Helicity Decomposition of Evolution of Incompressible Turbulence. III. Direct-Interaction Approximation and Test-Field Models of Two- and Three-Dimensional Homogeneous and Free-Slip Channel Cases

Leaf Turner
Theoretical Division
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

Abstract

A structure function for the incompressible Navier-Stokes equation is defined using a Helmholtz decomposition of the velocity field. At the nub of such a decomposition is a helicity basis which obviates the cumbersome use of solenoidal projection operators. The evolution equations of the two standard closures, the direct-interaction approximation (DIA) and the test-field model (TFM), used to describe three-dimensional, homogeneous turbulence, are specified in terms of the structure function. Analogous results for the inhomogeneous, free-slip channel turbulence problem are derived using a random phase approximation. Finally, the evolution equations of the two closures for two-dimensional incompressible homogeneous turbulence are extracted readily from the three-dimensional case. The close resemblence of the spectral evolution equations for different geometries is exhibited.

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I. INTRODUCTION

In this manuscript, we shall explore the dependence of the mathematical structure of Kraichnan's direct-interaction approximation (DIA) [Kraichnan 1959; Orszag 1973] and his test-field model (TFM) (Orszag 1973; Kraichnan 1971, 1972; Leith & Kraichnan 1972) closures on the fundamental *structure functions* of incompressible, Navier-Stokes turbulence. These functions embody the complete physical and geometric information of the Navier-Stokes equation in the context of a presumably complete set of solenoidal basis vectors. This set of vectors is used to represent the fluid velocity. The vectors satisfy certain boundary conditions, thereby incorporating geometric information. We shall demonstrate that such a set can be utilized felicitously to extract the DIA and TFM evolution equations of incompressible fluid turbulence in a variety of geometries, thereby obviating the use of ungainly solenoidal projection operators.

The DIA and TFM are especially interesting models of Navier-Stokes turbulence. The DIA describes the evolution of a two-time energy spectrum and does so with a closure both on the Navier-Stokes equation and on the equation describing the evolution of an associated Green's function. It is a fully self-consistent analytical turbulence theory. Computations with it are difficult, and it fails to satisfy Galilean invariance, a consequence of which is its inability to yield the anticipated Kolmogorov $k^{-5/3}$-behavior of the homogeneous, isotropic energy spectrum on the inertial range. The TFM, being a single-time model for the evolution of the energy spectrum, leads to somewhat more tractable computations. It is similar to the EDQNM model, but is more fundamental in that the eddy-damping functions are themselves determined by the TFM's equations.

In Turner 1996a and 1996b, which we shall refer to as Papers I and II, we have already presented the analogous equations using the eddy-damped quasi-normal Markovian (EDQNM) closure (Orszag 1973). However, here our emphasis will be on presenting the unifying and simplifying aspects of using the structure function, the defining element of the physics and geometry of the dynamics, as the fundamental entity of the closure approximations. Given the structure functions, one can write down immediately the evolution equations of the DIA and TFM closures. Indeed, one gracefully can obtain two-dimensional homogeneous turbulence results from the three-dimensional case!

In Sec. II, we shall write down the helicity basis vectors of the Helmholtz decomposition that we shall be using to describe a three-dimensional turbulence that is statistically homogeneous. We shall then write down the Navier-Stokes evolution equation of the fluid in terms of the evolution equations of the spectral coefficients using this helicity decomposition. As a result, we shall be able to define the structure function. The DIA equations describing this three-dimensional, statistically homogeneous turbulence follow directly. We then simplify and reduce the number of DIA equations by restricting our attention to the case when the turbulence is also statistically isotropic and mirror-symmetric.

For this statistically isotropic and mirror-symmetric case, in Sec. III we shall derive the closely related TFM equations through the additional use of two auxiliary structure functions. These two structure functions are derived from the equation for the time-evolution of the solenoidal component of a test field due to spatial gradients of its irrotational component along the flow field and from the equation for the time-evolution of the irrotational component of the test field due to spatial gradients of its solenoidal component along the flow field, each equation ignoring a pressure term. The use of this passive convection scheme of Kraichnan has received motivational discussions by him and others (Orszag 1973; Kraichnan 1971, 1972; Leith & Kraichnan 1972).

In Sec. IV, we shall present the basis vectors for flow in a channel bounded by two infinite free-slip planes. Following the methodology of Sec. II and using the restricted random phase approximation (RPA) introduced in Paper II, we shall extract the structure function and then directly write down the DIA equations for the turbulent evolution of the energy spectrum.

In Sec. V, we shall augment the methodology of Sec. III with the RPA to obtain the equations of the TFM closure for the turbulent evolution of the energy spectrum for a channel flow.

In Secs. VI and VII, we shall draw upon the results of the previous sections to extract directly the equations of the DIA and TFM closures for the evolution of an energy spectrum of an arbitrary statistically homogeneous, two-dimensional turbulence. We then specialize these to the case of isotropic turbulence.

In Sec. VIII, we shall summarize the main points of our analysis. These include having demonstrated the close resemblence of spectral evolution equations for different geometries.

II. Three-Dimensional DIA for Homogeneous Turbulence

We have shown in Paper I that we can express the solenoidal (i.e., incompressible) velocity field, $\mathbf{u}(\mathbf{r}, t)$, in an unbounded geometry as a sum over a set of states that are specified by a wave vector, $\mathbf{k}$, and a helicity, s_i, where $s_\pm = \pm 1$. Thus,

$$\mathbf{u}(\mathbf{r},t) = \sum_{i=\pm} \int d^3k\, c_{s_i}(\mathbf{k},t) \vec{\xi}_{s_i}(\mathbf{k},\mathbf{r})\,.$$

The basis vectors of this *helicity* decomposition, $\vec{\xi}_{s_i}(\mathbf{r}, \mathbf{k})$, are simply the product of an exponential associated with a wave vector $\mathbf{k}$ with a helicity vector, $\hat{\chi}_{s_i}(\mathbf{k})$, perpendicular to that wave vector:

$$\vec{\xi}_{s_i}(\mathbf{k},\mathbf{r}) \equiv \hat{\chi}_{s_i}(\mathbf{k}) \exp(i\,\mathbf{k}\cdot\mathbf{r})\,.$$

We are using the notation ˆ to denote unit vectors. If we define the right-handed triad of orthonormal basis vectors,

$$\hat{\mathbf{e}}^{(1)}(\mathbf{k}) \equiv \frac{\hat{\mathbf{z}}\times\mathbf{k}}{|\hat{\mathbf{z}}\times\mathbf{k}|}\,,\ \hat{\mathbf{e}}^{(2)}(\mathbf{k}) \equiv \hat{\mathbf{k}}\times\hat{\mathbf{e}}^{(1)}(\mathbf{k})\,,\ \hat{\mathbf{e}}^{(3)}(\mathbf{k}) \equiv \hat{\mathbf{k}}\,; \tag{1}$$

then the helicity vectors are defined by

$$\hat{\chi}_s(\mathbf{k}) \equiv \frac{\hat{\mathbf{e}}^{(1)}(\mathbf{k}) + i\,s\,\hat{\mathbf{e}}^{(2)}(\mathbf{k})}{(2)^{\frac{1}{2}}\,i}\,.$$

Observe that

$$\hat{\chi}^*_s(\mathbf{k})\cdot\hat{\chi}_{s'}(\mathbf{k}) = \delta_{ss'}\,.$$

A subscripted delta, δ_{ij}, is being used to mean a Kronecker delta function of i and j. Using these definitions, one can verify that

$$\frac{1}{(2\pi)^3}\int d^3r\, \vec{\xi}^*_i(\mathbf{k},\mathbf{r})\cdot\vec{\xi}_j(\mathbf{k}',\mathbf{r}) = \delta_{ij}\,\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,$$

$$\vec{\xi}^*_i(\mathbf{k},\mathbf{r}) = \vec{\xi}_i(-\mathbf{k},\mathbf{r})\,;$$

so that the reality of the velocity field, $\mathbf{u}(\mathbf{r},t)$, imposes the reality condition,

$$c_i^*(\mathbf{k},t) = c_i(-\mathbf{k},t)\,, \tag{2}$$

on the spectral coefficients. The spatial integrals are here taken over all of space.

The incompressible Navier-Stokes equation yields the following evolution equation for the spectral coefficients in this helicity decomposition:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) c_i(\mathbf{k},t) = \int d^3p\, d^3q \sum_{l,m=\pm} g_{lmi}(\mathbf{p},\mathbf{q},-\mathbf{k}) c_l(\mathbf{p},t) c_m(\mathbf{q},t)\,, \tag{3}$$

where

$$g_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = \tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\,, \tag{4}$$

and, where, by the reality condition, Eq. (2):

$$g_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = g^*_{lmi}(-\mathbf{p},-\mathbf{q},-\mathbf{k}).$$

The integrations are over all of wave-vector space.

We shall be using $\delta^{(n)}(\mathbf{k})$ to refer to the n-dimensional Dirac delta function of $\mathbf{k}$. If the superscript, n, is dropped, the default will be a one-dimensional delta function. We shall refer to $g_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})$ as the structure function for the dynamics, and $\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})$ as the reduced structure function. For the case at hand; i.e., $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$, we have shown in Paper I that the reduced structure function satisfies (Waleffe 1992)

$$\begin{aligned}\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\frac{i s_i s_l s_m}{2^{\frac{3}{2}}}\left[\frac{A(k,p,q)}{k\,p\,q}\right]\times \\ \exp\left[i(s_i\phi_{\mathbf{k}} + s_l\phi_{\mathbf{p}} + s_m\phi_{\mathbf{q}})_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\right](s_m q - s_l p)(s_i\,k + s_l\,p + s_m\,q)\,.\end{aligned} \tag{5}$$

Here, $A(k,p,q)$, is the area of the triangle formed from the wave vectors and is function of only their magnitudes. The reason for the subscript on the argument of the exponential is that the angles, $\phi_{\mathbf{k}}, \phi_{\mathbf{p}}, \phi_{\mathbf{q}}$, depend upon the unit normal to the plane of the triangle defined by:

$$\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q}) \equiv \frac{\mathbf{k}\times\mathbf{p}}{|\mathbf{k}\times\mathbf{p}|} = \frac{\mathbf{p}\times\mathbf{q}}{|\mathbf{p}\times\mathbf{q}|} = \frac{\mathbf{q}\times\mathbf{k}}{|\mathbf{q}\times\mathbf{k}|}\,. \tag{6}$$

We have shown in Paper I that for each wave vector, $\mathbf{k}$, $\mathbf{p}$, or $\mathbf{q}$, the cosine and sine of the associated angle above, such as $\phi_{\mathbf{k}}$, satisfy

$$\cos(\phi_{\mathbf{k}}) = \hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q}) \cdot \hat{\mathbf{e}}^{(1)}(\mathbf{k}), \tag{7a}$$

$$\sin(\phi_{\mathbf{k}}) = \hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q}) \cdot \hat{\mathbf{e}}^{(2)}(\mathbf{k}). \tag{7b}$$

One can verify that

$$\tilde{g}_{ilm}(\mathbf{k},\mathbf{p},\mathbf{q}) + \tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) + \tilde{g}_{mil}(\mathbf{q},\mathbf{k},\mathbf{p}) = 0. \tag{8}$$

Observing that the normal vector changes sign under exchange of any two of its arguments, one also can derive the following symmetry property of the reduced structure function:

$$\tilde{g}_{ilm}(\mathbf{k},\mathbf{p},\mathbf{q}) = \tilde{g}_{lmi}(\mathbf{p},\mathbf{k},\mathbf{q}). \tag{9}$$

An additional useful property of the reduced structure function that follows from Eq. (5) is

$$\tilde{g}_{ilm}(\mathbf{k},-\mathbf{k},0) = 0. \tag{10}$$

Letting $\mathbf{w}$ refer to either $\mathbf{k}$, $\mathbf{p}$, or $\mathbf{q}$, we shall find it useful to define (Turner 1996a, Waleffe 1992)

$$\begin{aligned} \hat{\mathbf{o}}^{(1)}(\mathbf{w}) &\equiv \hat{\mathbf{n}}\,(\mathbf{k},\mathbf{p},\mathbf{q}), \\ \hat{\mathbf{o}}^{(2)}(\mathbf{w}) &\equiv \hat{\mathbf{w}} \times \hat{\mathbf{n}}\,(\mathbf{k},\mathbf{p},\mathbf{q}), \\ \hat{\mathbf{o}}^{(3)}(\mathbf{w}) &\equiv \hat{\mathbf{w}}. \end{aligned} \tag{11}$$

Then

$$\hat{\chi}_l(\mathbf{w}) = \exp\left(i\, s_l \phi_{\mathbf{w}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} \hat{\Xi}_l(\mathbf{w}), \tag{12}$$

where

$$\hat{\Xi}_l(\mathbf{w}) \equiv \frac{\hat{\sigma}^{(1)}(\mathbf{w}) + i s_l\, \hat{\sigma}^{(2)}(\mathbf{w})}{(2)^{\frac{1}{2}} i}. \tag{13}$$

The turbulent spectral tensor, $U_{ij}(\mathbf{k},t,t')$, is defined using the translational symmetry property of homogeneous turbulence, which is the crucial property that simplifies the mathematical treatment of such turbulence:

$$< c_i(\mathbf{k},t)c_j(\mathbf{k}',t') > \equiv U_{ij}(\mathbf{k},t,t')\delta^{(3)}(\mathbf{k}+\mathbf{k}')\,. \tag{14}$$

The brackets, $< >$, denote an ensemble average; i.e., the spectral tensor is defined to be an ensemble average of the quadratic products of the spectral coefficients. According to Eqs. (2) and (14), this spectral tensor satisfies the following symmetry properties:

$$U_{ij}(\mathbf{k},t,t') = U_{ji}(-\mathbf{k},t',t) = U^*_{ij}(-\mathbf{k},t,t')\,. \tag{15}$$

(In the case of channel flow, which is inhomogeneous, we shall be invoking a random phase approximation to obtain the analogous simplification.)

Fortified with these properties of the structure functions and of the spectral tensor, one can derive readily the coupled equations that govern the evolution of the spectral tensor in the direct-interaction approximation:

$$\eta_{il}(\mathbf{k},t,s) =$$

$$-4 \sum_{l',m',m,n=\pm} \int_{\mathbf{p}=\mathbf{k}-\mathbf{q}} d^3q\, G_{mn}(\mathbf{q},t,s)U_{l'm'}(\mathbf{p},t,s)\tilde{g}^*_{m'ln}(\mathbf{p},-\mathbf{k},\mathbf{q})\tilde{g}_{ml'i}(\mathbf{q},\mathbf{p},-\mathbf{k})\,, \tag{16a}$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U_{ij}(\mathbf{k},t,t') + \sum_{l=\pm}\int_0^t ds\, \eta_{il}(\mathbf{k},t,s)U_{lj}(\mathbf{k},s,t')$$

$$= 2 \sum_{l,m,l',m',n=\pm} \int_{\mathbf{p}=\mathbf{k}-\mathbf{q},} d^3q\; \tilde{g}^*_{m'l'n}(\mathbf{q},\mathbf{p},-\mathbf{k})\, \tilde{g}_{mli}(\mathbf{q},\mathbf{p},-\mathbf{k})\times \tag{16b}$$

$$\int_0^{t'} ds\, G_{jn}(-\mathbf{k},t',s)U_{ll'}(\mathbf{p},t,s)U_{mm'}(\mathbf{q},t,s)\,,$$

and

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) G_{ij}(\mathbf{k},t,t') + \sum_{l=\pm}\int_{t'}^t ds\, \eta_{il}(\mathbf{k},t,s)G_{lj}(\mathbf{k},s,t') \;=\; \delta_{ij}\,\delta(t-t')\,, \tag{16c}$$

where the Green's function, $G_{ij}(\mathbf{k}, t, t')$, satisfies the following conditions:

$$G_{ij}(\mathbf{k},t+0^+,t) = \delta_{ij},$$

$$G_{ij}(\mathbf{k},t,t') = 0\,, t < t'\,. \tag{17}$$

Equations (16) and (17) are the DIA equations that describe the evolution of the spectral tensor for a completely arbitrary incompressible, statistically homogeneous Navier-Stokes fluid turbulence; a turbulence that need not be statistically either isotropic or mirror symmetric.

It is interesting to verify that these equations reduce to the well-known DIA equations for the special case when the turbulence is both statistically isotropic and mirror symmetric. In such a case, as shown in Paper I, our tensors become scalar functions of wave-vector magnitudes:

$$\eta_{ij}(\mathbf{k},t,s) \;=\delta_{ij}\eta(k;t,s)\,,$$

$$U_{ij}(\mathbf{k},t,s) \;=\delta_{ij}U(k;t,s)\,, \tag{18}$$

$$G_{ij}(\mathbf{k},t,s) \;=\; \delta_{ij}G(k;t,s)\,.$$

For this statistically homogeneous case, therefore, we may evaluate the fluid's kinetic energy in terms of a spectral density, $U_E(k;t,s)$, that satisfies:

$$\frac{< u^2(\mathbf{r},t) >}{2} \;=\frac{1}{2}\int\; d^3k\,[2\,U(k;t,t)] = 2\pi \int\; k^2\,dk\,U_E(k;t,t)\,. \tag{19}$$

We thus shall define the energy spectrum for this isotropic, mirror-symmetric case by setting:

$$U_E(k;t,s) \equiv 2\,U(k;t,s)\,. \tag{20}$$

We shall now simplify Eqs. (16) and (17) for this isotropic, mirror-symmetric case. First, we insert Eqs. (18) into these equations to obtain for fixed vector $\mathbf{k}$:

$$\eta(k;t,s)\delta_{il} = \,-2 \int\limits_{\mathbf{p}=\mathbf{k}-\mathbf{q}} d^3q\,G(q;,t,s)U_E(p;t,s) \sum_{l',m=\pm} \tilde{g}^*_{l'lm}(\mathbf{p},-\mathbf{k},\mathbf{q})\tilde{g}_{ml'i}(\mathbf{q},\mathbf{p},-\mathbf{k})\,, \tag{21a}$$

$$\left[\left(\frac{\partial}{\partial t}+\nu k^2\right)U_E(k;t,t')+\int_0^t ds\,\eta(k;t,s)U_E(k;s,t')\right]\delta_{ij}$$

$$=\int\limits_{\mathbf{p}=\mathbf{k}-\mathbf{q},} d^3q\int_0^{t'} ds\,G(k;t',s)U_E(p;t,s)U_E(q;t,s)\times \tag{21b}$$

$$\sum_{l,m=\pm}\tilde{g}^*_{lmj}(\mathbf{p},\mathbf{q},-\mathbf{k})\,\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},-\mathbf{k}),$$

and

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)G(k;t,t')+\int_{t'}^t ds\,\eta(k;t,s)G(k;s,t')\;=\;\delta(t-t'), \tag{21c}$$

where the Green's function, $G(k;t,t')$, satisfies

$$G(k;t+0^+,t)=1,$$

$$G(k;t,t')=0\;,\;t<t'\,. \tag{22}$$

We now need only to evaluate the sum over helicities on the right-hand side of the equations for $\eta(k;t,s)$ and for the evolution of $U_E(k;t,t')$. We shall first evaluate the former sum:

$$\sum_{l',m=\pm}\tilde{g}^*_{l'lm}(\mathbf{p},-\mathbf{k},\mathbf{q})\tilde{g}_{ml'i}(\mathbf{q},\mathbf{p},-\mathbf{k})=$$

$$\sum_{l',m=\pm}\left[\frac{A^2(k,p,q)}{2\,k^2p^2q^2}\right]\exp\left\{i[(s_i-s_l)\phi_{-\mathbf{k}}]_{\hat{\mathbf{n}}(\mathbf{p},-\mathbf{k},\mathbf{q})}\right\}\times$$

$$\frac{s_is_l[(s_lk-s_{l'}p)(s_lk+s_{l'}p+s_mq)][(s_{l'}p-s_mq)(s_{l'}p+s_mq+s_ik)]}{4}= \tag{23}$$

$$s_is_l\exp\left\{i[(s_i-s_l)\phi_{-\mathbf{k}}]_{\hat{\mathbf{n}}(\mathbf{p},-\mathbf{k},\mathbf{q})}\right\}\left[\frac{A^2(k,p,q)}{2\,k^2p^2q^2}\right]\left[\left(k^2-p^2\right)\left(p^2-q^2\right)-s_is_lk^2q^2\right].$$

In Eq. (23), only the factor, $\exp\left\{i[(s_i-s_l)\phi_{-\mathbf{k}}]_{\hat{\mathbf{n}}(\mathbf{p},-\mathbf{k},\mathbf{q})}\right\}$, contains the angle, $(\phi_{-\mathbf{k}})_{\hat{\mathbf{n}}(\mathbf{p},-\mathbf{k},\mathbf{q})}$, which is an azimuthal angle of the wave vector, $\mathbf{q}$, about the fixed wave vector, $-\mathbf{k}$. (See Paper I.) The magnitude of the wave vector, $\mathbf{p}$, contains the only other dependence on an angle; namely, the cosine of the angle between $\mathbf{q}$ and $-\mathbf{k}$. As a result, one can immediately execute the integration over the azimuthal angle leading to the factor, $2\pi\delta_{il}$. Then when $i = l$, we use the change of variables:

$$\int\limits_{\mathbf{p}=\mathbf{k}-\mathbf{q}} d^3q \;\rightarrow\; 2\pi \int\limits_{\mathbf{p}+\mathbf{q}=\mathbf{k}} dpdq\, \frac{q\,p}{k}\,. \tag{24}$$

Finally, we observe that

$$\frac{A^2(k,p,q)}{k^2p^2q^2} = \frac{\sin^2(\alpha_k)}{4\,k^2}\,, \tag{25}$$

where α_k, α_p, and α_q are the angles opposite the wave vectors, $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$, respectively, in the triangle bounded by these wave vectors. Inserting these results into Eq. (21a), we arrive at

$$\eta(k;t,s) = \frac{\pi}{2\,k^3} \int\limits_{\mathbf{p}+\mathbf{q}=\mathbf{k}} dp\,dq\,q\,p\;\sin^2(\alpha_k)\left[\left(p^2-q^2\right)\left(k^2-q^2\right)+k^2p^2\right] G(p;t,s)U_E(q;t,s)\,. \tag{21a$'$}$$

We can massage the sum over helicities on the right-hand side of Eq. (21b) in a similar manner:

$$\begin{aligned}
&\sum_{l,m=\pm} \tilde{g}^*_{lmj}(\mathbf{p},\mathbf{q},-\mathbf{k})\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},-\mathbf{k}) = \\
&\sum_{l,m=\pm}\left[\frac{A^2(k,p,q)}{2\,k^2p^2q^2}\right]\exp\left\{i[(s_i-s_j)\,\phi_{-\mathbf{k}}]_{\hat{\mathbf{n}}(\mathbf{p},\mathbf{q},-\mathbf{k})}\right\}\times \\
&\frac{s_is_j\left[(s_mq-s_lp)\,(s_mq+s_lp+s_jk)\right]\left[(s_mq-s_lp)(s_mq+s_lp+s_ik)\right]}{4} = \\
&s_is_j\exp\left\{i[(s_i-s_j)\,\phi_{-\mathbf{k}}]_{\hat{\mathbf{n}}(\mathbf{p},\mathbf{q},-\mathbf{k})}\right\}\left[\frac{A^2(k,p,q)}{2\,k^2p^2q^2}\right]\left[\left(q^2-p^2\right)^2+s_is_jk^2\left(q^2+p^2\right)\right].
\end{aligned} \tag{26}$$

As in our earlier case, the only dependences on angle in the integrand of Eq. (26) are on the cosine of the angle between $\mathbf{q}$ and $\mathbf{k}$ (through the magnitude of the wave vector, $\mathbf{p}$) and on $(\phi_{-\mathbf{k}})_{\hat{\mathbf{n}}(\mathbf{p},\mathbf{q},-\mathbf{k})}$, an azimuthal angle of the wave vector, $\mathbf{q}$, about wave vector, $-\mathbf{k}$.

By our earlier reasoning, therefore, the integrand vanishes unless $i = j$, yielding:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U_E(k;t,t') + \int_0^t ds\, \eta(k;t,s) U_E(k;s,t') \; =$$

$$\frac{1}{8}\int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k}) \frac{\sin^{\mathbf{2}}(\alpha_k)}{k^2}\left[\left(q^2-p^2\right)^2 + k^2\left(q^2+p^2\right)\right] \times$$

$$\int_0^{t'} ds\, G(k;t',s) U_E(q;t,s) U_E(p;t,s)\,.$$

Using the symmetry of the integrand under exchange of $\mathbf{q}$ and $\mathbf{p}$, we may replace the factor, $\left(q^2-p^2\right)^2 + k^2\left(q^2+p^2\right)$, by $2\left[\left(k^2-q^2\right)\left(p^2-q^2\right) + k^2p^2\right]$ without altering the value of the integral on the right-hand side of this equation. We now perform integration over $\mathbf{p}$ and use the ansatz, Eq. (24), to obtain the final equation for the evolution of the energy spectrum:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U_E(k;t,t') + \int_0^t ds\, \eta(k;t,s) U_E(k;s,t') \; =$$

$$\frac{\pi}{2\,k^3}\int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq\, p\, q \sin^{\mathbf{2}}(\alpha_k)\left[\left(p^2-q^2\right)\left(k^2-q^2\right) + k^2p^2\right] \times \tag{21b'}$$

$$\int_0^{t'} ds\, G(k;t',s) U_E(q;t,s) U_E(p;t,s)\,.$$

Equations (21a',b',c) and (22) constitute the DIA system of equations describing a three-dimensional, statistically homogeneous, isotropic, mirror-symmetric, incompressible Navier-Stokes turbulence (Kraichnan 1976).

III. Three-Dimensional TFM for Homogeneous, Isotropic, Reflection-Invariant Turbulence

Unlike the DIA closure, the test-field model closure constitutes a single-time closure for the spectral density of homogeneous turbulence, $U_{ij}(\mathbf{k},t)$, where

$$< c_i(\mathbf{k},t) c_j(\mathbf{k}',t) > \equiv U_{ij}(\mathbf{k},t)\delta^{(3)}(\mathbf{k}+\mathbf{k}')\,. \tag{27}$$

The test-field model's method (Leith & Kraichnan 1972) of closing the second-order moment equation obtained from Eq. (3) in order to derive the time-evolution of $U_{ij}(\mathbf{k},t)$ requires two more structure functions that have no apparent relevance to Eq. (3). Indeed, we distill them from the test-field model's prescription of focusing on the time-evolution of a test-field's solenoidal component, $\mathbf{v}^s(\mathbf{r},t)$ due to spatial gradients of its irrotational component along the fluid velocity, $\mathbf{u}(\mathbf{r},t)$, and on the time-evolution of the test-field's irrotational component, $\mathbf{v}^c(\mathbf{r},t)$, due to spatial gradients of its solenoidal component along the fluid velocity, where the test-field's velocity $\mathbf{v}$ is the sum of $\mathbf{v}^s + \mathbf{v}^c$. This passive convection scheme ignores any pressure field.

We shall represent the solenoidal and irrotational components by:

$$\mathbf{v}^s(\mathbf{r},t) = \sum_{i=\pm} \int d^3k\, \tilde{c}_i(\mathbf{k},t)\vec{\xi}_i(\mathbf{k},\mathbf{r})\,, \tag{28a}$$

$$\mathbf{v}^c(\mathbf{r},t) = \int d^3k\, \tilde{d}(\mathbf{k},t)\nabla\Psi(\mathbf{k},\mathbf{r})\,; \tag{28b}$$

where $\tilde{c}_i(\mathbf{k},t)$ and $\tilde{d}(\mathbf{k},t)$ are the spectral coefficients. The scalar potential $\Psi(\mathbf{k},\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})/k$, so that

$$\nabla\Psi(\mathbf{k},\mathbf{r}) = i\,\frac{\mathbf{k}}{k}\exp(i\,\mathbf{k}\cdot\mathbf{r})\,.$$

One can verify easily that

$$\frac{1}{(2\pi)^3}\int d^3r\, \nabla\Psi^*(\mathbf{k},\mathbf{r})\cdot\nabla\Psi(\mathbf{k}',\mathbf{r}) = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,$$

and also, by virtue of the boundary conditions and the solenoidality of the $\vec{\xi}$'s, that for any $\mathbf{k}$ and $\mathbf{k}'$:

$$\int d^3r\, \nabla\Psi^*(\mathbf{k},\mathbf{r})\cdot\vec{\xi}_\pm(\mathbf{k}',\mathbf{r}) = \int d^3r\, \nabla\cdot\left[\Psi^*(\mathbf{k},\mathbf{r})\vec{\xi}_\pm(\mathbf{k}',\mathbf{r})\right] = 0\,.$$

We next insert Eqs. (28) into the equation for the passive convection of the test-field and ignore any pressure contribution,

$$\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \mathbf{u}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t) = \nu\,\nabla^2\mathbf{v}(\mathbf{r},t)\,. \tag{29}$$

Using the orthonormality of the solenoidal and irrotational basis vectors, we can isolate the contribution to $(\partial/\partial t + \nu k^2)\tilde{c}_i(\mathbf{k},t)$ produced by $\tilde{d}(\mathbf{k},t)$ and, conversely, the contribution to $(\partial/\partial t + \nu k^2)\tilde{d}(\mathbf{k},t)$ produced by $\tilde{c}_i(\mathbf{k},t)$. It is these contributions that determine the required additional structure functions. After some manipulation, one obtains:

$$\frac{\partial \tilde{c}_i(\mathbf{k},t)}{\partial t} + \nu k^2 \tilde{c}_i(\mathbf{k},t)$$

$$= \int d^3q\, d^3p \frac{\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})}{q} \left[\mathbf{p}\cdot\hat{\chi}_i(-\mathbf{k})\right] \sum_{l=\pm} \left[-\mathbf{k}\cdot\hat{\chi}_l(\mathbf{p})\right] c_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t) \qquad (30a)$$

$$\equiv \int d^3q\, d^3p \sum_{l=\pm} g^s_{li}(\mathbf{p},\mathbf{q},-\mathbf{k}) c_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t)\ ,$$

$$\frac{\partial \tilde{d}(\mathbf{k},t)}{\partial t} + \nu k^2 \tilde{d}(\mathbf{k},t)$$

$$= -\int d^3q\, d^3p \frac{\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})}{k} \sum_{l,m=\pm} \left[\mathbf{p}\cdot\hat{\chi}_m(\mathbf{q})\right]\left[\mathbf{q}\cdot\hat{\chi}_l(\mathbf{p})\right] c_l(\mathbf{p},t)\tilde{c}_m(\mathbf{q},t) \qquad (30b)$$

$$\equiv \int d^3q\, d^3p \sum_{l,m=\pm} g^c_{lm}(\mathbf{p},\mathbf{q},-\mathbf{k}) c_l(\mathbf{p},t)\tilde{c}_m(\mathbf{q},t)\,.$$

In strict analogy with Eq. (3), we therefore can define reduced irrotational and solenoidal structure functions, $\tilde{g}^c_{ij}(\mathbf{k},\mathbf{p},\mathbf{q})$ and $\tilde{g}^s_{ij}(\mathbf{k},\mathbf{p},\mathbf{q})$. Thus

$$\tilde{g}^s_{li}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \frac{\left[\mathbf{p}\cdot\hat{\chi}_i(\mathbf{k})\right]\left[\mathbf{k}\cdot\hat{\chi}_l(\mathbf{p})\right]}{q} = \tilde{g}^s_{il}(\mathbf{k},\mathbf{q},\mathbf{p})\,, \qquad (31a)$$

$$\tilde{g}^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv -\frac{\left[\mathbf{p}\cdot\hat{\chi}_m(\mathbf{q})\right]\left[\mathbf{q}\cdot\hat{\chi}_l(\mathbf{p})\right]}{k} = \tilde{g}^c_{ml}(\mathbf{q},\mathbf{p},\mathbf{k})\,. \qquad (31b)$$

Notice also that reality imposes the following conditions on the structure functions:

$$g^s_{li}(\mathbf{p},\mathbf{q},\mathbf{k}) = g^{s*}_{li}(-\mathbf{p},-\mathbf{q},-\mathbf{k})\,,\ g^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k}) = g^{c*}_{lm}(-\mathbf{p},-\mathbf{q},-\mathbf{k})\,.$$

These structure functions can be evaluated readily with the use of Eqs. (6) and (11) - (13)

above. For example, we find

$$\mathbf{p}\cdot\hat{\chi}_i(\mathbf{k}) = \exp\left(is_i\phi_{\mathbf{k}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\,\mathbf{p}\cdot\hat{\Xi}_i(\mathbf{k}) = \exp\left(is_i\phi_{\mathbf{k}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\mathbf{p}\cdot\left[\frac{i\,s_i\,\mathbf{k}\times\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}{k\,(2)^{\frac{1}{2}}\,i}\right]$$

$$= -s_i\,\frac{\exp\left(is_i\phi_{\mathbf{k}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}}{k}\,\frac{(\mathbf{k}\times\mathbf{p})\cdot\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}{(2)^{\frac{1}{2}}} = -(2)^{\frac{1}{2}}\,\frac{A(k,p,q)}{k}\,s_i\;\exp\left(is_i\phi_{\mathbf{k}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\,. \tag{32}$$

Replacing the spin index i with l, interchanging $\mathbf{k}$ with $\mathbf{p}$, and using the above-mentioned antisymmetry of $\hat{\mathbf{n}}$ under interchange of any two of its arguments, we find similarly that

$$\mathbf{k}\cdot\hat{\chi}_l(\mathbf{p}) = -(2)^{\frac{1}{2}}\,\frac{A(k,p,q)}{p}\,s_l\;\exp\left(is_l\phi_{\mathbf{p}}\right)_{\hat{\mathbf{n}}(\mathbf{p},\mathbf{k},\mathbf{q})} = (2)^{\frac{1}{2}}\,\frac{A(k,p,q)}{p}\,s_l\;\exp\left(is_l\phi_{\mathbf{p}}\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\,. \tag{33}$$

From Eqs. (31a), (32), and (33), we obtain the reduced solenoidal structure function:

$$\tilde{g}^s_{li}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\frac{2A^2(k,p,q)}{k\,p\,q}\,s_i s_l \exp\left[i(s_i\phi_{\mathbf{k}} + s_l\phi_{\mathbf{p}})_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\right]\,. \tag{34}$$

Analogously, we can extract immediately the reduced irrotational structure function:

$$\tilde{g}^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{2A^2(k,p,q)}{k\,p\,q}\,s_l s_m \exp\left[i(s_l\phi_{\mathbf{p}} + s_m\phi_{\mathbf{q}})_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\right]\,. \tag{35}$$

From the antisymmetry of $\hat{\mathbf{n}}$ under interchange of any two of its arguments, we observe that the irrotational and solenoidal structure functions are related by:

$$\tilde{g}^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\tilde{g}^s_{lm}(\mathbf{p},\mathbf{k},\mathbf{q})\,. \tag{36}$$

Using Eqs. (30) and (36), one may verify that

$$\int d^3k\left[\left|\tilde{d}(\mathbf{k},t)\right|^2 + \sum_{i=\pm}\left|\tilde{c}_i(\mathbf{k},t)\right|^2\right]$$

is conserved in the absence of viscosity (Kraichnan 1971, 1972; Leith & Kraichnan 1972). Equations (30) and (34) - (36) constitute the necessary additional foundation for obtaining the equations of the TFM closure for the homogeneous turbulence.

Thus far in our TFM development we have been considering an arbitrary homogeneous turbulence, we shall now specialize to the case of a turbulence that is isotropic and mirror symmetric. As in Paper I, we do so by setting

$$< c_i(\mathbf{k},t)c_j(\mathbf{k}',t) > \equiv \delta_{ij}\, \delta^{(3)}(\mathbf{k}+\mathbf{k}')U(k,t)\,,$$

$$< \tilde{c}_i(\mathbf{k},t)\tilde{c}_j(\mathbf{k}',t) > \equiv \delta_{ij}\, \delta^{(3)}(\mathbf{k}+\mathbf{k}')U^s(k,t)\,, \tag{37}$$

$$< \tilde{d}(\mathbf{k},t)\tilde{d}(\mathbf{k}',t) > \equiv \delta_{ij}\, \delta^{(3)}(\mathbf{k}+\mathbf{k}')U^c(k,t)\,.$$

To evaluate the time-evolution of $U(k,t)$ and the solenoidal and irrotational spectral densities, $U^s(k,t)$ and $U^c(k,t)$, respectively, we use Eqs. (3) and (30) to obtain:

$$\left[\frac{\partial}{\partial t} + \nu\left(k^2 + k'^2\right)\right] < c_i(\mathbf{k},t)c_j(\mathbf{k}',t) > =$$

$$\int d^3q\, d^3p \sum_{l,m=\pm} g_{lmi}(\mathbf{p},\mathbf{q},-\mathbf{k}) < c_l(\mathbf{p},t)c_m(\mathbf{q},t)c_j(\mathbf{k}',t) > +$$

$$\int d^3q\, d^3p \sum_{l,m=\pm} g_{lmj}(\mathbf{p},\mathbf{q},-\mathbf{k}') < c_l(\mathbf{p},t)c_m(\mathbf{q},t)c_i(\mathbf{k},t) > \;, \tag{38a}$$

$$\left[\frac{\partial}{\partial t} + \nu\left(k^2 + k'^2\right)\right] < \tilde{c}_i(\mathbf{k},t)\tilde{c}_j(\mathbf{k}',t) > =$$

$$\int d^3q\, d^3p \sum_{l=\pm} g^s_{li}(\mathbf{p},\mathbf{q},-\mathbf{k}) < c_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t)\tilde{c}_j(\mathbf{k}',t) > + \tag{38b}$$

$$\int d^3q\, d^3p \sum_{l=\pm} g^s_{lj}(\mathbf{p},\mathbf{q},-\mathbf{k}') < c_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t)\tilde{c}_i(\mathbf{k},t) > \;,$$

$$\left[\frac{\partial}{\partial t} + \nu\left(k^2 + k'^2\right)\right] < \tilde{d}(\mathbf{k},t)\tilde{d}(\mathbf{k}',t) > =$$

$$\int d^3q\, d^3p \sum_{l,m=\pm} g^c_{lm}(\mathbf{p},\mathbf{q},-\mathbf{k}) < c_l(\mathbf{p},t)\tilde{c}_m(\mathbf{q},t)\tilde{d}(\mathbf{k}',t) > + \tag{38c}$$

$$\int d^3q\, d^3p \sum_{l,m=\pm} g^c_{lm}(\mathbf{p},\mathbf{q},-\mathbf{k}') < c_l(\mathbf{p},t)\tilde{c}_m(\mathbf{q},t)\tilde{d}(\mathbf{k},t) > .$$

One now repeats this procedure to calculate the time-evolution of that triple correlation on the right-hand side of Eqs. (38b,c) intrinsic to the TFM:

$$\left[\frac{\partial}{\partial t} + \nu\left(p^2 + q^2 + k'^2\right)\right] < c_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t)\tilde{c}_i(\mathbf{k}',t) > =$$

$$\int d^3q'\, d^3p' \sum_{l',m'=\pm} g_{l'm'l}(\mathbf{p}',\mathbf{q}',-\mathbf{p}) < c_{l'}(\mathbf{p}',t)c_{m'}(\mathbf{q}',t)\tilde{d}(\mathbf{q},t)\tilde{c}_i(\mathbf{k}',t) > +$$

$$\int d^3q'\, d^3p' \sum_{l',m'=\pm} g^c_{l'm'}(\mathbf{p}',\mathbf{q}',-\mathbf{q}) < c_l(\mathbf{p},t)c_{l'}(\mathbf{p}',t)\tilde{c}_{m'}(\mathbf{q}',t)\tilde{c}_i(\mathbf{k}',t) > +$$

$$\int d^3q'\, d^3p' \sum_{l'=\pm} g^s_{l'i}(\mathbf{p}',\mathbf{q}',-\mathbf{k}') < c_l(\mathbf{p},t)c_{l'}(\mathbf{p}',t)\tilde{d}(\mathbf{q}',t)\tilde{d}(\mathbf{q},t) > . \tag{39}$$

After making the standard quasinormal approximation of the correlation functions on the right-hand side of this equation, neglecting possible correlations of the form $< c\tilde{c} >$ and $< c\tilde{d} >$, and recalling that Eq. (5) implies that $g_{l'm'l}(\mathbf{p}', -\mathbf{p}'; 0)$ vanishes, we find that only the last two terms yield a nonvanishing contribution to the right-hand side of this equation.

Using the definitions of Eqs. (37) for the isotropic and mirror-symmetric turbulence and following the TFM ansatz of setting the correlation on the left-hand side of Eq. (39) equal to its right-hand side multiplied by a time-dependent function, $\tilde{\theta}(k, p, q; t)$, whose evolution is determined by the TFM as will be seen below, we obtain:

$$< c_i(\mathbf{k},t)\tilde{c}_l(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > \; = \; \tilde{\theta}(k,p,q;t)\times$$

$$[g^c_{il}(-\mathbf{k},-\mathbf{p},-\mathbf{q})U^s(p,t) \; + g^s_{il}(-\mathbf{k},-\mathbf{q},-\mathbf{p})U^c(q,t)]\,U(k,t)\,. \tag{40}$$

A similar evaluation of the triple correlation function on the right-hand side of Eq. (38a), which also takes into account Eqs. (9) and (10), yields:

$$< c_i(\mathbf{k},t)c_l(\mathbf{p},t)c_m(\mathbf{q},t) >= 2\,\theta\,(k,p,q;t)\;[g_{lmi}(-\mathbf{p},-\mathbf{q},-\mathbf{k})U(p,t)U(q,t)+$$

$$g_{mil}(-\mathbf{q},-\mathbf{k},-\mathbf{p})U(q,t)U(k,t) + g_{ilm}(-\mathbf{k},-\mathbf{p},-\mathbf{q})U(k,t)U(p,t)]\,. \tag{41}$$

The function, $\theta(k,p,q;t)$, necessarily a real function that is totally symmetric under exchange of any two of its wave-vector arguments, will be determined by the TFM as seen below.

We now insert Eq. (40) into the right-hand sides of Eqs. (38b) and (38c) following the TFM prescription of keeping only the term proportional to the solenoidal spectral density in the first case and the irrotational spectral density in the second case. We use Eqs. (31) and (36) and then replace dummy variables $\mathbf{q}$ and $\mathbf{p}$ by their negatives in terms proportional to $\delta^{(3)}(\mathbf{q}+\mathbf{p}-\mathbf{k})$ to find:

$$\left(\frac{\partial}{\partial t}+2\nu\,k^2\right)U^s(k,t)\delta_{ij} = U^s(k,t)\int d^3q\,d^3p\,\tilde{\theta}(p,k,q;t)U(p,t)\times$$

$$\sum_{l=\pm}[\tilde{g}^c_{il}(-\mathbf{k},-\mathbf{p},-\mathbf{q})\tilde{g}^s_{lj}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})+$$

$$\tilde{g}^c_{jl}(\mathbf{k},-\mathbf{p},-\mathbf{q})\tilde{g}^s_{li}(\mathbf{p},\mathbf{q},-\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})] = -2\,U^s(k,t)\times$$

$$\int d^3q\,d^3p\,\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,k,q;t)U(p,t)\sum_{l=\pm}\tilde{g}^c_{il}(-\mathbf{k},-\mathbf{p},-\mathbf{q})\tilde{g}^c_{lj}(\mathbf{p},\mathbf{k},\mathbf{q}) \tag{42a}$$

$$= -2\,U^s(k,t)\int d^3q\,d^3p\,\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,k,q;t)U(p,t)\left[\tilde{\mathbf{g}}^{c\dagger}(\mathbf{p},\mathbf{k},\mathbf{q})\tilde{\mathbf{g}}^c(\mathbf{p},\mathbf{k},\mathbf{q})\right]_{ij}$$

$$= -2\,U^s(k,t)\int d^3q\,d^3p\,\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,k,q;t)U(p,t)\left[\tilde{\mathbf{g}}^{s\dagger}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{\mathbf{g}}^s(\mathbf{p},\mathbf{q},\mathbf{k})\right]_{ij}\,,$$

$$\left(\frac{\partial}{\partial t}+2\nu\, k^2\right) U^c(k,t) = \; U^c(k,t)\int d^3q\, d^3p\, \tilde\theta(p,q,k;t) U(p,t)\times$$

$$\sum_{l,m=\pm}\, [\tilde g^s_{lm}(-\mathbf{p},-\mathbf{k},-\mathbf{q})\tilde g^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})+$$

$$\tilde g^s_{lm}(-\mathbf{p},\mathbf{k},-\mathbf{q})\tilde g^c_{lm}(\mathbf{p},\mathbf{q},-\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})]$$

$$= -\; U^c(k,t)\int d^3q\, d^3p\, \tilde\theta(p,q,k;t) U(p,t)\times$$

$$\sum_{l,m=\pm}\, [\tilde g^c_{lm}(-\mathbf{p},-\mathbf{q},-\mathbf{k})\tilde g^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})+ \tag{42b}$$

$$\tilde g^c_{lm}(-\mathbf{p},-\mathbf{q},\mathbf{k})\tilde g^c_{lm}(\mathbf{p},\mathbf{q},-\mathbf{k})\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})]$$

$$= -2\, U^c(k,t)\int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde\theta(p,q,k;t) U(p,t)\times$$

$$\sum_{l,m=\pm}\, |\tilde g^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k})|^2$$

$$= -2\, U^c(k,t)\int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde\theta(p,q,k;t) U(p,t) Tr\left[\tilde{\mathbf{g}}^{c\dagger}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{\mathbf{g}}^c(\mathbf{p},\mathbf{q},\mathbf{k})\right]$$

$$= -2\, U^c(k,t)\int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde\theta(p,q,k;t) U(p,t) Tr\left[\tilde{\mathbf{g}}^{s\dagger}(\mathbf{p},\mathbf{k},\mathbf{q})\tilde{\mathbf{g}}^{\mathbf{s}}(\mathbf{p},\mathbf{k},\mathbf{q})\right]\, .$$

In our final expressions, Eqs (42), we have defined the matrices,

$$[\tilde{\mathbf{g}}^c(\mathbf{k},\mathbf{p},\mathbf{q})]_{ij} \equiv \tilde g^c_{ij}(\mathbf{k},\mathbf{p},\mathbf{q})\, ,\;\; [\tilde{\mathbf{g}}^s(\mathbf{k},\mathbf{p},\mathbf{q})]_{ij} \equiv \tilde g^s_{ij}(\mathbf{k},\mathbf{p},\mathbf{q})\, .$$

Following Leith and Kraichnan (1972), we define the turbulent eddy damping factors:

$$\mu^s_{ij}(k,t) \equiv \int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,k,q;t)U(p,t)\left[\tilde{\mathbf{g}}^{c\dagger}(\mathbf{p},\mathbf{k},\mathbf{q})\tilde{\mathbf{g}}^{c}(\mathbf{p},\mathbf{k},\mathbf{q})\right]_{ij}$$

$$= \int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,k,q;t)U(p,t)\left[\tilde{\mathbf{g}}^{s\dagger}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{\mathbf{g}}^{s}(\mathbf{p},\mathbf{q},\mathbf{k})\right]_{ij}, \tag{43a}$$

$$\mu^c(k,t) \equiv \int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,q,k;t)U(p,t)Tr\left[\tilde{\mathbf{g}}^{s\dagger}(\mathbf{p},\mathbf{k},\mathbf{q})\tilde{\mathbf{g}}^{s}(\mathbf{p},\mathbf{k},\mathbf{q})\right]$$

$$= \int d^3q\, d^3p\, \delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\tilde{\theta}(p,q,k;t)U(p,t)Tr\left[\tilde{\mathbf{g}}^{c\dagger}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{\mathbf{g}}^{c}(\mathbf{p},\mathbf{q},\mathbf{k})\right] . \tag{43b}$$

By virtue of the assumed isotropic character of the turbulence, these eddy damping factors must be scalar functions of only the magnitude of the $\mathbf{k}$-vector and of time. We shall show next that $\mu^s_{ij}(k,t)$ is indeed of the diagonal form $\mu^s(k,t)\delta_{ij}$. The evolution equations for U^s and U^c, Eqs. (42), are seen to have damping terms, $\nu k^2 + \mu^s(k,t)$ and $\nu k^2 + \mu^c(k,t)$, respectively. Unlike the B^G of Eq. (12) of Leith & Kraichnan 1972, whose nonnegative character is shrouded therein, the nonnegative nature of the coefficients of $\tilde{\theta}$ in the above integrands of $\mu^s(k,t)$ and $\mu^c(k,t)$ is manifest in the helicity decomposition.

We now shall evaluate these eddy damping factors. Using Eqs. (34) and (35), we immediately note that

$$\sum_{l=\pm} \tilde{g}^{s*}_{li}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{g}^{s}_{lj}(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{4A(k,p,q)^4}{k^2p^2q^2}\, s_i s_j \,\exp\left[i\left(s_j - s_i\right)\phi_{\mathbf{k}}\right]_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} \sum_{l=\pm} 1$$

$$= \frac{8A(k,p,q)^4}{k^2p^2q^2}\, s_i s_j \,\exp\left[i\left(s_j - s_i\right)\phi_{\mathbf{k}}\right]_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})},$$

$$Tr\left[\tilde{\mathbf{g}}^{c\dagger}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{\mathbf{g}}^{c}(\mathbf{p},\mathbf{q},\mathbf{k})\right] = \frac{4A(k,p,q)^4}{k^2p^2q^2} \sum_{l,m=\pm} 1 = \frac{16A(k,p,q)^4}{k^2p^2q^2} .$$

Inserting these results back into our expressions for the turbulent eddy damping factors and using the same reasoning that we earlier used in Sec. II, one readily finds the following forms for these damping factors:

$$\mu^s_{ij}(k,t) \;=\; \mu^s(k,t)\delta_{ij}\,,$$

$$\mu^s(k,t) = \pi \int\limits_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} dq\, dp \left(\frac{q\,p}{k}\right)^3 \sin^4(\alpha_k)\ \tilde{\theta}(p,k,q;t) U(p,t)$$

$$= \frac{\pi}{2} \int\limits_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} dq\, dp \left(\frac{q\,p}{k}\right)^3 \sin^4(\alpha_k)\ \tilde{\theta}(p,k,q;t) U_E(p,t)\,, \tag{44a}$$

$$\mu^c(k,t) = 2\pi \int\limits_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} dq\, dp \left(\frac{q\,p}{k}\right)^3 \sin^4(\alpha_k)\ \tilde{\theta}(p,q,k;t) U(p,t)$$

$$= \pi \int\limits_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} dq\, dp \left(\frac{q\,p}{k}\right)^3 \sin^4(\alpha_k)\ \tilde{\theta}(p,q,k;t) U_E(p,t)\,; \tag{44b}$$

where we are letting α_k be the angle between the p and q sides of the kpq triangle, so that $A(k,p,q) = pq\sin(\alpha_k)/2$, and where [motivated by Eq. (20)] we are setting $U_E(k,t) \equiv 2U(k,t)$.

Finally, one can use the symmetry properties satisfied by the $\tilde{g}$ structure functions, Eqs. (8) and (9), to show in a similar fashion that, after we insert Eq. (41) into the right-hand side of Eq. (38a) and use Eq. (37), we obtain the following equation that describes the evolution of the energy spectrum, $U_E(k,t)$:

$$\left(\frac{\partial}{\partial t} + 2\nu\, k^2\right) U_E(k,t)\delta_{ij} \;=\; 2\int d^3q\, d^3p\, \theta(p,q,k;t)\delta^{(3)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\times$$

$$[U_E(k,t) - U_E(q,t)]U_E(p,t)\times \tag{45}$$

$$\sum_{l,m=\pm} [\tilde{g}^*_{ilm}(\mathbf{k},\mathbf{p},\mathbf{q})\tilde{g}_{lmj}(\mathbf{p},\mathbf{q},\mathbf{k}) + \tilde{g}_{jlm}(\mathbf{k},\mathbf{p},\mathbf{q})\tilde{g}^*_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})]\ .$$

To evaluate the sum, we use Eq. (5) and observe again that the argument of Sec. II leads to nonvanishing integrals on the right-hand side from only the diagonal elements; i.e., those elements for which $i = j$. Focusing our attention upon only the diagonal elements, we observe that

$$\sum_{l,m=\pm} \tilde{g}^*_{ilm}(\mathbf{k},\mathbf{p},\mathbf{q})\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = \sum_{l,m=\pm} \frac{1}{2^3}\left[\frac{A(k,p,q)}{k\,p\,q}\right]^2 \times$$

$$[(s_l p - s_i k)(s_i k + s_l p + s_m q)][(s_m q - s_l p)(s_i k + s_l p + s_m q)] = \sum_{l,m=\pm} \frac{1}{2^3}\left[\frac{A(k,p,q)}{k\,p\,q}\right]^2 \times$$

$$\left[(p^2 - k^2) + s_m q(s_l p - s_i k)\right]\left[(q^2 - p^2) + s_i k(s_m q - s_l p)\right] = \sum_{l,m=\pm} \frac{1}{2^3}\left[\frac{A(k,p,q)}{k\,p\,q}\right]^2 \times$$

$$\left[(p^2 - k^2)\left(q^2 - p^2\right) + s_m q\, s_i k(s_l p - s_i k)(s_m q - s_l p)\right] =$$

$$\frac{1}{2}\left[\frac{A(k,p,q)}{k\,p\,q}\right]^2 \left[(p^2 - k^2)\left(q^2 - p^2\right) - k^2 q^2\right] .$$

Inserting this result back into Eq. (45) and following the procedure of Sec. II, we obtain the final form for the evolution equation governing the energy spectrum, $U_E(k,t)$:

$$\left(\frac{\partial}{\partial t} + 2\nu\, k^2\right) U_E(k,t) \;=\; \frac{\pi}{k^3} \int\limits_{\mathbf{p}+\mathbf{q}+\mathbf{k}=0} dq\, dp\; q\, p\, \sin^2(\alpha_k)\, \theta(p,q,k;t) \times \tag{46}$$

$$[U_E(q,t) - U_E(k,t)]U_E(p,t)\left[(k^2 - p^2)\left(q^2 - p^2\right) + k^2 q^2\right] .$$

The final equations of the TFM for homogeneous, isotropic, mirror-invariant turbulence that close the system of equations are:

$$\left[\frac{\partial}{\partial t} + \nu\,\left(k^2 + p^2 + q^2\right)\right]\theta(k,p,q;t) = 1 - [\mu^s(k,t) + \mu^s(p,t) + \mu^s(q,t)]\theta(k,p,q;t)\,,$$

$$\left[\frac{\partial}{\partial t} + \nu\,\left(k^2 + p^2 + q^2\right)\right]\tilde{\theta}(k,p,q;t) = 1 - [\mu^s(k,t) + \mu^s(p,t) + \mu^c(q,t)]\tilde{\theta}(k,p,q;t); \tag{47}$$

along with the initial conditions:

$$\theta(k,p,q;0) \;=\; \tilde{\theta}(k,p,q;0) \;=\; 0 \;. \tag{48}$$

Equations (44), (46)-(48) constitute the three-dimensional TFM model of homogeneous, isotropic, mirror-symmetric, Navier-Stokes turbulence (Kraichnan 1976).

IV. Three-Dimensional DIA for Free-Slip Channel Turbulence

In Paper II, we have presented the eddy-damped quasinormal Markovian closure for the Navier-Stokes equation describing a fluid velocity, $\mathbf{u}(\mathbf{r},t)$, confined within two parallel, infinite, free-slip (stress-free), planar boundaries at $y = 0$ and at $y = L_y$. We take the periodicity lengths parallel to the boundaries to be L_x and L_z in the x- and z-directions, respectively.

We represent the fluid velocity and associated vorticity by:

$$\mathbf{u}(\mathbf{r},t) = V_0\hat{\mathbf{z}} + \sum_{l,m,n=-\infty}^{\infty} c_{lmn}(t)\vec{\Delta}_{lmn}(\mathbf{r}),$$

$$\vec{\omega}(\mathbf{r},t) = \sum_{l,m,n=-\infty}^{\infty} \lambda_{lmn}\, c_{lmn}(t)\vec{\sigma}_{lmn}(\mathbf{r});$$

where $\hat{\mathbf{z}}$ is taken to be the direction of the net flux specified by V_0, and where the fluxless vector basis functions, $\vec{\Delta}_{lmn}(\mathbf{r})$, and the vector basis functions, $\vec{\sigma}_{lmn}(\mathbf{r})$, are specified by:

$$\vec{\Delta}_{lmn}(\mathbf{r}) \equiv \left[\frac{\vec{\xi}_+(\mathbf{k}_+,\mathbf{r}) - \vec{\xi}_-(\mathbf{k}_-,\mathbf{r})}{2}\right],$$

$$\vec{\sigma}_{lmn}(\mathbf{r}) \equiv \left[\frac{\vec{\xi}_+(\mathbf{k}_+,\mathbf{r}) + \vec{\xi}_-(\mathbf{k}_-,\mathbf{r})}{2}\right].$$

These sets of basis functions are related to each other by:

$$\nabla \times \vec{\Delta}_{lmn}(\mathbf{r}) = k\,\vec{\sigma}_{lmn}(\mathbf{r}),$$

$$\nabla \times \vec{\sigma}_{lmn}(\mathbf{r}) = k\,\vec{\Delta}_{lmn}(\mathbf{r}).$$

Also observe that

$$\vec{\sigma}^*_{lmn}(\mathbf{r}) = \vec{\sigma}_{-l-m-n}(\mathbf{r}) \ ,\ \vec{\Delta}^*_{lmn}(\mathbf{r}) = \vec{\Delta}_{-l-m-n}(\mathbf{r}).$$

When $k = 0$, the associated basis vectors, $\vec{\Delta}_{000}$ and $\vec{\sigma}_{000}$ are defined to be zero. The wave vectors, $\mathbf{k}_{\pm}$, functions of l, m, and n, are defined by:

$$\mathbf{k}_{+} \equiv \mathbf{k} \equiv \left(\frac{2\pi l}{L_x}, \frac{\pi m}{L_y}, \frac{2\pi n}{L_z} \right) \; ; \; \mathbf{k}_{-} \equiv \left(\frac{2\pi l}{L_x}, -\frac{\pi m}{L_y}, \frac{2\pi n}{L_z} \right) .$$

With these definitions, each set of basis vectors satisfies orthonormality,

$$\frac{2}{L_x L_y L_z} \int d^3 r \, \vec{\Delta}^*_{lmn}(\mathbf{r}) \cdot \vec{\Delta}_{l'm'n'}(\mathbf{r}) = \frac{2}{L_x L_y L_z} \int d^3 r \, \vec{\sigma}^*_{lmn}(\mathbf{r}) \cdot \vec{\sigma}_{l'm'n'}(\mathbf{r}) = \delta_{ll'} \delta_{mm'} \delta_{nn'} \;\; ,$$

and the free-slip (stressless) boundary conditions:

$$\vec{\Delta}_{lmn}(\mathbf{r}) \cdot \hat{\mathbf{y}} \mid_{y=0,L_y} = 0 \, ,$$

$$\vec{\sigma}_{lmn}(\mathbf{r}) \times \hat{\mathbf{y}} \mid_{y=0,L_y} = 0 \, .$$

Without loss of generality, we shall choose $V_0 = 0$. For further information, see Paper II.

In this section and the next, which treat the free-slip channel flow case, all 3-dimensional spatial integrals are taken over the domain: $0 \leq x \leq L_x$, $0 \leq y \leq L_y$, $0 \leq z \leq L_z$. One then obtains the following equation of spectral evolution:

$$\left[\frac{\partial}{\partial t} + \nu k^2 \right] c(\mathbf{k},t) = \sum_{\mathbf{k}_1,\mathbf{k}_2} c(\mathbf{k}_1,t) c(\mathbf{k}_2,t) g(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}) \, , \tag{49}$$

in which the structure function, $g(\mathbf{k},\mathbf{p},\mathbf{q})$, is given by:

$$g(\mathbf{k},\mathbf{p},\mathbf{q}) = \frac{\delta_{k_1+p_1+q_1,0} \, \delta_{k_3+p_3+q_3,0}}{4} \left[\frac{-i \, A(k,p,q)}{2^{\frac{1}{2}} \, k \, p \, q} \right] \times$$

$$\Big\{ \delta_{p_2+k_2+q_2,0} \, \exp\left[i \big(\phi_{\mathbf{k}_+} + \phi_{\mathbf{p}_+} + \phi_{\mathbf{q}_+} \big)_{\hat{\mathbf{n}}(\mathbf{k}_+,\mathbf{p}_+,\mathbf{q}_+)} \right] (p+k+q)(p-k)$$

$$+ \delta_{p_2+k_2-q_2,0} \, \exp\left[i \big(\phi_{\mathbf{k}_+} + \phi_{\mathbf{p}_+} - \phi_{\mathbf{q}_-} \big)_{\hat{\mathbf{n}}(\mathbf{k}_+,\mathbf{p}_+,\mathbf{q}_-)} \right] (p+k-q)(p-k)$$

$$+ \delta_{p_2-k_2-q_2,0} \, \exp\left[i \big(\phi_{\mathbf{k}_+} - \phi_{\mathbf{p}_-} + \phi_{\mathbf{q}_+} \big)_{\hat{\mathbf{n}}(\mathbf{k}_+,\mathbf{p}_-,\mathbf{q}_+)} \right] (p-k-q)(p+k)$$

$$+ \, \delta_{p_2-k_2+q_2,0} \, \exp\left[i \Big(-\phi_{\mathbf{k}_-} + \phi_{\mathbf{p}_+} + \phi_{\mathbf{q}_+} \Big)_{\hat{\mathbf{n}}(\mathbf{k}_-,\mathbf{p}_+,\mathbf{q}_+)} \right] (p-k+q)(p+k) \Big\} \; ; \tag{50}$$

and satisfies the following properties:

$$g(\mathbf{k},\mathbf{p},\mathbf{q}) \; = \; g(\mathbf{p},\mathbf{k},\mathbf{q}) \, , \tag{51a}$$

$$g(\mathbf{k},\mathbf{p},\mathbf{q}) + g(\mathbf{p},\mathbf{q},\mathbf{k}) + g(\mathbf{q},\mathbf{k},\mathbf{p}) \; = \; 0\,, \tag{51b}$$

$$g^*(\mathbf{k},\mathbf{p},\mathbf{q}) = g(-\mathbf{k},-\mathbf{p},-\mathbf{q})\,. \tag{51c}$$

The (three-dimensional) Kronecker delta symbols of the form, $\delta_{\mathbf{v}_1,\mathbf{v}_2}$, to be used here equals unity when $\mathbf{v}_1 = \mathbf{v}_2$ and otherwise equals zero. The wave-vector sums are taken over all values of $\mathbf{k}$; namely, over $-\infty \; < \; l \; < \; \infty$, $-\infty \; < \; m \; < \; \infty$, $-\infty \; < \; n \; < ;\infty$.

Having completed these preliminaries, we shall merely indicate that one again can follow the standard procedure to obtain the final DIA equations for the free-slip channel flow. There are only a few differences.

First, unlike the earlier homogeneous case, the spectral coefficients of this case depend only on wave number, not on helicity index. This makes the formal manipulations of this case easier than those of the homogeneous three-dimensional case. However, the structure function is somewhat more complicated for this channel flow case.

Second, we cannot call upon the critical simplifying feature that translational symmetry imposes on homogeneous turbulence; namely that $< c(\mathbf{k},t)c(\mathbf{k}',t) >$ vanishes when $\mathbf{k} \; + \; \mathbf{k}' \; \neq \; 0$. Instead, as a result of phase-mixing of different modes of the Navier-Stokes fluid due to the nonlinear nature of its dynamics, we invoke the restricted *random phase approximation* (RPA), discussed in Paper II, which leads to the same simplification:

$$< c(\mathbf{k},t)c(\mathbf{k}',t) > = \; \delta_{\mathbf{k}+\mathbf{k}',0} U(\mathbf{k},t,t') \;\; .$$

In analogy with Eq. (15), one readily can note that

$$U(\mathbf{k},t,t') \; = \; U(-\mathbf{k},t',t) = U^*(-\mathbf{k},t,t')\,.$$

Third, if one takes the ensemble-averaged value of Eq. (49) and uses the RPA, one obtains:

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) < c(\mathbf{k},t) > = \sum_{\mathbf{p}} U(\mathbf{p},t,t) g(\mathbf{p},-\mathbf{p},-\mathbf{k}) \; = \; 0\,. \tag{52}$$

If U satisfies $U(\mathbf{p}_+,t,t) = U(\mathbf{p}_-,t,t)$, then Eq. (52) will be satisfied. (See Paper II.) The vanishing of the sum on the right-hand side of Eq. (52) is an ingredient of the derivation of the DIA closure summarized below.

To make contact with the physical energy spectrum when $t \to t'$, we observe that:

$$\frac{1}{2\,L_x L_y L_z} \int d^3r\, u^2(\mathbf{r},t) = \frac{V_0^2}{2} + \sum_{\mathbf{k}} \frac{U(\mathbf{k},t,t)}{4},$$

and therefore define $U_E(\mathbf{k},t,t') \equiv U(\mathbf{k},t,t')/2$, which is the discretized version of what was meant by $U_E(\mathbf{k},t,t')$ in Eq. (19).

Using the above three properties, we can derive the DIA closure for the free-slip channel:

$$\eta(\mathbf{k},t,s) = -8 \sum_{\mathbf{q},\mathbf{p}} G(\mathbf{q},t,s) U_E(\mathbf{p},t,s) g(\mathbf{p},\mathbf{q},-\mathbf{k}) g^*(-\mathbf{k},\mathbf{p},\mathbf{q})\,, \tag{53a}$$

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) U_E(\mathbf{k},t,t') + \int_0^t ds\, \eta(\mathbf{k},t,s) U_E(\mathbf{k},s,t')$$
$$= 4 \sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q},\mathbf{p},-\mathbf{k})|^2 \int_0^{t'} ds\, G(-\mathbf{k},t',s)\, U_E(\mathbf{q},t,s) U_E(\mathbf{p},t,s)\,, \tag{53b}$$

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) G(\mathbf{k},t,t') + \int_{t'}^t ds\, \eta(\mathbf{k},t,s) G(\mathbf{k},s,t') = \delta(t-t')\,. \tag{53c}$$

Again the Green's function, $G(\mathbf{k},t,t')$ satisfies:

$$G(\mathbf{k},t+0^+,t) = 1,$$
$$G(\mathbf{k},t,t') = 0\ ,\ t < t'\,. \tag{54}$$

Using the structure function's properties, Eq. (51a,b), and the symmetry on $\mathbf{q}$ and $\mathbf{p}$ of the integrand of Eq. (53b), we also can express Eq. (53b) in the form:

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) U_E\,(\mathbf{k},t,t') + \int_0^t ds\, \eta\,(\mathbf{k},t,s)\, U_E\,(\mathbf{k},s,t')$$
$$= -8 \sum_{\mathbf{q},\mathbf{p}} g\,(\mathbf{q},\mathbf{p},-\mathbf{k})\, g^*\,(-\mathbf{k},\mathbf{q},\mathbf{p}) \int_0^{t'} ds\, G\,(-\mathbf{k},t',s)\; U_E\,(\mathbf{q},t,s)\, U_E\,(\mathbf{p},t,s)\,. \tag{53a'}$$

Equations (53) and (54) constitute the DIA closure for the free-slip channel flow.

In order that Eq. (52) remain satisfied, it suffices to show that if the spectral density initially satisfies the wave-vector-space symmetry, $U(\mathbf{k}_+, 0, 0) = U(\mathbf{k}_-, 0, 0)$, then the DIA equations preserve this symmetry. Since the time-evolution equation for the Green's function trivially preserves this symmetry, it is only necessary to check that the time-evolution equation for $U(\mathbf{k}, t, t')$ also preserves it. It suffices to show that

$$\sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q},\mathbf{p}, -\mathbf{k}_+)|^2 F(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q},\mathbf{p}, -\mathbf{k}_-)|^2 F(\mathbf{q},\mathbf{p})$$

where F is any function satisfying $F(\mathbf{q}_+, \mathbf{p}_+) = F(\mathbf{q}_-, \mathbf{p}_-)$. To do so, we first use the property, derived in Paper II, that

$$g(\mathbf{q}_+,\mathbf{p}_+, -\mathbf{k}_+) = -g(-\mathbf{q}_-,-\mathbf{p}_-,\mathbf{k}_-)\,.$$

Thus, making use of the reality property of the structure functions, Eq. (51c), we perform the following transformations of the sum:

$$\sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q},\mathbf{p}, -\mathbf{k}_+)|^2 F(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{q},\mathbf{p}} |g(-\mathbf{q}_-,-\mathbf{p}_-,\mathbf{k}_-)|^2 F(\mathbf{q}_-,\mathbf{p}_-) =$$

$$\sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q}_-,\mathbf{p}_-, -\mathbf{k}_-)|^2 F(\mathbf{q}_-,\mathbf{p}_-) = \sum_{\mathbf{q},\mathbf{p}} |g(\mathbf{q},\mathbf{p}, -\mathbf{k}_-)|^2 F(\mathbf{q},\mathbf{p})\,.$$

Thus, our sufficient condition is verified. As a result, the condition, Eq. (52), necessary for the validity of the DIA equations describing turbulence about at most only a trivial (i.e., time-independent, spatially constant) mean flow, is preserved by the DIA equations.

Using the assumed symmetry (which we've now shown is maintained by the DIA evolution equations) that the G's, the η's, and the U_E's are invariant when any of the wave vectors arguments are reflected about the $x - z$ plane, we find that Eqs. (53a) and

(53b) can be re-expressed as shown in Eqs. (55a) and (55b):

$$\eta(\mathbf{k};t,s) = \frac{1}{4\,k^2} \sum_{\mathbf{p},\mathbf{q} \ni p_2 q_2 \neq 0} \delta_{\mathbf{q}+\mathbf{p},\mathbf{k}} \sin^2(\alpha_k) \left[\left(p^2 - q^2\right)\left(k^2 - q^2\right) + k^2 p^2\right] G(\mathbf{p};t,s) U_E(\mathbf{q};t,s)$$

$$+ \frac{1}{k^2} \delta_{k_2,0} \sum_{\mathbf{q},\mathbf{p} \ni q_2 = p_2 = 0} \delta_{k_1,p_1+q_1} \delta_{k_3,p_3+q_3} \sin^2(\alpha_k) \left[\left(k^2 - p^2\right)\left(q^2 - p^2\right)\right] G(\mathbf{q};t,s) U_E(\mathbf{p};t,s)$$

$$+ \textit{ hybrid contributions}\,, \tag{55a}$$

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right) U_E(\mathbf{k};t,t') + \int_0^t ds\, \eta(\mathbf{k};t,s) U_E(\mathbf{k};s,t') =$$

$$\frac{1}{4\,k^2} \sum_{\mathbf{p},\mathbf{q} \ni p_2 q_2 \neq 0} \delta_{\mathbf{q}+\mathbf{p},\mathbf{k}} \sin^2(\alpha_k) \left[\left(p^2 - q^2\right)\left(k^2 - q^2\right) + k^2 p^2\right] \times$$

$$\int_0^{t'} ds\, G(\mathbf{k};t',s) U_E(\mathbf{q};t,s) U_E(\mathbf{p};t,s)$$

$$+ \frac{1}{k^2} \delta_{k_2,0} \sum_{\mathbf{q},\mathbf{p} \ni q_2 = p_2 = 0} \delta_{q_1+p_1,k_1} \delta_{q_3+p_3,k_3} \sin^2(\alpha_k) \left[\left(k^2 - p^2\right)\left(q^2 - p^2\right)\right] \times$$

$$\int_0^{t'} ds\, G(\mathbf{k};t',s) U_E(\mathbf{q};t,s) U_E(\mathbf{p};t,s) +$$

$$\textit{hybrid contributions}\,, \tag{55b}$$

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right) G(\mathbf{k};t,t') + \int_t^{t'} ds\, \eta(\mathbf{k};t,s) G(\mathbf{k};s,t') = \delta(t - t')\,.$$

where $G(\mathbf{k};t,t')$ satisfies Eqs. (54).

The three terms on the right-hand sides of Eqs. (55a) and (55b) have a particularly tantalizing nature. The first of these terms, those associated with $\mathbf{k}$, $\mathbf{p}$, and $\mathbf{q}$ wave vectors all of which have nonvanishing y-components, demonstrate a three-dimensional aspect. They are virtually identical to the right-hand sides of Eqs. (21a) and (21b), respectively, which refer to the evolution of isotropic, mirror-symmetric, homogeneous three-dimensional turbulence. To obtain this correspondence, merely employ Eq. (24) when comparing the sum over the discrete states with the integral over the continuum of states.

The second of these terms, those associated with $\mathbf{k}, \mathbf{p}$, and $\mathbf{q}$ wave vectors all of which have vanishing y-components, demonstrate a two-dimensional aspect. To observe this correspondence, we shall have to look ahead in Sec. VI. These terms are virtually identical to the right-hand sides of Eqs. (90a) and (90b), respectively, which refer to the evolution of isotropic, mirror-symmetric, homogeneous two-dimensional turbulence. To obtain this correspondence, merely employ Equation (89) when comparing the sum over the discrete states with the integral over the continuum of states and utilize Eq. (20) relating U to U_E.

The final term, labeled *hybrid contribution* merely refers to the remaining terms, terms that are associated with $\mathbf{k}, \mathbf{p}$, and $\mathbf{q}$ wave vectors only one of which has a vanishing y-component.

As in Paper II, we wish to emphasize that even when our spectrum, $U(\mathbf{k}, t)$, is isotropic in wave-vector space, the mapping from wave-vector space back to physical space will be neither homogeneous nor isotropic!

V. Three-Dimensional TFM for Free-Slip Channel Turbulence

We turn now to the TFM model for this free-slip channel. Again utilizing the random phase approximation, we observe that the TFM model provides a single-time closure for the spectrum, $U(\mathbf{k}, t)$, where

$$< c(\mathbf{k}, t) c(\mathbf{k}', t) > \equiv U(\mathbf{k}, t) \delta_{\mathbf{k}+\mathbf{k}',0} \quad . \tag{56}$$

As in Sec. III, we need the additional set of the appropriate irrotational basis vectors required for representing an arbitrary (i.e., not necessarily incompressible) velocity field confined within the two free-slip boundaries described in the preceding section. If we define the scalar potential

$$\beta_{lmn}(\mathbf{r}) \equiv \frac{\epsilon_m}{k} \cos\left(\frac{m \pi y}{L_y}\right) \exp\left[2\pi i \left(\frac{l x}{L_x} + \frac{n z}{L_z}\right)\right] ,$$

where

$$\epsilon_m \equiv \begin{cases} 1\,, m\ \neq 0\,, \\ \\ \dfrac{1}{(2)^{\frac{1}{2}}}\,, m\ =\ 0\,; \end{cases}$$

we obtain the desired orthonormal irrotational set of basis vectors,

$$\vec{\kappa}_{lmn}(\mathbf{r}) \equiv \nabla\beta_{lmn}(\mathbf{r}) = \frac{\epsilon_m}{k}\ \exp\left[2\pi i\left(\frac{l\,x}{L_x}+\frac{n\,z}{L_z}\right)\right]\times$$

$$\left[\frac{2\,\pi\,i\,l}{L_x}\cos\left(\frac{m\,\pi\,y}{L_y}\right)\hat{\mathbf{x}} - \frac{m\,\pi}{L_y}\sin\left(\frac{m\,\pi\,y}{L_y}\right)\hat{\mathbf{y}} + \frac{2\,\pi\,i\,n}{L_z}\cos\left(\frac{m\,\pi\,y}{L_y}\right)\hat{\mathbf{z}}\right]\,.$$

Observe that $\beta_{lmn}(\mathbf{r})$ satisfies the Helmholtz equation,

$$\nabla^2\beta_{lmn}(\mathbf{r}) + k^2\beta_{lmn}(\mathbf{r})\ =\ 0. \tag{57}$$

One may verify, using the periodicity of $\vec{\kappa}_{lmn}(\mathbf{r})$ in the $\hat{\mathbf{x}}$- and $\hat{\mathbf{z}}$-directions, respectively, using the boundary condition that at the planar boundaries the normal component of $\kappa_{lmn}(\mathbf{r})$ vanishes, and using Eq. (57) that for $m, m' \geq 0$

$$\frac{2}{L_x L_y L_z}\int d^3r\,\vec{\kappa}^*_{lmn}(\mathbf{r})\cdot\vec{\kappa}_{l'm'n'}(\mathbf{r}) = \delta_{ll'}\,\delta_{mm'}\,\delta_{nn'}\,,$$

$$\frac{2}{L_x L_y L_z}\int d^3r\,\vec{\kappa}^*_{lmn}(\mathbf{r})\cdot\vec{\Delta}_{l'm'n'}(\mathbf{r})\ =\ 0\,.$$

To have concise notation for facilitating upcoming evaluations of triple products, it will be convenient to define $\beta(\mathbf{k},\mathbf{r})$ as follows:

$$\beta(\mathbf{k},\mathbf{r}) \equiv \beta_{lmn}(\mathbf{r}) = \frac{\epsilon_m}{2\,k}[\exp(i\,\mathbf{k}_+\cdot\mathbf{r}) + \exp(i\,\mathbf{k}_-\cdot\mathbf{r})] = \frac{\epsilon_m}{2\,k}\sum_{s_l=\pm}\exp(i\,\mathbf{k}_{s_l}\cdot\mathbf{r}).$$

We obtain similarly

$$\vec{\kappa}(\mathbf{k},\mathbf{r}) \equiv \vec{\kappa}_{lmn}(\mathbf{r}) = \frac{\epsilon_m}{2\,k}\sum_{s_l=\pm} i\,\mathbf{k}_{s_l}\ \exp(i\,\mathbf{k}_{s_l}\cdot\mathbf{r}).$$

Notice also that $\vec{\kappa}(\mathbf{k},\mathbf{r})$ satisfies the reality condition:

$$\vec{\kappa}(\mathbf{k},\mathbf{r}) = \vec{\kappa}^*(-\mathbf{k},\mathbf{r}).$$

Using analogous notation, we shall also set

$$\vec{\Delta}(\mathbf{k},\mathbf{r}) \equiv \vec{\Delta}_{lmn}(\mathbf{r}) = \frac{1}{2}\sum_{l} s_l \vec{\chi}_{s_l}(\mathbf{k}_{s_l}) \exp(i\,\mathbf{k}_{s_l}\cdot\mathbf{r})\,.$$

For this case of a free-slip channel, we set the solenoidal $\mathbf{v}^s(\mathbf{r},t)$ and irrotational components $\mathbf{v}^c(\mathbf{r},t)$ of the test field test-field, $\mathbf{v}(\mathbf{r},t)$, equal to the following expressions:

$$\mathbf{v}^s(\mathbf{r},t) = \sum_{\mathbf{k}} \tilde{c}(\mathbf{k},t)\vec{\Delta}(\mathbf{k},\mathbf{r})\,,\; \mathbf{v}^c(\mathbf{r},t) = \sum_{\mathbf{k}\ni m_k\geq 0} \tilde{d}(\mathbf{k},t)\vec{\kappa}(\mathbf{k},\mathbf{r})\,;$$

where

$$\mathbf{v}(\mathbf{r},t) \;=\; \mathbf{v}^s(\mathbf{r},t) + \,\mathbf{v}^c(\mathbf{r},t).$$

We again implement the procedure discussed following Eq. (27), and use the following decomposition of $\mathbf{u}(\mathbf{r},t)$,

$$\mathbf{u}(\mathbf{r},t) = \sum_{\mathbf{k}} c(\mathbf{k},t)\vec{\Delta}(\mathbf{k},\mathbf{r})\,,$$

to obtain the equations analogous to Eqs. (30) and (31):

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right)\tilde{c}(\mathbf{k},t) = \sum_{\mathbf{p},\mathbf{q}\ni m_q\geq 0} g^s(\mathbf{p},\mathbf{q},-\mathbf{k})c(\mathbf{p},t)\tilde{d}\,(\mathbf{q},t)\,, \tag{58a}$$

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right)\tilde{d}(\mathbf{k},t) = \sum_{\mathbf{q},\mathbf{p}} g^c(\mathbf{p},\mathbf{q},-\mathbf{k})c(\mathbf{p},t)\tilde{c}\,(\mathbf{q},t)\,; \tag{58b}$$

where

$$g^s(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{\epsilon_{m_q}}{4q} \sum_{s_k,s_p,s_q=\pm} \delta_{\mathbf{k}_{s_k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\, s_k s_p \left[\mathbf{p}_{s_p}\cdot\hat{\chi}_{s_k}(\mathbf{k}_{s_k})\right]\left[\mathbf{k}_{s_k}\cdot\hat{\chi}_{s_p}\left(\mathbf{p}_{s_p}\right)\right]\,, \tag{59a}$$

$$g^c(\mathbf{p},\mathbf{q},\mathbf{k}) = - \frac{\epsilon_{m_k}}{4k} \sum_{s_k,s_p,s_q=\pm} \delta_{\mathbf{k}_{s_k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\, s_p s_q \left[\mathbf{q}_{s_q}\cdot\hat{\chi}_{s_p}\left(\mathbf{p}_{s_p}\right)\right]\left[\mathbf{p}_{s_p}\cdot\hat{\chi}_{s_q}\left(\mathbf{q}_{s_q}\right)\right]\,; \tag{59b}$$

and where m_q and m_k refer to the m-value associated with the y-components of the $\mathbf{q}$ and $\mathbf{k}$ wave vectors.

Again paralleling the development of Sec. III, these structure functions can be simplified still further:

$$g^s(\mathbf{p},\mathbf{q},\mathbf{k}) = -\frac{\epsilon_{m_q} A^2(k,p,q)}{2\,k\,p\,q}\times$$

$$\sum_{s_k,s_p,s_q=\pm} \delta_{\mathbf{k}_{s_k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\ \exp\left[i\Big(s_p\phi_{\mathbf{p}_{s_p}} + s_k\phi_{\mathbf{k}_{s_k}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k}_{s_k},\mathbf{p}_{s_p},\mathbf{q}_{s_q}\right)}\right],$$

$$g^c(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{\epsilon_{m_k} A^2(k,p,q)}{2\,k\,p\,q}\times$$

$$\sum_{s_k,s_p,s_q=\pm} \delta_{\mathbf{k}_{s_k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\ \exp\left[i\Big(s_p\phi_{\mathbf{p}_{s_p}} + s_q\phi_{\mathbf{q}_{s_q}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k}_{s_k},\mathbf{p}_{s_p},\mathbf{q}_{s_q}\right)}\right].$$

Then making use of the symmetry derived in Paper II:

$$\exp\left[i\Big(\phi_{\mathbf{k}_{s_k}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k}_{s_k},\mathbf{p}_{s_p},\mathbf{q}_{s_q}\right)}\right] = \exp\left[-i\Big(\phi_{\mathbf{k}_{-s_k}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k}_{-s_k},\mathbf{p}_{-s_p},\mathbf{q}_{-s_q}\right)}\right],$$

we obtain the final form of the these two structure functions:

$$g^s(\mathbf{p},\mathbf{q},\mathbf{k}) = -\frac{\epsilon_{m_q} A^2(k,p,q)}{k\,p\,q}\times$$

$$\sum_{s_p,s_q=\pm} \delta_{\mathbf{k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\ \exp\left[i\Big(s_p\phi_{\mathbf{p}_{s_p}} + \phi_{\mathbf{k}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k},\mathbf{p}_{s_p},\mathbf{q}_{s_q}\right)}\right], \tag{60a}$$

$$g^c(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{\epsilon_{m_k} A^2(k,p,q)}{k\,p\,q}\times$$

$$\sum_{s_p,s_q=\pm} \delta_{\mathbf{k}+\mathbf{p}_{s_p}+\mathbf{q}_{s_q},0}\ \exp\left[i\Big(s_p\phi_{\mathbf{p}_{s_p}} + s_q\phi_{\mathbf{q}_{s_q}}\Big)_{\hat{\mathbf{n}}\left(\mathbf{k},\mathbf{p}_{s_p},\mathbf{q}_{s_q}\right)}\right]. \tag{60b}$$

The following symmetry properties of these functions, which are the analogs of those presented in Eqs. (31) and (36), follow immediately from Eqs. (59):

$$g^s(\mathbf{p},\mathbf{q},\mathbf{k}) = g^s(\mathbf{k},\mathbf{q},\mathbf{p})\,,\ g^c(\mathbf{p},\mathbf{q},\mathbf{k}) = g^c(\mathbf{q},\mathbf{p},\mathbf{k})\,,$$

$$g^c(\mathbf{p},\mathbf{q},\mathbf{k}) = -\,g^s(\mathbf{p},\mathbf{k},\mathbf{q})\,.$$

As a result, we may define

$$\breve{g}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv g^{s}(\mathbf{p},\mathbf{q},\mathbf{k}) = -g^{c}(\mathbf{p},\mathbf{k},\mathbf{q})\,. \tag{61}$$

Finally observe that $\breve{g}(\mathbf{p},\mathbf{q},\mathbf{k})$ satisfies the reality property that

$$\breve{g}(\mathbf{p},\mathbf{q},\mathbf{k}) = \breve{g}^{*}(-\mathbf{p},-\mathbf{q},-\mathbf{k})\,. \tag{62}$$

Using Eqs. (58) and (61), one can verify that $\sum_{\mathbf{k}\ni m_k\geq 0}\left|\tilde{d}(\mathbf{k},t)\right|^2 + \sum_{\mathbf{k}}|\tilde{c}(\mathbf{k},t)|^2$ is conserved in the absence of viscosity (Kraichnan 1971, 1972; Leith & Kraichnan 1972).

As in Sec. III, we turn to the evaluation of the time-evolution of the spectral densities, $U^s(\mathbf{k},t)$, $U^c(\mathbf{k},t)$, and $U(\mathbf{k},t)$. The latter spectrum has already been defined by Eq. (56). Using the RPA, the two former spectra are defined by:

$$< \tilde{c}(\mathbf{k},t)\tilde{c}(\mathbf{k}',t) > \equiv \delta_{\mathbf{k}+\mathbf{k}',0}\,U^s(\mathbf{k},t)\,,\ < \tilde{d}(\mathbf{k},t)\tilde{d}(\mathbf{k}',t) > \equiv \delta_{\mathbf{k}+\mathbf{k}',0}\,U^c(\mathbf{k},t)\,. \tag{63}$$

One should also note that

$$U(\mathbf{k},t) = U(-\mathbf{k},t) = U^{*}(\mathbf{k},t), U^{s}(\mathbf{k},t) = U^{s}(-\mathbf{k},t) = U^{s*}(\mathbf{k},t),$$

$$U^{c}(\mathbf{k},t) = U^{c}(-\mathbf{k},t) = U^{c*}(\mathbf{k},t)\,; \tag{64}$$

To evaluate the time-evolution of $U(\mathbf{k},t)$, $U^s(\mathbf{k},t)$ and $U^c(\mathbf{k},t)$, we first observe that

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)U(\mathbf{k},t) = \sum_{\mathbf{p},\mathbf{q}} g(\mathbf{p},\mathbf{q},\mathbf{k}) < c(\mathbf{p},t)c(\mathbf{q},t)c(\mathbf{k},t) > +(\mathbf{k} \Leftrightarrow -\mathbf{k})\,, \tag{65a}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)U^s(\mathbf{k},t) = \sum_{\mathbf{p},\mathbf{q}\ni m_q\geq 0} g^{s}(\mathbf{p},\mathbf{q},\mathbf{k}) < c(\mathbf{p},t)\tilde{d}(\mathbf{q},t)\tilde{c}(\mathbf{k},t) > +(\mathbf{k} \Leftrightarrow -\mathbf{k})\,, \tag{65b}$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) U^c(\mathbf{k},t) =$$

$$\sum_{\mathbf{p},\mathbf{q}} g^c(\mathbf{p},\mathbf{q},\mathbf{k}) < c(\mathbf{p},t)\tilde{c}(\mathbf{q},t)\tilde{d}(\mathbf{k},t) > +(\mathbf{k} \Leftrightarrow -\mathbf{k})\,. \tag{65c}$$

In order to evaluate the triple correlation on the right-hand side of the last two of these equations, we first extract its time-evolution by using Eq. (49) and (58):

$$\left[\frac{\partial}{\partial t} + \nu\left(p^2 + q^2 + k^2\right)\right] < c(\mathbf{k},t)\tilde{c}(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > =$$

$$\sum_{\mathbf{p}',\mathbf{q}'} g(\mathbf{p}',\mathbf{q}',-\mathbf{k}) < c(\mathbf{p}',t)c(\mathbf{q}',t)\tilde{c}(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > +$$

$$\sum_{\mathbf{p}',\mathbf{q}'} g^c(\mathbf{p}',\mathbf{q}',-\mathbf{q}) < c(\mathbf{p}',t)\tilde{c}(\mathbf{q}',t)c(\mathbf{k},t)\tilde{c}(\mathbf{p},t) > +$$

$$\sum_{\mathbf{p}',\mathbf{q}' \ni m_{q'} \geq 0} g^s(\mathbf{p}',\mathbf{q}',-\mathbf{p}) < c(\mathbf{p}',t)\tilde{d}(\mathbf{q}',t)c(\mathbf{k},t)\tilde{d}(\mathbf{q},t) > \,.$$

In analogy with our derivation for the homogeneous TFM, we now set the triple correlation on the left-hand side of this equation equal to the right-hand side multiplied by $\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)$:

$$< c(\mathbf{k},t)\tilde{c}(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > = \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)\times$$

$$\left(\sum_{\mathbf{p}',\mathbf{q}'} g(\mathbf{p}',\mathbf{q}',-\mathbf{k}) < c(\mathbf{p}',t)c(\mathbf{q}',t)\tilde{c}(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > + \right.$$

$$\sum_{\mathbf{p}',\mathbf{q}'} g^c(\mathbf{p}',\mathbf{q}',-\mathbf{q}) < c(\mathbf{p}',t)\tilde{c}(\mathbf{q}',t)c(\mathbf{k},t)\tilde{c}(\mathbf{p},t) > +$$

$$\left.\sum_{\mathbf{p}',\mathbf{q}' \ni m_{q'} \geq 0} g^s(\mathbf{p}',\mathbf{q}',-\mathbf{p}) < c(\mathbf{p}',t)\tilde{d}(\mathbf{q}',t)c(\mathbf{k},t)\tilde{d}(\mathbf{q},t) > \right)\,. \tag{66}$$

The function, $\tilde{\theta}$ must satisfy the reality condition:

$$\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t) = \tilde{\theta}^*(-\mathbf{k},-\mathbf{p},-\mathbf{q},t)\,. \tag{67}$$

The fourth-order correlations are then reduced to bilinear products of spectral densities by invoking the usual quasinormal approximation and using the random-phase approximations embodied by Eqs. (56) and (63). In standard TFM fashion, we ignore second-order correlations of the c's with either the $\tilde{c}$'s or the $\tilde{d}$'s. Finally, in precise analogy with our discussion of the DIA for this channel flow problem, we observe we must require that

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) < c(\mathbf{k},t) > = \sum_{\mathbf{p}} U(\mathbf{p},t) g(\mathbf{p},-\mathbf{p},-\mathbf{k}) \;=\; 0\,. \tag{68}$$

As discussed in Paper II, the vanishing of this sum is assured when

$$U(\mathbf{p}_+,t) = U(\mathbf{p}_-,t)\,.$$

(After we have obtained the equations of the TFM closure, we shall demonstrate that this symmetry is indeed maintained by the equations.) We then find that Eq. (66) reduces to:

$$< c(\mathbf{k},t)\tilde{c}(\mathbf{p},t)\tilde{d}(\mathbf{q},t) > \;=\; \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)\times$$

$$[g^c(-\mathbf{k},-\mathbf{p},-\mathbf{q})U^s(\mathbf{p},t) + g^s(-\mathbf{k},-\mathbf{q},-\mathbf{p})U^c(\mathbf{q},t)]U(\mathbf{k},t)\,.$$

Starting from Eq. (49), one similarly finds that

$$< c(\mathbf{k},t)c(\mathbf{p},t)c(\mathbf{q},t) >= 2\,\theta\,(\mathbf{k},\mathbf{p},\mathbf{q},t)\ [g(-\mathbf{p},-\mathbf{q},-\mathbf{k})U(\mathbf{p},t)U(\mathbf{q},t)+$$

$$g(-\mathbf{q},-\mathbf{k},-\mathbf{p})U(\mathbf{q},t)U(\mathbf{k};t) + g(-\mathbf{k},-\mathbf{p},-\mathbf{q})U(\mathbf{k},t)U(\mathbf{p},t)]\,. \tag{69}$$

The function θ must be symmetric under the exchange of any two of its wave-vector arguments. Additionally, it must satisfy the reality condition,

$$\theta\,(\mathbf{k},\mathbf{p},\mathbf{q};t) = \theta^*\,(-\mathbf{k},-\mathbf{p},-\mathbf{q};t)\ .$$

We shall be defining θ and $\tilde{\theta}$ such that they are real functions; namely, such that they satisfy:

$$\theta\,(\mathbf{k},\mathbf{p},\mathbf{q},t) = \theta^*(\mathbf{k},\mathbf{p},\mathbf{q},t)\,,\ \tilde{\theta}\,(\mathbf{k},\mathbf{p},\mathbf{q},t) = \tilde{\theta}^*(\mathbf{k},\mathbf{p},\mathbf{q},t)\,. \tag{70}$$

Inserting Eq. (69) into Eq. (65a) and taking note of the symmetry properties of the structure function given by Eqs. (51a,b) yield our final equation for the time-evolution of the spectrum $U(\mathbf{k}, t)$:

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] U(\mathbf{k},t) \; =$$

$$4\sum_{\mathbf{p},\mathbf{q}} \{\theta(\mathbf{k},\mathbf{p},\mathbf{q},t) g(\mathbf{p},\mathbf{q},\mathbf{k}) g(-\mathbf{p},-\mathbf{k},-\mathbf{q})[U(\mathbf{k},t) - U(\mathbf{q},t)]U(\mathbf{p},t)\} + (\mathbf{k} \Leftrightarrow -\mathbf{k}) \, .$$

Repeating the argument of the preceding section, we set $U_E(\mathbf{k}, t) \equiv U(\mathbf{k}, t)/2$ and rewrite this equation as

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] U_E(\mathbf{k},t) \; =$$

$$8\sum_{\mathbf{p},\mathbf{q}} \{\theta(\mathbf{k},\mathbf{p},\mathbf{q},t) g(\mathbf{p},\mathbf{q},\mathbf{k}) g(-\mathbf{p},-\mathbf{k},-\mathbf{q})[U_E(\mathbf{k},t) - U_E(\mathbf{q},t)]U_E(\mathbf{p},t)\} + (\mathbf{k} \Leftrightarrow -\mathbf{k}) \, . \tag{71}$$

As in Sec. III, we obtain the evolution equations for the solenoidal and irrotational spectral densities, U^s and U^c, respectively, in accordance with the TFM prescription of keeping only the term proportional to the solenoidal spectral density in the first case and the irrotational spectral density in the second case. Using the reality conditions satisfied by $\tilde{\theta}$, g^s, g^c, U^s, U^c, and U [Eqs. (61), (62), (64), and (67)], using Eq. (61) that relates g^s and g^c to $\breve{g}$ and, finally, taking advantage of the fact that we shall be defining $\tilde{\theta}$ such that it satisfies Eq. (70) yield:

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) U^s(\mathbf{k},t) \; = -2\mu^s(\mathbf{k},t) U^s(\mathbf{k},t) \, ,$$

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) U^c(\mathbf{k},t) \; = -2\mu^c(\mathbf{k},t) U^c(\mathbf{k},t) \, ;$$

where, again following Leith and Kraichnan (1972), we have defined the turbulent eddy damping factors, μ^s and μ^c:

$$\mu^s(\mathbf{k},t) \equiv \sum_{\mathbf{p},\mathbf{q}\ni m_q \geq 0} \tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q},t)|\breve{g}(\mathbf{p},\mathbf{q},\mathbf{k})|^2 U(\mathbf{p},t)$$

$$= 2 \sum_{\mathbf{p},\mathbf{q}\ni m_q \geq 0} \tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q},t)|\breve{g}(\mathbf{p},\mathbf{q},\mathbf{k})|^2 U_E(\mathbf{p},t)\,, \tag{72a}$$

$$\mu^c(\mathbf{k},t) \equiv \sum_{\mathbf{p},\mathbf{q}} \tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k},t)|\breve{g}(\mathbf{p},\mathbf{k},\mathbf{q})|^2 U(\mathbf{p},t) = 2\sum_{\mathbf{p},\mathbf{q}} \tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k},t)|\breve{g}(\mathbf{p},\mathbf{k},\mathbf{q})|^2 U_E(\mathbf{p},t)\,. \tag{72b}$$

The same property that was observed in Eq. (43), the manifest nonnegative character of the coefficients of the $\tilde{\theta}$ in the above summands, is again present. Furthermore, these two turbulent eddy damping factors are clearly real functions and satisfy

$$\mu^s(\mathbf{k},t) = \mu^s(-\mathbf{k},t)\,,\; \mu^c(\mathbf{k},t) = \mu^c(-\mathbf{k},t)\,.$$

Equations (71), (72), as well as

$$\left[\frac{\partial}{\partial t} + \nu\,\left(k^2+p^2+q^2\right)\right]\theta(\mathbf{k},\mathbf{p},\mathbf{q},t) = 1 - [\mu^s(\mathbf{k},t) + \mu^s(\mathbf{p},t) + \mu^s(\mathbf{q},t)]\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)\,,$$

$$\left[\frac{\partial}{\partial t} + \nu\,\left(k^2+p^2+q^2\right)\right]\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t) = 1 - [\mu^s(\mathbf{k},t) + \mu^s(\mathbf{p},t) + \mu^c(\mathbf{q},t)]\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t); \tag{73}$$

constitute the equations of the TFM closure for the free-slip channel flow. The initial condition for the spectral evolution equation, Eq. (71), is the given initial spectrum, $U(\mathbf{k},0)$. The initial conditions for the last two equations are:

$$\theta(\mathbf{k},\mathbf{p},\mathbf{q},0) = \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},0) = 0\;\;. \tag{74}$$

Observe that these equations will yield real functions for θ and $\tilde{\theta}$.

One now may verify that Eq. (68) is consistent with the TFM equations. By the argument of Paper II, one can see that the structure function, $\breve{g}$, given by Eqs. (60) and (61), obeys the symmetry condition:

$$\breve{g}(\mathbf{k}_+,\mathbf{p}_+,\mathbf{q}_+) = \breve{g}^*(\mathbf{k}_-,\mathbf{p}_-,\mathbf{q}_-) = \breve{g}(-\mathbf{k}_-,-\mathbf{p}_-,-\mathbf{q}_-)\,. \tag{75}$$

Using Eq. (75), we see that Eq. (68) will be consistent with the equations of the TFM closure if initially,

$$U_E(\mathbf{k}_+,0) = U_E(\mathbf{k}_-,0)\,.$$

(See also the analogous argument detailed for the EDQNM model of channel flow in a slab presented in Paper II.)

As in the previous section, we make use of the assumed symmetry of the spectrum, U_E, under reflection of its wave-vector argument about the $x-z$ plane (a symmetry that is maintained by the TFM dynamical evolution equations) to rewrite Eqs. (71), (72), (73), and (74) as follows:

$$\begin{aligned}\left[\frac{\partial}{\partial t} + 2\nu k^2\right] U_E(\mathbf{k},t) = & \frac{1}{2k^2} \sum_{\mathbf{p},\mathbf{q}\ni p_2 q_2 \neq 0} \delta_{\mathbf{q}+\mathbf{p},\mathbf{k}} \sin^2(\alpha_k)\theta(\mathbf{k},\mathbf{p},\mathbf{q};t)\times \\ & \left[\left(k^2-p^2\right)\left(q^2-p^2\right)+k^2q^2\right]\left[U_E(\mathbf{q},t)-U_E(\mathbf{k},t)\right]U_E(\mathbf{p},t)+ \\ & \frac{2}{k^2}\delta_{k_2,0} \sum_{\mathbf{q},\mathbf{p}\ni q_2=p_2=0} \delta_{q_1+p_1,k_1}\delta_{q_3+p_3,k_3} \sin^2(\alpha_k)\theta(\mathbf{k},\mathbf{p},\mathbf{q};t)\times \\ & \left(k^2-p^2\right)\left(q^2-p^2\right)\left[U_E(\mathbf{q},t)-U_E(\mathbf{k},t)\right]U_E(\mathbf{p},t)+ \\ & \mathit{hybrid\,contributions}\,;\end{aligned} \tag{76}$$

$$\begin{aligned}\mu^s(\mathbf{k},t) = & \frac{1}{2} \sum_{\mathbf{p},\mathbf{q}\ni m_q>0,p_2q_2\neq 0} \delta_{\mathbf{q}+\mathbf{p},\mathbf{k}} \left(\frac{pq}{k}\right)^2 \sin^4(\alpha_k)\tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q};t)U_E(\mathbf{p},t) + \\ & \delta_{k_2,0} \sum_{\mathbf{q},\mathbf{p}\ni q_2=p_2=0} \delta_{q_1+p_1,k_1}\delta_{q_3+p_3,k_3} \left(\frac{pq}{k}\right)^2 \sin^4(\alpha_k)\tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q};t)U_E(\mathbf{p},t) + \\ & \mathit{hybrid\,contributions}\,,\end{aligned} \tag{77a}$$

$$\mu^c(\mathbf{k},t) = \frac{1}{2} \sum_{\mathbf{p},\mathbf{q} \ni p_2 q_2 \neq 0} \delta_{\mathbf{q}+\mathbf{p},\mathbf{k}} \left(\frac{p\,q}{k}\right)^2 \sin^4(\alpha_k) \tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k};t) U_E(\mathbf{p},t) +$$

$$\delta_{k_2,0} \sum_{\mathbf{q},\mathbf{p} \ni q_2 = p_2 = 0} \delta_{q_1+p_1,k_1} \delta_{q_3+p_3,k_3} \left(\frac{p\,q}{k}\right)^2 \sin^4(\alpha_k) \tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k};t) U_E(\mathbf{p},t) + \quad (77b)$$

hybrid contributions ;

$$\left[\frac{\partial}{\partial t} + \nu \left(k^2 + p^2 + q^2\right)\right] \theta(\mathbf{k},\mathbf{p},\mathbf{q};t) = 1 - [\mu^s(\mathbf{k},t) + \mu^s(\mathbf{p},t) + \mu^s(\mathbf{q},t)] \theta(\mathbf{k},\mathbf{p},\mathbf{q};t)\,,$$

$$\left[\frac{\partial}{\partial t} + \nu \left(k^2 + p^2 + q^2\right)\right] \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q};t) = 1 - [\mu^s(\mathbf{k},t) + \mu^s(\mathbf{p},t) + \mu^c(\mathbf{q},t)] \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q};t); \quad (78)$$

where we are given the initial spectrum, $U(\mathbf{k},0)$, and where we impose the initial conditions:

$$\theta(\mathbf{k},\mathbf{p},\mathbf{q};0) = \tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q};0) = 0\,. \quad (79)$$

As in Sec. IV, we again observe that the right-hand sides of Eqs. (76) and (77) decompose into three tantalizing sets of terms. The first set of these terms, those associated with $\mathbf{k}, \mathbf{p}$, and $\mathbf{q}$ wave vectors all of which have nonvanishing y-components, demonstrates a three-dimensional aspect. They are virtually identical to the right-hand sides of Eqs. (46) and (44), respectively, which refer to the evolution of isotropic, mirror-symmetric, homogeneous three-dimensional turbulence. To obtain this correspondence, merely employ Eq. (24) when comparing the sum over the discrete states with the integral over the continuum of states.

The second set of these terms, those associated with $\mathbf{k}, \mathbf{p}$, and $\mathbf{q}$ wave vectors all of which have vanishing y-components, demonstrates a two-dimensional aspect. To observe this correspondence, we shall have to look ahead in Sec. VII. These terms are virtually identical to the right-hand sides of Eqs. (100) and (101), which refer to the evolution of isotropic, mirror-symmetric, homogeneous two-dimensional turbulence. To obtain this correspondence, merely employ Eq. (89) when comparing the sum over the discrete states with the integral over the continuum of states.

The final set of terms labeled *hybrid contribution* is the collection of remaining terms, those associated with $\mathbf{k}, \mathbf{p}$, and $\mathbf{q}$ wave vectors only one of which has a vanishing y-component.

We wish to re-emphasize that even when our spectrum, $U(\mathbf{k}, t)$, is isotropic in wave-vector space, the mapping from wave-vector space back to physical space will be neither homogeneous nor isotropic!

VI. Two-Dimensional DIA for Homogeneous Turbulence

The main purpose of this section and the next one is to derive the two-dimensional DIA and TFM closures for homogeneous turbulence directly from the three-dimensional versions discussed in Secs II and III. The ease with which we perform these derivations originates from our use of a helicity decomposition. Of course, one can similarly derive two-dimensional EDQNM from the three-dimensional version of Part I. See Cambon, Mansour, & Godeferd 1997.

We first set forth the set of basis vectors appropriate for the description of statistically homogeneous two-dimensional turbulence. Here the velocity is taken to be just in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. The z-coordinate is ignorable. Thus, we must make two fundamental modifications of our treatment of the three-dimensional case. First, we must choose all of our wave-vectors to have no z-component. Second, instead of using the basis vectors, $\vec{\xi}_{\pm}(\mathbf{k}, \mathbf{r})$, we must rid ourselves of any z-component in our basis vectors. To do this, we take as our solenoidal basis vectors, the toroidal vectors:

$$\vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r}) \equiv \frac{\vec{\xi}_{+}(\mathbf{k},\mathbf{r}) + \vec{\xi}_{-}(\mathbf{k},\mathbf{r})}{(2)^{\frac{1}{2}}} = \vec{\sigma}^{*}_{(2)}(-\mathbf{k},\mathbf{r})\,. \tag{80}$$

We are subscripting these structure functions with the label (2) to distinguish these functions of homogeneous, two-dimensional turbulence from those of channel flow.

It follows from the definition, that these basis vectors satisfy orthonormality:

$$\frac{1}{(2\pi)^2} \int d^2r\, \vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r}) \cdot \vec{\sigma}^{*}_{(2)}(\mathbf{k}',\mathbf{r}) = \delta^{(2)}(\mathbf{k} - \mathbf{k}')\,. \tag{81}$$

The integral is taken over the infinite $x-y$ plane. The vectors provide a complete basis for representing the incompressible flow, $\vec{\mathbf{u}}$:

$$\mathbf{u}(\mathbf{r},t) = \int d^2k\, c(\mathbf{k},t)\vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r})\,, \tag{82}$$

where the integral is taken over the infinite $k_x - k_y$ plane. Reality of the velocity field requires that the spectral coefficients satisfy

$$c(\mathbf{k},t) = c^*(-\mathbf{k},t)\ .$$

From the assumption of statistical homogeneity of the turbulence, we observe that a spectral density, $U(\mathbf{k},t,t')$, may be defined by

$$< c(\mathbf{k},t)c^*(\mathbf{k}',t') > \equiv \delta^{(2)}(\mathbf{k}-\mathbf{k}')U(\mathbf{k},t,t')\,. \tag{83}$$

This spectral density satisfies the following symmetry properties:

$$U(\mathbf{k},t,t') \;=\; U^*(-\mathbf{k},t,t') = U(-\mathbf{k},t',t)\ .$$

Using Eqs. (80), (82), and (83), the fluid's kinetic energy density can be evaluated in terms of this spectral density:

$$\frac{< u^2(\mathbf{r}) >}{2} \;=\; \frac{1}{2}\int d^2k\, U(\mathbf{k},t,t)\,.$$

The vorticity corresponding to the flow velocity, Eq. (82), is given by:

$$\vec{\omega}(\mathbf{r},t) = \int d^2k\;\, k\, c(\mathbf{k},t)\vec{\Delta}_{(2)}(\mathbf{k},\mathbf{r}).$$

The $\{\vec{\Delta}_{(2)}(\mathbf{k},\mathbf{r})\}$, which provide a complete (solenoidal) set of orthonormal vectors (poloidal, in nature) for representing the vorticity under these free-slip boundary conditions, are defined by:

$$\vec{\Delta}_{(2)}(\mathbf{k},\mathbf{r}) \equiv \frac{\vec{\xi}_+(\mathbf{k},\mathbf{r}) - \vec{\xi}_-(\mathbf{k},\mathbf{r})}{(2)^{\frac{1}{2}}}\,.$$

These vectors satisfy

$$\frac{1}{(2\pi)^2}\int d^2r\, \vec{\Delta}_{(2)}(\mathbf{k},\mathbf{r})\cdot\vec{\Delta}^*_{(2)}(\mathbf{k}',\mathbf{r}) = \delta^{(2)}(\mathbf{k}-\mathbf{k}')$$

and

$$\nabla \times \vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r}) = k\,\vec{\Delta_{(2)}}(\mathbf{k},\mathbf{r})\ ,\ \nabla \times \vec{\Delta_{(2)}}(\mathbf{k},\mathbf{r}) = k\,\vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r})\,.$$

Observe also that $\vec{\Delta}_{(2)}$ is a vector perpendicular to the $x-y$ plane.

With these tools, we write down our evolution equation, Eq. (3), now in two-dimensions:

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right) c_i(\mathbf{k},t) = \sum_{l,m=\pm} \int d^2q\,d^2p\,g_{lmi}(\mathbf{p},\mathbf{q},-\mathbf{k})c_l(\mathbf{p},t)c_m(\mathbf{q},t)\,.$$

Then, using Eqs. (80) and (82), we observe that

$$c_\pm(\mathbf{k},t) = \frac{c(\mathbf{k},t)}{(2)^{\frac{1}{2}}}\,.$$

Inserting this result into the evolution equation, we find that

$$\left(\frac{\partial}{\partial t} + \nu\,k^2\right) c(\mathbf{k},t) = \int d^2q\,d^2p\,g_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k})c(\mathbf{p},t)c(\mathbf{q},t)\,, \tag{84}$$

where the structure function, $g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})$, is given by

$$g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{1}{(2)^{\frac{1}{2}}} \sum_{l,m=\pm} g_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})$$

with

$$g_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = \delta^{(2)}(\mathbf{k}+\mathbf{p}+\mathbf{q})\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k})\,.$$

Recall

$$\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\frac{is_is_ls_m}{2^{\frac{3}{2}}}\left[\frac{A(k,p,q)}{k\,p\,q}\right] \times$$

$$\exp\left[i(s_i\phi_{\mathbf{k}} + s_l\phi_{\mathbf{p}} + s_m\phi_{\mathbf{q}})_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})}\right](s_mq - s_lp)(s_i\,k + s_l\,p + s_m\,q)\,.$$

For this two-dimensional case in which the wave vectors are orthogonal to the z-direction, Eqs. (1) and (7a) imply that

$$\cos\left(\phi_k\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = \cos\left(\phi_p\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = \cos\left(\phi_p\right)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = 0\,,$$

since the $\hat{\mathbf{e}}^{(1)}$ vector is in the plane of the wave vectors. Observing that $\hat{\mathbf{e}}^{(2)} = \hat{\mathbf{z}}$ for any of the three wave vectors, Eq. (1) and (7b) yield

$$\sin(\phi_k)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = \sin(\phi_p)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = \sin(\phi_q)_{\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})} = \pm 1 \equiv s_H(\mathbf{k},\mathbf{p},\mathbf{q}),$$

in which the upper sign is taken when $\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})$ points along the $+\hat{\mathbf{z}}$-direction, and the lower sign when $\hat{\mathbf{n}}(\mathbf{k},\mathbf{p},\mathbf{q})$ points along the $-\hat{\mathbf{z}}$-direction. This serves to define the new variable, $s_H(\mathbf{k},\mathbf{p},\mathbf{q})$. We remark that s_H changes sign under interchange of any two of its wave-vector arguments and also that $s_H(\mathbf{k},\mathbf{p},\mathbf{q}) = s_H(-\mathbf{k},-\mathbf{p},-\mathbf{q})$.

Thus the expression for $\tilde{g}_{lmi}(\mathbf{k},\mathbf{p},\mathbf{q})$ simplifies to:

$$\begin{aligned}\tilde{g}_{lmi}(\mathbf{p},\mathbf{q},\mathbf{k}) &= -\frac{s_H(\mathbf{k},\mathbf{p},\mathbf{q})}{2^{\frac{3}{2}}}\left[\frac{A(k,p,q)}{k\,p\,q}\right](s_m q - s_l p)(s_i\,k + s_l\,p + s_m\,q) \\ &= -\frac{s_H(\mathbf{k},\mathbf{p},\mathbf{q})}{2^{\frac{3}{2}}}\left[\frac{A(k,p,q)}{k\,p\,q}\right]\left[q^2 - p^2 + s_i k(s_m q - s_l p)\right] .\end{aligned}$$

Summing over the l and m helicity indices, we obtain $g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})$:

$$g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = -s_H(\mathbf{k},\mathbf{p},\mathbf{q})\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\left[\frac{A(k,p,q)}{k\,p\,q}\right]\left(q^2 - p^2\right) . \tag{85}$$

Note that in two-dimensions $g_{(2)}$ is a real function having the following properties:

$$g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = g_{(2)}(-\mathbf{p},-\mathbf{q},-\mathbf{k})\,,\, g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = g_{(2)}(\mathbf{q},\mathbf{p},\mathbf{k}),$$

$$g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) + g_{(2)}(\mathbf{q},\mathbf{k},\mathbf{p}) + g_{(2)}(\mathbf{k},\mathbf{p},\mathbf{q}) = 0\,.$$

Again we define the reduced structure function, $\tilde{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})$ by

$$g_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \tilde{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k}). \tag{86}$$

Equation (49) of Sec. IV is an evolution equation of the same form as our current evolution equation. Thus, we can take the DIA equations of that section, with some minor obvious adaptations for the two-dimensional integrations over a continuum of states.

Remembering that in Sec. IV, $U_E(\mathbf{k},t,t') = U(\mathbf{k},t,t')/2 =< c(\mathbf{k},t)c(\mathbf{k}',t') > /2$, we immediately can write down the DIA equations for an arbitrary two-dimensional, statistically homogeneous turbulence:

$$\eta(\mathbf{k},t,s) = -4 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\ G(\mathbf{q},t,s)U(\mathbf{p},t,s)\tilde{g}_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k})\tilde{g}_{(2)}(-\mathbf{p},\mathbf{k},-\mathbf{q})$$

$$= 4 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\ \frac{A^2(k,p,q)}{k^2\,p^2\,q^2}\left(q^2-p^2\right)\left(k^2-p^2\right)G(\mathbf{q},t,s)U(\mathbf{p},t,s), \tag{87a}$$

$$\left(\frac{\partial}{\partial t}+\nu\,k^2\right)U(\mathbf{k},t,t') + \int_0^t ds\,\eta(\mathbf{k},t,s)U(\mathbf{k},s,t')$$

$$= -4 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\,\tilde{g}_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k})\tilde{g}_{(2)}(-\mathbf{p},\mathbf{k},-\mathbf{q})\int_0^{t'} ds\,G(-\mathbf{k},t',s)\,U(\mathbf{q},t,s)U(\mathbf{p},t,s)$$

$$= 2 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\left|\,\tilde{g}_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k})\right|^2\int_0^{t'} ds\,G(-\mathbf{k},t',s)\,U(\mathbf{q},t,s)U(\mathbf{p},t,s) \tag{87b}$$

$$= 4 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\,\frac{A^2(k,p,q)}{k^2\,p^2\,q^2}\left(q^2-p^2\right)\left(k^2-p^2\right)\int_0^{t'} ds\,G(-\mathbf{k},t',s)\,U(\mathbf{q},t,s)U(\mathbf{p},t,s)$$

$$= 2 \int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p\frac{A^2(k,p,q)}{k^2\,p^2\,q^2}\left(q^2-p^2\right)^2\int_0^{t'} ds\,G(-\mathbf{k},t',s)\,U(\mathbf{q},t,s)U(\mathbf{p},t,s)\,,$$

$$\left(\frac{\partial}{\partial t}+\nu\,k^2\right)G(\mathbf{k},t,t') + \int_{t'}^t ds\,\eta(\mathbf{k},t,s)G(\mathbf{k},s,t') = \delta(t-t')\,. \tag{87c}$$

Again the Green's function, $G(\mathbf{k},t,t')$ satisfies:

$$G(\mathbf{k},t+0^+,t) = 1,$$

$$G(\mathbf{k},t,t') = 0\ ,\ t < t'\,. \tag{88}$$

Observing that

$$\int\limits_{\mathbf{q}=\mathbf{k}-\mathbf{p}} d^2p \to \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} \frac{dp\,dq}{\sin(\alpha_k)}\,, \tag{89}$$

we can re-express the DIA equations for the case of two-dimensional *isotropic* turbulence as:

$$\eta(k;t,s) = \frac{1}{k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq \;\; \sin(\alpha_k)\, \left(k^2 - p^2\right) \left(q^2 - p^2\right) G(q;t,s) U(p;t,s), \tag{90a}$$

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) U(k;t,t') + \int_0^t ds\, \eta(k;t,s) U(k;s,t')$$

$$= \frac{1}{2\, k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq \;\; \sin(\alpha_k) \left(q^2 - p^2\right)^2 \int_0^{t'} ds\, G(-k;t',s)\, U(q;t,s) U(p;t,s) \tag{90b}$$

$$= \frac{1}{k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq \;\; \sin(\alpha_k)\, \left(k^2 - p^2\right) \left(q^2 - p^2\right) \int_0^{t'} ds\, G(-k;t',s)\, U(q;t,s) U(p;t,s),$$

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) G(k;t,t') + \int_{t'}^t ds\, \eta(k;t,s) G(k;s,t') = \delta(t - t')\,; \tag{90c}$$

where the Green's function, $G(k;t,t')$ satisfies:

$$G(k;t + 0^+,t) = 1,$$
$$G(k;t,t') = 0 \;,\; t < t'\,. \tag{91}$$

Equations (90) and (91) are the DIA equations describing a statistically homogeneous, isotropic, two-dimensional Navier-Stokes turbulence (Kraichnan 1976).

VII. Two-Dimensional TFM for Homogeneous Turbulence

The derivation of the TFM equations for two-dimensional, statistically homogeneous turbulence parallels the DIA derivation very closely. We again observe that all wave vectors have no z-component. Furthermore, we again use the set of basis vectors, $\{\vec{\sigma}_{(2)}(\mathbf{k}, r)\}$ [see Eq. (80)], to form the decomposition for the fluid velocity:

$$\mathbf{u}(\mathbf{r},t) = \int d^2k\, c(\mathbf{k},t) \vec{\sigma}_{(2)}(\mathbf{k},\mathbf{r})\,.$$

As we observed in the last section, we can think of the spectral coefficients, $\{c(\mathbf{k},t)\}$, as resulting from the requirement that the coefficients of the helical states be chosen as

$$c_i(\mathbf{k},t) = \frac{c(\mathbf{k},t)}{(2)^{\frac{1}{2}}} \; .$$

Similarly, we obtain the solenoidal component of the test field by setting

$$\tilde{c}_i(\mathbf{k},t) = \frac{\tilde{c}(\mathbf{k},t)}{(2)^{\frac{1}{2}}}$$

in Eq. (28a). This yields

$$\mathbf{v}^s(\mathbf{r},t) = \int d^2k \, \tilde{c}(\mathbf{k},t) \vec{\sigma_{(2)}}(\mathbf{k},\mathbf{r}) \, . \tag{92}$$

We expand the irrotational component of the test field in terms of the basis vectors formed from the gradient of the scalar potential, $\Psi(\mathbf{k},\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})/k$, namely,

$$\nabla\Psi(\mathbf{k},\mathbf{r}) = i\frac{\mathbf{k}}{k}\exp(i\mathbf{k}\cdot\mathbf{r}) \; .$$

These basis vectors satisfy the orthonormality condition that

$$\frac{1}{(2\pi)^2}\int d^2r \, \nabla\Psi^*(\mathbf{k},\mathbf{r}) \cdot \nabla\Psi(\mathbf{k}',\mathbf{r}) = \delta^{(2)}(\mathbf{k}-\mathbf{k}') \, .$$

Thus the irrotational component of the test field velocity can be represented by:

$$\mathbf{v}^c(\mathbf{r},t) = \int d^2k \, \tilde{d}(\mathbf{k},t) \nabla\Psi(\mathbf{k},\mathbf{r}) \, . \tag{93}$$

The spectral coefficients of the solenoidal and irrotational components of the test velocity field are given by the $\tilde{c}(\mathbf{k},t)$'s and the $\tilde{d}(\mathbf{k},t)$'s, respectively.

If we now insert Eqs. (92) and (93) into Eq. (29) and use the arguments of the previous section for extracting the desired equations of the two-dimensional geometry from the corresponding equations of the original three-dimensional geometry, Eqs. (30), we obtain the following equations that prescribe the time-evolution of the spectral coefficients, $\tilde{c}(\mathbf{k},t)$, and $\tilde{d}(\mathbf{k},t)$,

$$\left(\frac{\partial}{\partial t} + \nu \, k^2\right) \tilde{c}(\mathbf{k},t) = \int d^2q \, d^2p \; g^s_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k}) c(\mathbf{p},t) \tilde{d}(\mathbf{q},t) \, , \tag{94a}$$

$$\left(\frac{\partial}{\partial t} + \nu\, k^2\right) \tilde{d}(\mathbf{k},t) = \int d^2q\, d^2p\; g^c_{(2)}(\mathbf{p},\mathbf{q},-\mathbf{k})c(\mathbf{p},t)\tilde{c}(\mathbf{q},t)\,; \tag{94b}$$

where

$$g^s_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \sum_{l=\pm} g^s_{li}(\mathbf{p},\mathbf{q},\mathbf{k})\,,$$

$$g^c_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \frac{1}{2} \sum_{l,m=\pm} g^c_{lm}(\mathbf{p},\mathbf{q},\mathbf{k})\,.$$

These equations supplement Eqs. (84) and (85) of the previous section that prescribe the time-evolution of $c(\mathbf{k},t)$.

Using Eqs. (34) and (35), we find

$$\tilde{g}^s_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\,\frac{2\,A^2(k,p,q)}{k\,p\,q} \sum_{l=\pm} s_i s_l [i\, s_i\, s_H(\mathbf{k},\mathbf{p},\mathbf{q})][i\, s_l\, s_H(\mathbf{k},\mathbf{p},\mathbf{q})] = \frac{4\,A^2(k,p,q)}{k\,p\,q}\,, \tag{95a}$$

$$\tilde{g}^c_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{A^2(k,p,q)}{k\,p\,q} \sum_{l,m=\pm} s_l s_m [i\, s_l s_H(\mathbf{k},\mathbf{p},\mathbf{q})][i\, s_m\, s_H(\mathbf{k},\mathbf{p},\mathbf{q})] = -\frac{4\,A^2(k,p,q)}{k\,p\,q}\,; \tag{95b}$$

where we have again used the nomenclature that

$$g^s_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \tilde{g}^s_2(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\,,$$

$$g^c_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \tilde{g}^c_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\,.$$

Thus we may define

$$\breve{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \tilde{g}^s_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) = -\,\tilde{g}^c_{(2)}(\mathbf{p},\mathbf{k},\mathbf{q})\,,$$

where these structure functions are totally symmetric functions of the magnitudes of their wave-vector arguments and satisfy

$$\breve{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k}) \equiv \breve{g}^*_{(2)}(-\mathbf{p},-\mathbf{q},-\mathbf{k}).$$

Using Eqs. (94) and (95), one again may verify that $\int d^2 k \left[\left|\tilde{d}(\mathbf{k},t)\right|^2 + |\tilde{c}(\mathbf{k},t)|^2\right]$ is conserved in the absence of viscosity (Kraichnan 1971, 1972; Leith & Kraichnan 1972).

We can now proceed rapidly to the final equations of the TFM closure for the two-dimensional statistically homogeneous turbulence. We proceed precisely as in Sec. V to obtain these final equations. Defining

$$< c(\mathbf{k},t)c(\mathbf{k}',t) > \equiv \delta^{(2)}(\mathbf{k}+\mathbf{k}')U(\mathbf{k},t)\,,$$

where

$$U(\mathbf{k},t) = U(-\mathbf{k},t) \ ,$$

we obtain the time-evolution equation for the spectrum, $U(\mathbf{k},t)$:

$$\left(\frac{\partial}{\partial t} + 2\nu\, k^2\right) U(\mathbf{k},t) = 4\int d^2p\, d^2q\, [\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\times$$
$$\left\{\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)\tilde{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})\tilde{g}_{(2)}(-\mathbf{p},-\mathbf{k},-\mathbf{q})[U(\mathbf{k},t)-U(\mathbf{q},t)]U(\mathbf{p},t)\right\}$$
$$+(\mathbf{k} \Leftrightarrow -\mathbf{k})\,].$$

As in Sec. V, the function θ is a totally symmetric function of its wave-vector arguments satisfying $\theta(\mathbf{k},\mathbf{p},\mathbf{q},t) = \theta^*(-\mathbf{k},-\mathbf{p},-\mathbf{q},t)$. We shall be choosing θ to be a real function; i.e., $\theta(\mathbf{k},\mathbf{p},\mathbf{q},t) = \theta^*(\mathbf{k},\mathbf{p},\mathbf{q},t)$. Then using Eqs. (85) and (86), this spectral evolution equation also can be written as

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) U(\mathbf{k},t) = -4\int d^2p\, d^2q\, \delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\frac{A^2(k,p,q)}{k^2p^2q^2}\left(q^2-p^2\right)\left(k^2-p^2\right)\times$$

$$\{\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)[U(\mathbf{k},t)-U(\mathbf{q},t)]U(\mathbf{p},t)\} + (\mathbf{k} \Leftrightarrow -\mathbf{k})$$

$$= -8\int d^2p\, d^2q\, \delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\frac{A^2(k,p,q)}{k^2p^2q^2}\left(q^2-p^2\right)\left(k^2-p^2\right)\times$$

$$\{\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)[U(\mathbf{k},t)-U(\mathbf{q},t)]U(\mathbf{p},t)\}\,. \tag{96}$$

We similarly define the spectra, $U^s(\mathbf{k},t)$ and $U^c(\mathbf{k},t)$ associated with the solenoidal and irrotational components of the test field in analogy with Sec. V:

$$< \tilde{c}(\mathbf{k},t)\tilde{c}(\mathbf{k}',t) > = \delta^{(2)}(\mathbf{k}+\mathbf{k}')U^{s}(\mathbf{k},t)\,,$$

$$< \tilde{d}(\mathbf{k},t)\tilde{d}(\mathbf{k}',t) > = \delta^{(2)}(\mathbf{k}+\mathbf{k}')U^{c}(\mathbf{k},t)\,.$$

Again implementing the procedure used in Sec. V, we find

$$\left(\frac{\partial}{\partial t}+2\,\nu\,k^2\right)U^s(\mathbf{k},t)=-2\,\mu^s_{(2)}(\mathbf{k},t)U^s(\mathbf{k},t)\,,$$

$$\left(\frac{\partial}{\partial t}+2\,\nu\,k^2\right)U^c(\mathbf{k},t)=-2\,\mu^c_{(2)}(\mathbf{k},t)U^c(\mathbf{k},t)\;;$$

where

$$\begin{aligned}\mu^s_{(2)}(\mathbf{k},t) &= \int d^2p\,d^2q\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\,\tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q},t)\left|\breve{g}_{(2)}(\mathbf{p},\mathbf{q},\mathbf{k})\right|^2U(\mathbf{p},t)\\ &= 16\int d^2p\,d^2q\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\frac{A^4(k,p,q)}{k^2\,p^2\,q^2}\,\tilde{\theta}(\mathbf{p},\mathbf{k},\mathbf{q},t)U(\mathbf{p},t),\end{aligned} \tag{97a}$$

$$\begin{aligned}\mu^c_{(2)}(\mathbf{k},t) &= \int d^2p\,d^2q\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\,\tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k},t)\left|\breve{g}_{(2)}(\mathbf{p},\mathbf{k},\mathbf{q})\right|^2U(\mathbf{p},t)\\ &= 16\int d^2p\,d^2q\delta^{(2)}(\mathbf{p}+\mathbf{q}+\mathbf{k})\frac{A^4(k,p,q)}{k^2\,p^2\,q^2}\,\tilde{\theta}(\mathbf{p},\mathbf{q},\mathbf{k},t)U(\mathbf{p},t).\end{aligned} \tag{97b}$$

The function, $\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)$, has been assumed to be a real function satisfying

$$\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)=\tilde{\theta}^*(-\mathbf{k},-\mathbf{p},-\mathbf{q},t)=\tilde{\theta}^*(\mathbf{k},\mathbf{p},\mathbf{q},t)\,.$$

Equations (96) and (97), along with

$$\left[\frac{\partial}{\partial t}+\nu\,\left(k^2+p^2+q^2\right)\right]\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)=1-\left[\mu^s_{(2)}(\mathbf{k},t)+\mu^s_{(2)}(\mathbf{p},t)+\mu^s_{(2)}(\mathbf{q},t)\right]\theta(\mathbf{k},\mathbf{p},\mathbf{q},t)\,,$$

$$\left[\frac{\partial}{\partial t}+\nu\,\left(k^2+p^2+q^2\right)\right]\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t)=1-\left[\mu^s_{(2)}(\mathbf{k},t)+\mu^s_{(2)}(\mathbf{p},t)+\mu^c_{(2)}(\mathbf{q},t)\right]\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},t); \tag{98}$$

satisfying the initial conditions:

$$\theta(\mathbf{k},\mathbf{p},\mathbf{q},0)\;=\;\tilde{\theta}(\mathbf{k},\mathbf{p},\mathbf{q},0)\;=\;0\;\;, \tag{99}$$

constitute the TFM closure for two-dimensional, statistically homogeneous turbulence.

Using Eq. (89), Eqs. (96) and (97) can be reduced further when the turbulence is isotropic (Kraichnan 1976):

$$\left(\frac{\partial}{\partial t} + 2\nu\, k^2\right) U(k,t) = \frac{2}{k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq\ \sin(\alpha_k)\, \theta(k,p,q;t)\ \times \tag{100}$$

$$\left(q^2 - p^2\right)\left(k^2 - p^2\right)[U(q,t) - U(k,t)]U(p,t)\,,$$

$$\mu^s_{(2)}(k,t) = \frac{1}{k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq\ \sin^3(\alpha_k)\, \tilde{\theta}(p,k,q;t) p^2\, q^2 U(p,t)\,, \tag{101a}$$

$$\mu^c_{(2)}(k,t) = \frac{1}{k^2} \int\limits_{\mathbf{q}+\mathbf{p}=\mathbf{k}} dp\, dq\ \sin^3(\alpha_k)\, \tilde{\theta}(p,q,k;t) p^2\, q^2 U(p,t)\,; \tag{101b}$$

where θ and $\tilde{\theta}$ satisfy

$$\left[\frac{\partial}{\partial t} + \nu\, \left(k^2 + p^2 + q^2\right)\right] \theta(k,p,q;t) = 1 - \left[\mu^s_{(2)}(k,t) + \mu^s_{(2)}(p,t) + \mu^s_{(2)}(q,t)\right] \theta(k,p,q;t)\,,$$

$$\left[\frac{\partial}{\partial t} + \nu\, \left(k^2 + p^2 + q^2\right)\right] \tilde{\theta}(k,p,q;t) = 1 - \left[\mu^s_{(2)}(k,t) + \mu^s_{(2)}(p,t) + \mu^c_{(2)}(q,t)\right] \tilde{\theta}(k,p,q;t); \tag{102}$$

and the initial conditions:

$$\theta(k,p,q;0) \ = \ \tilde{\theta}(k,p,q;0) \ = \ 0\ . \tag{103}$$

VIII. Summary and Conclusions

In this paper, we have shown that the use of a helicity decomposition for the representation of incompressible, Navier-Stokes turbulence greatly facilitates the analysis of turbulence closures. The associated compact notation, which obviates the use of solenoidal projection operators, clarifies the nonlinear coupling of the modes of the Navier-Stokes equation. This clarification is embodied in the structure function, which implicitly is a function of the Navier-Stokes modal dynamics as well as of the global geometry.

As a result, we obtained with relative ease the equations for the following closures and explored their relationships:

1) DIA for three-dimensional, homogeneous turbulence that need be neither isotropic, nor mirror-symmetric,

2) TFM for three-dimensional, homogeneous, isotropic, mirror-symmetric turbulence,

3) DIA for three-dimensional turbulence in a free-slip channel flow,

4) TFM for three-dimensional turbulence in a free-slip channel flow,

5) DIA for an arbitrary two-dimensional homogeneous turbulence,

6) TFM for an arbitrary two-dimensional homogeneous turbulence,

7) DIA for two-dimensional isotropic, homogeneous turbulence, and

8) TFM for two-dimensional isotropic, homogeneous turbulence.

These lead us to the following conclusions:

1) These DIA and TFM equations for the free-slip channel flow are entirely new results. They may be compared with the EDQNM model previously derived.

2) The evolution equations of turbulent spectra may have striking similarities in different geometries. Their different aspects in coordinate space arise entirely from the different bases used to represent the fluid velocity. These bases differ from each other due to differences in global geometry and boundary conditions.

3) Two-dimensional homogeneous turbulence closures are easily gleaned from the three-dimensional homogeneous closures. One merely extracts the structure functions associated with the two-dimensional case from their three-dimensional counterparts.

4) The physics of these decompositions is not obscured by the presence of cumbersome projection operators. Evidence is given by the clear positivity of the coefficients of the $\tilde{\theta}$ functions in all of the TFM expressions for the turbulent eddy damping factors, μ^s and μ^c.

The compact notational advantage of these decompositions allow for the study of turbulence in finite geometries and analysis of scalings in the associated nonisotropic and inhomogeneous turbulence environments. From there, one can develop theoretically-based engineering models. It will be interesting to study other types of closures using these decompositions.

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III. DIRECT-INTERACTION APPROXIMATION AND TEST-FIELD MODELS OF TWO- AND THREE-DIMENSIONAL HOMOGENEOUS AND FREE-SLIP CHANNEL CASES

Author(s): Leaf Turner, T-3

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