

Supplementary Information

Melting temperature changes during slip across subglacial cavities drive basal mass exchange

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This Supplementary Information includes brief further discussion of i) equilibrium phase behavior, ii) the vertical force balance, iii) variations in basal equilibrium temperature, iv) perturbed conductive temperature profiles, and v) the potential for englacial phase changes to modify thermal profiles.

Equilibrium Phase Behavior – Equations (1) and (2)

Equation (1) of the main text describes how the equilibrium temperature along an ice–liquid interface is affected by liquid pressure P and the normal component of the ice stress tensor at the interface σ_n . The somewhat involved arguments leading to this result for the general case of a non-hydrostatic ice stress state can be found in the referenced citations (e.g. Kamb, 1961; Paterson, 1973; Sekerka and Cahn, 2004) and will not be repeated here. However, it is both instructive and relatively straightforward to consider the idealized case of phase equilibrium conditions along an ice–liquid interface for which the homogeneous ice pressure is P_i . At equilibrium, the chemical potential μ (i.e. Gibbs free energy per molecule) of each phase must be the same. Considering small deviations of μ_i in the ice and μ_l in the liquid water from a reference state at temperature T_0 and pressure P_0 , the Gibbs–Duhem equation allows the change in chemical potential in each phase to be expressed using the leading-order terms in a Taylor series describing departures from those reference temperature and pressure conditions so that

$$\left. \frac{\partial \mu_l}{\partial T} \right|_{P_0} (T_{\text{eq}} - T_0) + \left. \frac{\partial \mu_l}{\partial P} \right|_{T_0} (P - P_0) \approx \left. \frac{\partial \mu_i}{\partial T} \right|_{P_0} (T_{\text{eq}} - T_0) + \left. \frac{\partial \mu_i}{\partial P_i} \right|_{T_0} (P_i - P_0) .$$

Here, the left side is the change in chemical potential in the liquid phase obtained by altering the temperature from T_0 to T_{eq} and the liquid pressure from P_0 to P , while the

right side is the change in chemical potential in the solid ice obtained by altering the temperature from T_0 to T_{eq} and the ice pressure from P_0 to P_i . The change in chemical potential with temperature at constant pressure is the specific entropy s , and the change in chemical potential with pressure at constant temperature is the specific volume, which is the inverse of density ρ . Making these substitutions and rearranging leads to

$$(s_l - s_i)(T_{\text{eq}} - T_0) \approx \frac{1}{\rho_i} (P_i - P_0) - \frac{1}{\rho_l} (P - P_0) ,$$

where the subscripts l and i refer to properties of liquid water and ice, respectively. The difference in specific entropy between liquid water and ice can be expressed as the ratio of the latent heat of fusion \mathcal{L} to the reference temperature T_0 , yielding (after further algebraic rearrangements)

$$T_{\text{eq}} - T_0 \approx \frac{T_0 (\rho_i - \rho_l)}{\rho_l \rho_i \mathcal{L}} \left[P - P_0 + \frac{\rho_l}{\rho_l - \rho_i} (P_i - P) \right] . \quad (\text{S1})$$

Following substitution of equation (2) from the main text for the Clapeyron slope C_0 and the ice normal stress σ_n for P_i , equation (S1) generalizes to equation (1) in the main text when the ice stress state at the interface cannot be approximated as hydrostatic. In non-hydrostatically stressed solids, the Gibbs free energy and its associated chemical potential are not well defined (e.g. Kamb, 1961). The careful analysis that leads to the result provided in equation (1) of the main text proceeds from the recognition that the dominant influence of the ice stress tensor on phase equilibrium conditions arises from the virtual work performed against σ_n as a result of the change in specific volume that accompanies the change in phase (Paterson, 1973, p. 366). Although the other principal components of the ice stress tensor do not have a significant influence on the equilibrium melting temperature, their departure from σ_n can have important dynamic consequences. Common glaciological examples of such non-hydrostatic stress states occur at the surface of a collapsing borehole (e.g. Nye, 1953) or an R-channel containing liquid at a pressure that is lower than the ice pressure (i.e. defined as one third the trace of the stress tensor) so that a deviatoric radial stress drives creep closure at a rate that is compensated in steady state by turbulent melting (Röthlisberger, 1972). Importantly, in such a system the force balance constraint requires that changes in liquid pressure be balanced by changes in normal stress and equation (1) correctly describes how this drives changes in the equilibrium melting temperature that follow the Clapeyron slope. Where temperature measurements coincide with liquid pressure measurements in boreholes that access the glacier bed, as recognized by Nye (1953), typically the normal stress in the ice is equal to the liquid pressure and transient pressure changes are expected to produce temperature changes that follow the Clapeyron slope (e.g. Andrews and others, 2014; Huss and others, 2007).

Vertical Force Balance – Equations (3) and (4)

Equation (3) of the main text contains a simple analysis of the vertical force balance over a representative portion of the glacier base, with horizontally projected area \mathcal{A} . Two key assumptions deserve mention. First, the area is chosen to be of sufficient size that the effects of vertical shear stresses on its boundaries (e.g. those that are associated with vertical components of ice flow) are negligible in comparison with the ice weight that must be supported over \mathcal{A} , namely

$$\int_{\mathcal{A}} \rho_i g H \, d\mathcal{A} . \quad (\text{S2})$$

Second, the entire basal surface is separated by liquid from underlying bed materials (referred to in the main text as minerals, noting that they may be comprised of both bedrock and unconsolidated sediments), and so it cannot support a shear stress. This implies that the vertical component of the suitably integrated stress in the ice normal to the basal surface can be equated with the glacier weight. With the basal ice surface at elevation $z = b(x, y)$ for some function b and horizontal Cartesian coordinates x and y , the unit outward normal to the basal ice surface can be expressed as

$$\hat{\mathbf{n}} = \frac{\left(\frac{\partial b}{\partial x} \hat{\mathbf{x}}, \frac{\partial b}{\partial y} \hat{\mathbf{y}}, -\hat{\mathbf{z}} \right)}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} ,$$

where the Cartesian unit vectors are $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. This implies that the magnitude of the vertical component (i.e. $\hat{\mathbf{z}}$) of normal stress on a basal area element is

$$\frac{\sigma_n}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} ,$$

and recognizing that the ratio between the size of a basal area element and its projection on the horizontal plane is

$$\frac{da}{d\mathcal{A}} = \sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1} ,$$

the magnitude of the vertical force arising from basal interactions can be written compactly as

$$\int_a \frac{\sigma_n}{\sqrt{\left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 + 1}} da = \int_{\mathcal{A}} \sigma_n \, d\mathcal{A} . \quad (\text{S3})$$

The liquid pressure and the ice normal stress are equal at the ice–liquid interface over macroscopic drainage elements. Away from such drainage elements, ice–mineral interactions generate the thermomolecular pressure that supports the portion of the overburden

that is not balanced by liquid pressure. A second distinct source of differences between σ_n and P arises from the effects of surface energy on curved ice–liquid interfaces that is known as the Gibbs–Thomson effect. The Gibbs–Thomson effect does not have a direct role in supporting the glacier weight, but is nevertheless important for maintaining the ice–liquid equilibrium temperature at a uniform level over mm-scale distances between roughness elements, for example in the pore throats between particles when the glacier bed is comprised of unconsolidated sediment. On an ice–liquid interface with unit normal $\hat{\mathbf{n}}$ directed outwards from the ice, the interfacial curvature is $\nabla \cdot \hat{\mathbf{n}}$ and the combined effects of the Gibbs–Thomson effect and the local thermomolecular pressure $p_T(\epsilon)$ give rise to an ice–liquid pressure difference of (e.g. Rempel, 2008, Eq. 2)

$$P_i - P \approx \sigma_n - P = \gamma_{il} \nabla \cdot \hat{\mathbf{n}} + p_T(\epsilon) ,$$

where $\gamma_{il} \approx 0.03 \text{ J m}^{-2}$ is the ice–liquid surface energy and ϵ is the local thickness of the premelted liquid films that separate the ice from the mineral substrate. Mathematically, the integral of curvature over a surface can be expressed as a line integral along its boundary, summing up components that are tangent to the surface along that boundary (e.g. Rempel and others, 2001). Hence, when integrated over an area along the basal surface that is chosen so that its tangent vector has only horizontal components on its boundaries (e.g. following the trace of the surface along the midpoints of pore throats), the net vertical force resulting from the thermomolecular pressure is exactly

$$\int_a \frac{p_T}{\sqrt{\left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 + 1}} da = \int_a \frac{\sigma_n - P}{\sqrt{\left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2 + 1}} da ;$$

the Gibbs–Thomson effect produces zero contribution to the integral on the right side. This is a case where rigorous mathematical description may prove less satisfying than simple physical reasoning. Demanding that surface–energy produce a net upwards force is akin to asking that the ice somehow be supported by a surface traction that has no means of exerting a net force against anything other than the adjacent ice on its boundaries (i.e. the glaciological equivalent of “pulling itself up by its bootstraps”).

In the main text we define $P_T = \sigma_n - P$ as the homogenized thermomolecular pressure that is the net force per unit macroscopic basal area arising from ice–mineral interactions; while P_T is suitably defined and may be regarded as uniform over macroscopic areas, over length scales of millimeters or smaller, the local thermomolecular pressure $p_T(\epsilon)$ that is responsible for generating this net pressure is itself made heterogeneous by local variations in ice–mineral separation ϵ and interfacial curvature $\nabla \cdot \hat{\mathbf{n}}$. (Further discussion of other glaciological consequences of such behavior is contained in Rempel, 2008; Meyer and others, 2018, 2019). Substituting the definition for P_T allows the integral of σ_n to be expressed as

$$\int_{\mathcal{A}} \sigma_n d\mathcal{A} = \int_{\mathcal{A}} (P + P_T) d\mathcal{A} . \tag{S4}$$

The vertical force balance is obtained by equating equation (S2) and (S4) to yield equation (3) in the main text.

To arrive at equation (4) in the main text, we define the average ice-equivalent thickness as

$$H_0 = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} H \, d\mathcal{A},$$

so that with the average liquid pressure defined as

$$\bar{P} = \frac{1}{\mathcal{A}} \int_{\mathcal{A}} P \, d\mathcal{A},$$

and the average thermomolecular pressure over the portion of the bed area that does not contain macroscopic drainage elements defined as

$$\bar{P}_T = \frac{1}{(1-\phi)\mathcal{A}} \int_{(1-\phi)\mathcal{A}} P_T \, d\mathcal{A},$$

the leftmost and rightmost sides of equation (3) can be expressed as

$$\rho_i g H_0 \mathcal{A} = \bar{P} \mathcal{A} + (1-\phi) \bar{P}_T \mathcal{A}. \quad (\text{S5})$$

For the idealized case considered, with a homogeneous liquid pressure $P = \bar{P}$ and with $P_T = \bar{P}_T$, equation (S5) simplifies to equation (4) in the main text.

Variations in Equilibrium Temperature – Equations (5)–(7)

The expressions in equations (5)–(7) of the main text are obtained by simple substitution into equation (1). It is worth emphasizing the differences in scale of changes in equilibrium temperature with changes in the controlling liquid pressure and ice normal stress. In particular, we have that at constant values of $P_T = \sigma_n - P$ (e.g. along the ice-walled boundaries of macroscopic drainage elements, with $P_T = 0$),

$$\left. \frac{\partial T_{\text{eq}}}{\partial P} \right|_{\text{constant } P_T} = -C_0.$$

This can be contrasted with the behavior that results from variations in P away from drainage elements, where P_T must also change to maintain the vertical force balance over representative projected areas \mathcal{A} that is described by equation (4) of the main text (or S5 above), so that

$$\left. \frac{\partial T_{\text{eq}}}{\partial P} \right|_{P_T=(P_0-P)/(1-\phi)} = \frac{C_0}{1-\phi} \frac{\rho_i + \phi(\rho_l - \rho_i)}{\rho_l - \rho_i}.$$

Because the density difference between the phases is about 10% of the densities of either the liquid or the solid, the equilibrium temperature over macroscopic drainage elements, where $P_T = 0$, tends to be about an order of magnitude less sensitive to liquid pressure changes than the equilibrium temperature over premelted regions.

Perturbed Temperature Profiles and Surface Gradients – Equations (8) and (13)

Equation (8) in the main text gives the perturbation to the temperature gradient along the sliding interface following a jump in equilibrium temperature of size ΔT ; this result can be found in standard reference texts, including Carslaw and Jaeger (1959). To further explore how the temperature profile responds to such a change in boundary temperature, consider the case in which the temperature profile within the ice immediately prior to the jump in temperature is steady at $\tilde{T}(z, 0) = \tilde{T}_i(z)$. We define the perturbed temperature as $T(z, t) = \tilde{T}(z, t) - \tilde{T}_i(z)$ so that when the heat transport can be considered to follow Fourier's law with a constant thermal conductivity, the evolution of T satisfies

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2},$$

where κ is the thermal diffusivity. Introducing the similarity variable $\eta = z/(2\sqrt{\kappa t})$, this can be written as

$$\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}.$$

Integrating twice and applying the boundary conditions that $T(0, t) = \Delta T$ and $T(\infty, t) = 0$ (with initial condition $T(z > 0, 0) = 0$) gives the perturbed temperature field as

$$T(z, t) = \Delta T \operatorname{erfc} \left(\frac{z}{2\sqrt{\kappa t}} \right),$$

and the perturbation to the temperature gradient as

$$\frac{\partial T}{\partial z} = -\frac{\Delta T}{\sqrt{\pi \kappa t}} \exp \left(\frac{-z^2}{4\kappa t} \right).$$

Equation (8) in the main text expresses the perturbed temperature gradient at $z = 0$.

The analysis is easily extended to consider the response of the temperature field to any number of subsequent changes in the boundary temperature at $z = 0$. For example, consider the case where, after applying a jump of ΔT at $t = 0$, the boundary temperature subsequently changes discontinuously again by $-\Delta T$ at $t = t_1$. The perturbed temperature field that we just determined satisfies the heat equation, so we can define $T_1(z, t \geq t_1) = \tilde{T}(z, t \geq t_1) - T(z, t \geq t_1)$, and look for a similarity solution in the same manner, finding that

$$T_1(z, t \geq t_1) = -\Delta T \operatorname{erfc} \left[\frac{z}{2\sqrt{\kappa(t - t_1)}} \right].$$

Hence the temperature field itself for $t \geq t_1$ is

$$\tilde{T}(z, t \geq t_1) = \tilde{T}_i + \Delta T \operatorname{erfc} \left(\frac{z}{2\sqrt{\kappa t}} \right) - \Delta T \operatorname{erfc} \left[\frac{z}{2\sqrt{\kappa(t - t_1)}} \right],$$

with gradient

$$\frac{\partial \tilde{T}}{\partial z} = -\frac{\Delta T}{\sqrt{\pi \kappa t}} \exp\left(\frac{-z^2}{4\kappa t}\right) + \frac{\Delta T}{\sqrt{\pi \kappa t}} \exp\left[\frac{-z^2}{4\kappa(t-t_1)}\right] + \frac{\partial \tilde{T}_i}{\partial z}.$$

For ice sliding at a fixed velocity u_s , the time taken to reach location x is x/u_s and the time elapsed since reaching a cavity's termination at $x = \ell$ is $t - t_1 = (x - \ell)/u_s$; making these two substitutions, the perturbation to the temperature gradient at $z = 0$ is given by equation (13) in the main text.

As noted in the main text, in our treatment we assume that the initial steady state temperature profile $\tilde{T}_i(z)$ reaches $T_{\text{premelting}}$ at the bed. This assumption is consistent with cases in which the fraction of the bed that is covered by macroscopic drainage elements ϕ is small. It is straightforward to consider alternative choices for $\tilde{T}_i(z)$. For example, as ϕ increases it may be sensible to instead set the initial basal temperature to an appropriately weighted combination of $T_{\text{premelting}}$ and T_{drainage} , with a uniform $\tilde{T}_i(z) = \phi T_{\text{drainage}} + (1 - \phi) T_{\text{premelting}} = T_{\text{premelting}} + \phi \Delta T$ as one obvious choice. Such a modification would reduce the size of the initial temperature step to $(1 - \phi)\Delta T$, with each subsequent upward and downward temperature step remaining unchanged at size ΔT . A disadvantage of this proposed alternative choice for $\tilde{T}_i(z)$ is that it does not match the boundary condition and reach $T_{\text{premelting}}$ at the bed. More generally, of course the assumption of any steady initial profile is likely to be violated in nature, with thermal conditions above the bed determined by the history of changes in basal temperature, surface temperature and internal dissipation.

Extension For High Englacial Liquid Concentrations

The Gibbs–Thomson effect, along with minor quantities of soluble impurities, enables liquid veins to persist along the triple junctions where three ice grains meet, connecting at nodes to form a liquid network that is in equilibrium with temperate ice (e.g. Nye and Frank, 1973). Estimates of the upper bound on liquid fraction at Blue Glacier by Raymond and Harrison (1975, Table III) suggest that values as high as 0.09% might have been present in fine-grained ice both near the surface and near the bed, and the question arises as to the effects of such liquid contents on the heat transport that enables basal freeze-on. To ascertain the likely importance of this complication, we first recognize that the liquid volume fraction itself n_l is expected to be a decreasing function of the temperature departure from bulk equilibrium (i.e. with $P = \sigma_n$). Including the effect of changes in n_l in the energy balance (while neglecting advective transport) leads to

$$\frac{\partial T}{\partial t} \left(1 - \frac{\mathcal{L}}{C_p} \frac{dn_l}{dT}\right) = \kappa \frac{\partial^2 T}{\partial z^2},$$

where \mathcal{L} is the latent heat and C_p is the specific heat capacity. Since englacial liquid contents are expected to become appreciable only in temperate ice, we anticipate that

dn_l/dT should produce a significant contribution to the left side of the equation over a fairly limited temperature range.

Although the precise form of the dependence of n_l on T is likely to depend on a number of highly variable factors, such as the ice crystal size distribution and typical aspect ratios, we can make progress by considering limiting cases. For example, if the liquid content owes its presence primarily to the colligative effects of minor quantities of impurities, we might expect that n_l should be well approximated over a limited range by a linear function of temperature. In this case, the results of the analysis above and in the main text remain functionally the same, but with a larger modified effective thermal diffusivity $\hat{\kappa}$ replacing its dry value as $\hat{\kappa} = \kappa/[1 - (\mathcal{L}/C)(dn_l/dT)]$. For example, this would be expected to reduce the thickness of freeze-on that accompanies transport over a single cavity from the expression given in equation (9) of the main text to

$$h_0 \approx \frac{2}{\sqrt{\pi}} \ell \frac{C_p \Delta T}{\mathcal{L}} \sqrt{\frac{\kappa}{u_s \ell \left(1 - \frac{\mathcal{L}}{C_p} \frac{dn_l}{dT}\right)}}.$$

For intuition, given a temperature change of ΔT , this modified formula produces a prediction for h_0 that is noticeably smaller than that from equation (9) of the main text if the change Δn_l is comparable to or larger than $C_p \Delta T / \mathcal{L}$, or about 0.06% when $\Delta T = 0.1$ °C. The reports from Blue Glacier suggest that such complications may be important in some portions of at least some temperate glaciers, but the prevalence of such conditions is not well constrained.

We note that in circumstances where the impurity content in the ice is sufficiently small that the liquid fraction is controlled primarily by the Gibbs–Thomson effect rather than by the temperature depression associated with colligative effects, the change in n_l with T might be expected to be nonlinear (e.g. Rempel, 2005), causing the sensitivity of diffusive heat transport to latent heat effects to decrease as the temperature cools. Moreover, at large values of n_l , the permeability of the ice to liquid transport may further affect the temperature field. As there is currently only very sparse quantitative data available to constrain the absolute levels of n_l in temperate basal ice, let alone changes in n_l with T , we leave further speculation over such potential effects to future work.

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