  $K_1(x)$  is the Macdonald function (D10) in (Rylyuk & Tkachenko).

The Poisson integral is

$$\int_{-\infty}^{\infty} \exp(-i\alpha p^2) dp = \exp(-i\pi/4) \sqrt{\frac{\pi}{\alpha}}. \quad (2.4)$$

The complex integrals are

$$\begin{aligned} J_1 &= \int_{-\infty}^{-\infty} \frac{\exp(i\alpha\xi)}{\exp(\xi) + 1} d\xi = -\frac{i\pi}{\sinh(\alpha\pi)}, \\ J_2 &= \int_{-\infty}^{-\infty} \frac{e^\xi}{(e^\xi + 1)^2} e^{i\alpha\xi} d\xi = \frac{\pi\alpha}{\sinh(\alpha\pi)}. \end{aligned} \quad (2.5)$$

The Fourier transformation is

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int d\mathbf{k} d\omega e^{-i(\omega t - \mathbf{k}\mathbf{r})} f(\mathbf{k}, \omega), \quad f(\mathbf{k}, \omega) = \int d\mathbf{r} dt e^{i(\omega t - \mathbf{k}\mathbf{r})} f(\mathbf{r}, t). \quad (2.6)$$

The Dirichlet formula is

$$\sum_{n,m=0}^{\infty} \mathcal{F}(n, m) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \mathcal{F}(n, m) + \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \mathcal{F}(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{\mathcal{F}(n, n+m) + \mathcal{F}(n+m, n)\} \quad (2.7)$$

and the functions  $\mathcal{F}_{n,n+m,\pm}(k_z)$  and  $\mathcal{F}_{n+m,n,\pm}(k_z)$  in (5.28) in (Rylyuk & Tkachenko) are

$$\begin{aligned} \mathcal{F}_{n,n+m,\pm}(k_z) &= \frac{\partial f_{\text{FD}}(\epsilon, k_z)}{\partial \epsilon} \Big|_{\epsilon=\epsilon_{n,\pm}} \frac{\nu_{\pm}}{\omega_{n+m,n,\pm}^2 + \nu_{\pm}^2} \langle n | j_{\mu} | n+m \rangle_{\pm} \langle n+m | j_{\nu} | n \rangle_{\pm}, \\ \mathcal{F}_{n+m,n,\pm}(k_z) &= \frac{\partial f_{\text{FD}}(\epsilon, k_z)}{\partial \epsilon} \Big|_{\epsilon=\epsilon_{n+m,\pm}} \frac{\nu_{\pm}}{\omega_{n,n+m,\pm}^2 + \nu_{\pm}^2} \langle n+m | j_{\mu} | n \rangle_{\pm} \langle n | j_{\nu} | n+m \rangle_{\pm}. \end{aligned} \quad (2.8)$$

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