Electromagnetism of One-Component Plasmas of Massless Fermions. Supplemental material

V. M. Rylyuk[1]†, I.M. Tkachenko[2,3]

[1]National Academy of Sciences of Ukraine "Center for Problems of Marine Geology, Geoecology and Sedimentary Ore Formation of the NAS of Ukraine, Kyiv, Ukraine"

[2]Departament de Matemàtica Aplicada, Universitat Polièecnica de València, Valencia, Spain

[3]Al-Farabi Kazakh National University, Almaty, Kazakhstan

(Received xx; revised xx; accepted xx)

1. Kramers-Kronig relations

By virtue of the Riesz-Herglotz theorem (Tkachenko et al. 2012) and directly from (5.21) in (Rylyuk & Tkachenko), in the upper-half plane Im $z > 0$, the following representation for the electrical susceptibility tensor holds

$$\hat{\kappa}(\boldsymbol{k}, z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_{\mathrm{H}}(\boldsymbol{k}, z)}{\omega - z} d\omega \tag{1.1}$$

and we arrive at the Kramers-Kronig relations,

$$\hat{\kappa}_{\mathrm{H}}(\boldsymbol{k}, \omega) = \frac{1}{\pi} \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_{\mathrm{AH}}(\boldsymbol{k}, \omega')}{\omega' - \omega} d\omega' \ ,$$
$$\hat{\kappa}_{\mathrm{AH}}(\boldsymbol{k}, \omega) = -\frac{1}{\pi} \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\kappa}_{\mathrm{H}}(\boldsymbol{k}, \omega')}{\omega' - \omega} d\omega' \ , \tag{1.2}$$

where the subscript AH denotes the anti-hermitian matrix. The fact that the function $\hat{\kappa}(\boldsymbol{k}, z)$ satisfies the Kramers-Kronig relations (1.2) is a direct consequence of the causal response of the plasma to an external perturbation. In the absence of the spatial dispersion ($\boldsymbol{k} = \boldsymbol{0}$) the conductivity tensor $\hat{\sigma}(z) = \lim_{\boldsymbol{k} \to \boldsymbol{0}} \hat{\sigma}(\boldsymbol{k}, z)$ is also a response function and it satisfies to the Kramers-Kronig relations

$$\hat{\sigma}_{\mathrm{H}}(\omega) = \frac{1}{\pi} \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\sigma}_{\mathrm{AH}}(\omega')}{\omega' - \omega} d\omega' \ ,$$
$$\hat{\sigma}_{\mathrm{AH}}(\omega) = -\frac{1}{\pi} \mathrm{V.P.} \int_{-\infty}^{\infty} \frac{\hat{\sigma}_{\mathrm{H}}(\omega')}{\omega' - \omega} d\omega' \ . \tag{1.3}$$

In this case for the conductivity tensor $\hat{\sigma}(z)$ the Kubo formula (5.22) in (Rylyuk & Tkachenko) is valid (Zubarev 1971).

The expressions in (5.21) (at $\boldsymbol{k} = \boldsymbol{0}$) and in (5.22) in (Rylyuk & Tkachenko) are the same in appearance, the only difference being in the meaning of the averaging operation $\langle ... \rangle$, that is, in how the Coulomb interaction is taken into account in the Hamiltonian - explicitly, or as a screening field. In the first case, the Hamiltonian contains the direct Coulomb interaction between Dirac fermions. In the second case, the Hamiltonian does

† Email address for correspondence: rvm@onu.edu.ua

not contain the Coulomb interaction, but the Coulomb interaction between the charges is taken into account by introducing a self-consistent screening field.

2. Some mathematical formulas

The Euler-Maclaurin sum formula is

$$\sum_{a\leqslant n<b} F(n) \simeq \int_a^b F(x)dx + B_1 F(x)|_a^b + \frac{B_2}{2}\left.\frac{\partial F(x)}{\partial x}\right|_a^b , \tag{2.1}$$

where B_n are the Bernoulli numbers, in particular, $B_1 = -1/2$, $B_2 = 1/6$.

The Poisson formula reads

$$\sum_{n=n_0}^{\infty} F(n) = \int_a^{\infty} F(x)dx + 2\mathrm{Re}\sum_{n=1}^{\infty}\int_a^{\infty} F(x)\exp(2\pi inx)\,dx \ , \quad n_0 - 1 < a < n_0 \tag{2.2}$$

and

$$J_{0,\pm} = -\hbar\int_x^{\infty} dn \int \left.\frac{\partial f_{\mathrm{FD}}(\epsilon)}{\partial\epsilon}\right|_{\epsilon=\epsilon_{n,\pm}} dk_z \simeq 4\pi^2\frac{n_e l^3}{v}K_1(x) \ , \tag{2.3}$$

where n_e is the electron number density, $l = \beta\hbar v$, $x = \sqrt{2}\beta\hbar\omega_{\mathrm{H}}$ and $K_1(x)$ is the Macdonald function (D10) in (Rylyuk & Tkachenko).

The Poisson integral is

$$\int_{-\infty}^{\infty}\exp\left(-i\alpha p^2\right)dp = \exp\left(-i\pi/4\right)\sqrt{\frac{\pi}{\alpha}} \ . \tag{2.4}$$

The complex integrals are

$$J_1 = \int_{-\infty}^{-\infty}\frac{\exp(i\alpha\xi)}{\exp(\xi)+1}d\xi = -\frac{i\pi}{\sinh(\alpha\pi)} \ ,$$
$$J_2 = \int_{-\infty}^{-\infty}\frac{\mathrm{e}^{\xi}}{(\mathrm{e}^{\xi}+1)^2}\mathrm{e}^{i\alpha\xi}d\xi = \frac{\pi\alpha}{\sinh(\alpha\pi)} \ . \tag{2.5}$$

The Fourier transformation is

$$f(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int d\boldsymbol{k}d\omega\mathrm{e}^{-i(\omega t-\boldsymbol{kr})}f(\boldsymbol{k},\omega) \ , \quad f(\boldsymbol{k},\omega) = \int d\boldsymbol{r}dt\mathrm{e}^{i(\omega t-\boldsymbol{kr})}f(\boldsymbol{r},t) \ . \tag{2.6}$$

The Dirichlet formula is

$$\sum_{n,m=0}^{\infty}\mathcal{F}(n,m) = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\mathcal{F}(n,m) + \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\mathcal{F}(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{\mathcal{F}(n,n+m)+\mathcal{F}(n+m,n)\} \tag{2.7}$$

and the functions $\mathcal{F}_{n,n+m,\pm}(k_z)$ and $\mathcal{F}_{n+m,n,\pm}(k_z)$ in (5.28) in (Rylyuk & Tkachenko) are

$$\mathcal{F}_{n,n+m,\pm}(k_z) = \left.\frac{\partial f_{\mathrm{FD}}(\epsilon,k_z)}{\partial\epsilon}\right|_{\epsilon=\epsilon_{n,\pm}}\frac{\nu_\pm}{\omega^2_{n+m,n,\pm}+\nu^2_\pm} < n|j_\mu|n+m >_\pm < n+m|j_\nu|n >_\pm \ ,$$
$$\mathcal{F}_{n+m,n,\pm}(k_z) = \left.\frac{\partial f_{\mathrm{FD}}(\epsilon,k_z)}{\partial\epsilon}\right|_{\epsilon=\epsilon_{n+m,\pm}}\frac{\nu_\pm}{\omega^2_{n,n+m,\pm}+\nu^2_\pm} < n+m|j_\mu|n >_\pm < n|j_\nu|n+m >_\pm \ . \tag{2.8}$$

REFERENCES

Tkachenko, Igor M., Arkhipov, Yuriy V., Askaruly, Adil 2012 The Method of Moments and its Applications in Plasma Physics, LAP LAMBERT Academic Publishing.

Rylyuk, V. M., Tkachenko, I.M. Electromagnetism of One-Component Plasmas of Massless Fermions.

Zubarev, D. N. 1971 Nonequilibrium Statistical Thermodynamics, Library of Congress Cataloging in Publication Data.