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Stretching functions and comparisons with a bench-mark solution

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1 Generation of nonuniform grids

There are numerous ways of generating grid for complicated geometries. The stretching transformations employed here are taken from the family of transformations proposed by Roberts (1971).

If the plume extends down to the bottom of the two-dimensional box, then the following two simple independent transformations are used to refine the grid along the plume and at the top and bottom of the chamber.

a. To refine along the plume, take

$$x = x_c \left\{ 1 + \frac{\sinh\left[\tau(\xi - B)\right]}{\sinh(\tau B)} \right\}, \tag{1}$$

where

$$B = \frac{1}{2\tau} \ln \left[\frac{1 + (\exp(\tau) - 1)\,(x_c/\Lambda)}{1 + (\exp(-\tau) - 1)(x_c/\Lambda)} \right] \qquad 0 < \tau < \infty.$$

Here x_c is a point between $x = 0$ and $x = \Lambda$, and τ is the stretching parameter which varies from $\tau = 0$ (no stretching) to large values which generate the greatest refinement near $x = x_c$.

b. To refine at the top and bottom of the chamber:

$$y = H/2 \frac{(\beta+1)\left[(\beta+1)/(\beta-1)\right]^{(2\eta-1)} - \beta + 1}{1 + \left[(\beta+1)/(\beta-1)\right]^{(2\eta-1)}} \qquad 1 < \beta < \infty. \tag{2}$$

This transformation refines the mesh equally near $y = 0$ and $y = H$. β is the stretching parameter related to non-dimensional boundary layer thickness. β varies from large values (no stretching) to 1 corresponding to most refinement. Figure 1 shows a stretched grid in the physical space when $H = 1$ and $\Lambda = 1$.

If the plume extends only down to $y = h > 0$, then the transformation given by (1) is used to refine along the plume and the following two transformations are

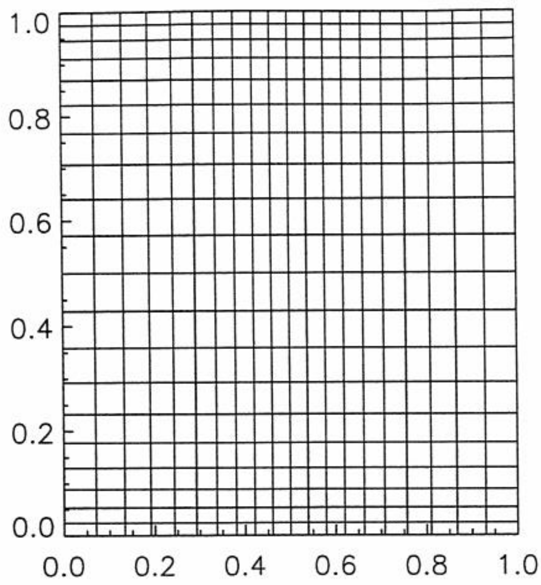

Figure 1: Grid refined along the vertical mid-plane and at the top and bottom of the chamber. The stretching parameters in (1), (2) are $x_c = 1/2$, $\tau = 2.5$, $\beta = 1.2$.

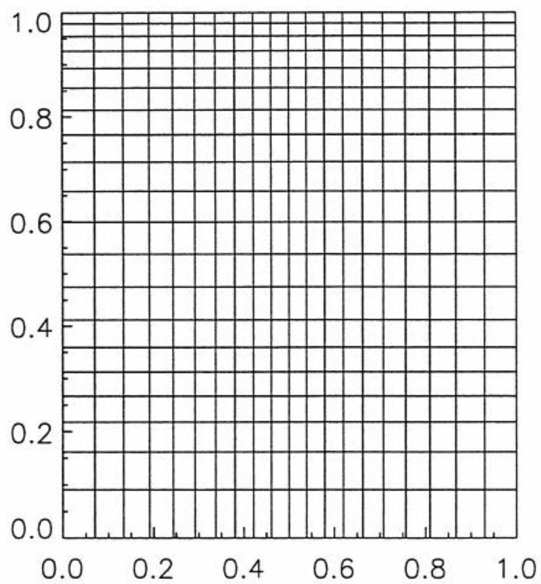

Figure 2: Grid refined along the vertical mid-plane and at the top ($y = 1$) and interior depth ($y = h$) of the chamber. The stretching parameters in (1), (3)–(4) are $x_c = 1/2$, $\tau = 2.5$, $\beta = 1.2$ and $h = 0.3$.

manipulated to give a mesh refined along the top and bottom of the plume:

$$y = H_1 \frac{\beta\,[(\beta+1)/(\beta-1)]^{\eta} - \beta}{1 + [(\beta+1)/(\beta-1)]^{\eta}} \quad 1 < \beta < \infty, \tag{3}$$

$$y = h\left\{1 + \frac{\sinh[\tau(\eta - B)]}{\sinh(\tau B)}\right\}, \tag{4}$$

where

$$B = \frac{1}{2\tau} \ln\left[\frac{1 + (\exp(\tau) - 1)\,(h/H_2)}{1 + (\exp(-\tau) - 1)\,(h/H_2)}\right] \quad 0 < \tau < \infty.$$

Here $H_1 + H_2 = H$. The stretching parameter β varies from large values (no stretching) to 1 corresponding to greatest refinement at $y = H$ and increasing τ refines the grid near $y = h$, the bottom of the plume. Figure 2 shows the refined grid for $H = 1$, $\Lambda = 1$ and $h = 0.3$. The stretching parameters τ for (1) and (4) are both taken as 2.5.

2 Code performance

It is clear that if the cell swimming speed is zero, then the equations of bioconvection reduce to the usual equations for heat convection problem. To validate our code, written in terms of stretched coordinates, the code has been run for the following problem with stretching and without stretching. The problem is that of two-dimensional heat convection in a rigid square cavity defined by $0 \le x \le 1$, $0 \le y \le 1$. All the variables are dimensionless. The governing equations are

$$\nabla^2\psi = -\zeta, \tag{5}$$

$$\nabla^2\zeta = P_r^{-1}\left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}\right) - R\frac{\partial T}{\partial x}, \tag{6}$$

$$\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}, \tag{7}$$

where R is the Rayleigh number, P_r is the Prandtl number and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If $\boldsymbol{n}$ is the outward drawn normal at any point on the unit square S, then the boundary conditions are

$$\begin{aligned} \psi = \frac{\partial \psi}{\partial n} &= 0, \\ T = 1 \quad & x = 0, \\ T = 0 \quad & x = 1, \\ \frac{\partial T}{\partial y} = 0 \quad & y = 0,\, 1. \end{aligned} \tag{8}$$

The stretching function given by (2) is chosen to refine the mesh near the four rigid boundaries. The grid geometry in the uniform computational space is shown in figure 3. The choice of the control volume automatically satisfies the boundary conditions on T.

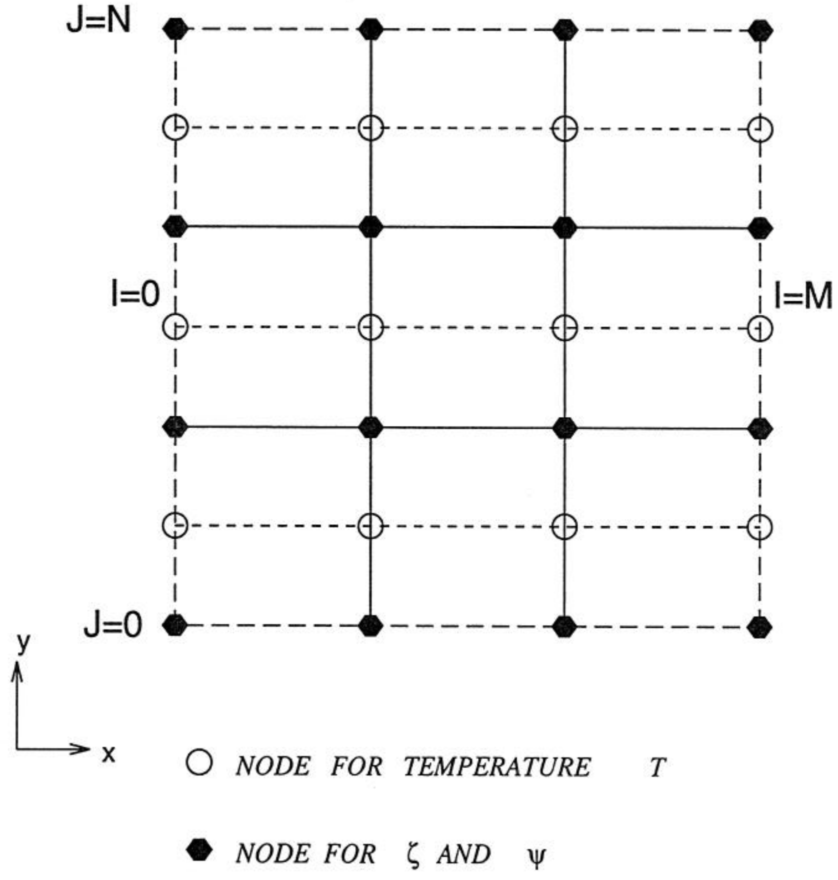

Figure 3: Staggered grid geometry for the heat convection problem. I = 0 and I = M represent the lateral walls. J=0 and J=N represent the horizontal walls.

Two set of runs were made with the following values of the stretching parameter

β subject to the convergence criterion

$$\max |T_{i+1} - T_i| \leq 10^{-8}.$$

a. $\beta = 10^5$, i.e. uniform grid.

b. $\beta = 1.5$, i.e. grid refined near the boundaries.

The properties of the solutions of heat convection are compared with the bench-mark solutions given by De Vahl Davis (1983) (see tables 1–6). In each table, the quantities given are

(i) the magnitude of the stream-function at the mid-point of the cavity

(ii) the maximum value of u on the vertical mid-plane ($x = 0.5$), together with its location.

(iii) the maximum value of v on horizontal mid-plane ($y = 0.5$), together with its location

(iv) the average Nusselt number Nu_0 on the vertical boundary x = 0

(v) the maximum and minimum values of local Nusselt number on $x = 0$ and its location.

The maximum and minimum values and their locations are computed by quadratic interpolating polynomial.

Figure 4 shows the streamlines for $R = 10^3$–10^5 and the corresponding maps of temperature are shown in figure 5. These figure are based on the finest uniform grid, 31×31.

At $R = 10^3$, streamlines are those of a single vortex with its centre in the centre of square cavity. The isotherms are almost parallel to the heated wall. The effect of convection is seen as the departure of the isotherms from the vertical. At $R = 10^4$, the central streamline is distorted to an elliptic shape and at $R = 10^5$, two secondary vortices appear with its long axis along the direction of the flow. The temperature gradient is severe along the wall but almost zero in the central region.

It is seen from the tables 1–6 that as the number of grid points increases, the solutions tend to bench-mark solution in both the stretched and unstretched cases. At higher Rayleigh numbers (see tables 5 and 6), the code with stretching performs better than that one without stretching, though the values of $Nu_{\max}$ and $Nu_{\min}$

Table 1: Properties of the solution of the heat convection problem using a uniform grid for the case $P_r = 0.71$ and $R = 10^3$

Nodes	$\lvert\psi_{\text{mid}}\rvert$	u_{max} $y(x=0.5)$	v_{max} $x(y=0.5)$	Nu_0	Nu_{max} $y(x=0)$	Nu_{min} $y(x=0)$
11× 11	1.169	3.616	3.552	1.126	1.537	0.690
		0.807	0.173		0.095	0.957
21× 21	1.170	3.625	3.665	1.118	1.510	0.693
		0.811	0.175		0.092	0.979
31× 31	1.172	3.636	3.681	1.118	1.507	0.692
		0.812	0.180		0.091	0.989
Bench-mark	1.174	3.649	3.697	1.117	1.505	0.692
solution		0.813	0.178		0.092	1

Table 2: Properties of the solution of the heat convection problem using a grid refined along the boundaries for the case $P_r = 0.71$ and $R = 10^3$

Nodes	$\lvert\psi_{\text{mid}}\rvert$	u_{max} $y(x=0.5)$	v_{max} $x(y=0.5)$	Nu_0	Nu_{max} $y(x=0)$	Nu_{min} $y(x=0)$
11× 11	1.191	3.707	3.649	1.067	1.458	0.652
		0.818	0.187		0.094	0.961
21× 21	1.179	3.652	3.694	1.088	1.470	0.672
		0.817	0.181		0.087	0.986
31× 31	1.177	3.651	3.697	1.098	1.481	0.679
		0.812	0.179		0.092	0.994
Bench-mark	1.174	3.649	3.697	1.117	1.505	0.692
solution		0.813	0.178		0.092	1

Table 3: Properties of the solution of the heat convection problem using a uniform grid for the case $P_r = 0.71$ and $R = 10^4$

Nodes	$\lvert\psi_{mid}\rvert$	u_{max}	v_{max}	Nu_0	Nu_{max}	Nu_{min}
		$y(x = 0.5)$	$x(y = 0.5)$		$y(x = 0)$	$y(x = 0)$
11× 11	5.135	16.183	18.577	2.508	4.187	0.582
		0.812	0.141		0.144	0.982
21× 21	5.069	16.071	19.442	2.296	3.701	0.589
		0.819	0.125		0.139	0.986
31× 31	5.068	16.130	19.537	2.261	3.596	0.588
		0.824	0.117		0.141	0.992
Bench-mark	5.071	16.178	19.617	2.238	3.528	0.586
solution		0.823	0.119		0.143	1

Table 4: Properties of the solution of the heat convection problem using a grid refined along the boundaries for the case $P_r = 0.71$ and $R = 10^4$

Nodes	$\lvert\psi_{mid}\rvert$	u_{max}	v_{max}	Nu_0	Nu_{max}	Nu_{min}
		$y(x = 0.5)$	$x(y = 0.5)$		$y(x = 0)$	$y(x = 0)$
11× 11	5.189	16.499	19.908	2.365	3.970	0.560
		0.823	0.117		0.143	0.977
21× 21	5.098	16.208	19.558	2.217	3.551	0.572
		0.825	0.121		0.140	0.990
31× 31	5.084	16.184	19.648	2.214	3.509	0.576
		0.825	0.117		0.143	0.996
Bench-mark	5.071	16.178	19.617	2.238	3.528	0.586
solution		0.823	0.119		0.143	1

Table 5: Properties of the solution of the heat convection problem using a uniform grid for the case $P_r = 0.71$ and $R = 10^5$

Nodes	$\lvert\psi_{mid}\rvert$	u_{max} $y(x=0.5)$	v_{max} $x(y=0.5)$	Nu_0	Nu_{max} $y(x=0)$	Nu_{min} $y(x=0)$
11× 11	10.431	37.21	76.19	5.900	10.181	0.611
		0.834	0.050		0.087	1
21× 21	9.264	34.87	67.62	5.048	9.356	0.701
		0.853	0.075		0.075	0.992
31× 31	9.141	34.58	67.28	4.736	8.532	0.723
		0.856	0.067		0.079	0.991
Bench-mark	9.111	34.73	68.59	4.509	7.717	0.729
solution		0.855	0.066		0.081	1

Table 6: Properties of the solution of the heat convection problem using a grid refined along the boundaries for the case $P_r = 0.71$ and $R = 10^5$

Nodes	$\lvert\psi_{mid}\rvert$	u_{max} $y(x=0.5)$	v_{max} $x(y=0.5)$	Nu_0	Nu_{max} $y(x=0)$	Nu_{min} $y(x=0)$
11× 11	9.755	37.02	61.08	5.760	10.682	0.585
		0.846	0.050		0.098	1
21× 21	9.181	34.75	67.68	4.734	8.665	0.702
		0.852	0.059		0.079	0.990
31× 31	9.133	34.68	68.51	4.552	8.043	0.714
		0.856	0.061		0.079	0.993
Bench-mark	9.111	34.73	68.59	4.509	7.717	0.729
solution		0.855	0.066		0.081	1

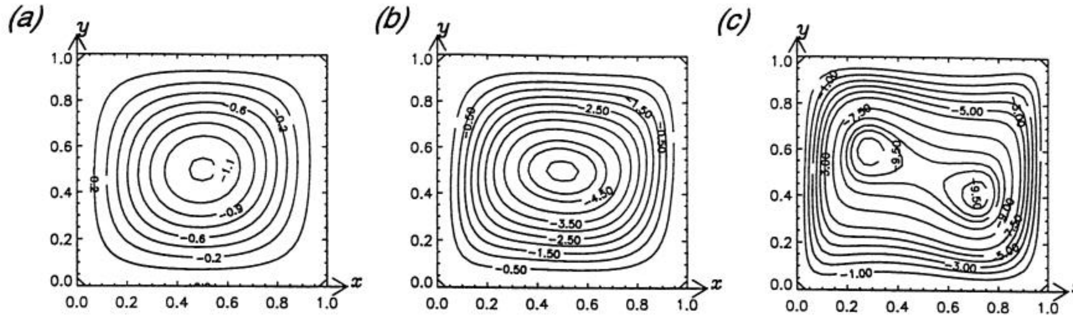

Figure 4: Streamlines for the heat convection problem at different Rayleigh numbers R ($Pr = 0.71$): (a) $R = 10^3$; (b) $R = 10^4$; (c) $R = 10^5$

for 11×11 grid are less accurate for the stretched grid. Thus at higher Rayleigh numbers boundary layer becomes important (evident from figures 4 and 5) and, to resolve the boundary layers, we need to use a stretching function if small number of grid points are used in the calculation.

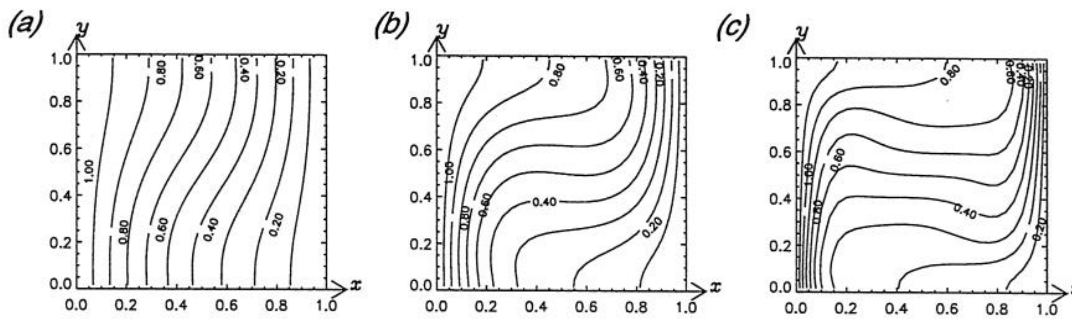

Figure 5: Isotherms for the heat convection problem at different Rayleigh numbers R ($Pr = 0.71$): (a) $R = 10^3$; (b) $R = 10^4$; (c) $R = 10^5$