FM 10505

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Stability of viscoelastic shear flows subjected to steady or oscillatory transverse flow

- referred to after equation (3.2)

Appendix B. Disturbance equations

The non-zero entries of $Wp\,\hat{\mathbf{A}}(t)$ in equation (3.2) for the CCMAC and DMAC cases are given below. The operator $\hat{D}$ represents the derivative w.r.t. the gap coordinate ρ. The matrix $\hat{\mathbf{E}}(t)$ has -1 on the diagonal corresponding to the constitutive equations and zeros elsewhere. The EVSS formulation (not shown here) is derived by substituting

(sloping up sigma without skew tilde)

$$
\begin{aligned}
\tilde{\tau}_{rr} &= \tilde{\Sigma}_{rr} + 2\,Wp\,\hat{D}\,\tilde{v}_r,\\
\tilde{\tau}_{r\theta} &= \tilde{\Sigma}_{r\theta} + Wp\,\hat{D}\,\tilde{v}_\theta,\\
\tilde{\tau}_{rz} &= \tilde{\Sigma}_{rz} + \mathrm{i}\alpha\,Wp\,\tilde{v}_r + Wp\,\hat{D}\,\tilde{v}_z,\\
\tilde{\tau}_{\theta\theta} &= \tilde{\Sigma}_{\theta\theta},\\
\tilde{\tau}_{\theta z} &= \tilde{\Sigma}_{\theta z} + Wp\,\tilde{v}_\theta,\\
\tilde{\tau}_{zz} &= \tilde{\Sigma}_{zz} + 2\mathrm{i}\alpha\,Wp\,\tilde{v}_z,
\end{aligned}
$$

where the components of $\mathbf{\Sigma}$ represent the elastic part of the extra stress tensor $\boldsymbol{\tau}$.

$$
\begin{aligned}
Wp\,A_{1,1} &= 1 + \mathrm{i}\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)\\
Wp\,A_{1,7} &= 2\mathrm{i}\alpha\,Wp\left(\frac{We_z\cos(\omega_1 t_1)}{(1+\omega_1^2 Wp^2)} + \frac{\omega_1 We_z Wp\sin(\omega_1 t_1)}{(1+\omega_1^2 Wp^2)(1+S)}\right) - 2Wp\,\hat{D}
\end{aligned}
$$

$$Wp\,A_{2,1} = Wp$$

$$Wp\,A_{2,2} = 1 + i\alpha We_z(1-\rho)\cos(\omega_1 t_1)$$

$$Wp\,A_{2,7} = -\frac{i\,\alpha\,Wp\,\left(2\,We_z\,Wp\,\cos(\omega_1\,t_1) + 3\,\omega_1\,We_z\,Wp^2\,\sin(\omega_1\,t_1) + \omega_1{}^3\,We_z\,Wp^4\,\sin(\omega_1\,t_1)\right)}{\left(1+\omega_1{}^2\,Wp^2\right)^2} + Wp^2\,D$$

$$Wp\,A_{2,8} = -i\,\alpha\,Wp\,\left(-\frac{We_z\,\cos(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)} - \frac{\omega_1\,We_z\,Wp\,\sin(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)}\right) - Wp\,D$$

$$Wp\,A_{3,1} = We_z\,\cos(\omega_1 t_1)$$

$$Wp\,A_{3,3} = 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)$$

$$Wp\,A_{3,7} = -\frac{i\,\alpha\,Wp\,\left(We_z{}^2 + 4\,\omega_1{}^2\,We_z{}^2\,Wp^2 + We_z{}^2\,\cos(2\,\omega_1\,t_1) - 2\,\omega_1{}^2\,We_z{}^2\,Wp^2\,\cos(2\,\omega_1\,t_1)\right)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)}$$
$$-\frac{3\,\omega_1\,We_z{}^2\,Wp^2\,\sin(2\,\omega_1\,t_1)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)} - i\,\alpha\,Wp$$

$$Wp\,A_{3,9} = -Wp\,D$$

$$Wp\,A_{4,2} = 2\,Wp$$

$$Wp\,A_{4,4} = 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)$$

$$Wp\,A_{4,8} = -\frac{2\,i\,\alpha\,Wp\,\left(2\,We_z\,Wp\,\cos(\omega_1\,t_1) + 3\,\omega_1\,We_z\,Wp^2\,\sin(\omega_1\,t_1) + \omega_1{}^3\,We_z\,Wp^4\,\sin(\omega_1\,t_1)\right)}{\left(1+\omega_1{}^2\,Wp^2\right)^2} + 2\,Wp^2\,D$$

$$Wp\,A_{5,2} = We_z\cos(\omega_1 t_1)$$

$$Wp\,A_{5,3} = Wp$$

$$Wp\,A_{5,5} = 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)$$

$$Wp\,A_{5,8} = -\frac{i\,\alpha\,Wp\,\left(We_z{}^2 + 4\,\omega_1{}^2\,We_z{}^2\,Wp^2 + We_z{}^2\,\cos(2\,\omega_1\,t_1) - 2\,\omega_1{}^2\,We_z{}^2\,Wp^2\,\cos(2\,\omega_1\,t_1)\right)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)}$$
$$-\frac{3\,\omega_1\,We_z{}^2\,Wp^2\,\sin(2\,\omega_1\,t_1)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)} - i\,\alpha\,Wp + Wp\,\left(\frac{We_z\,\cos(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)} + \frac{\omega_1\,We_z\,Wp\,\sin(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)}\right)\,D$$

$$Wp\,A_{5,9} = \frac{i\,\alpha\,\left(2\,We_z\,Wp\,\cos(\omega_1\,t_1) + 3\,\omega_1\,We_z\,Wp^2\,\sin(\omega_1\,t_1) + \omega_1{}^3\,We_z\,Wp^4\,\sin(\omega_1\,t_1)\right)}{\left(1+\omega_1{}^2\,Wp^2\right)^2} + Wp\,D$$

$$Wp\,A_{6,3} = 2\,We_z\cos(\omega_1 t_1)$$

$$Wp\,A_{6,6} = 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)$$

$$Wp\,A_{6,9} = -\frac{2\,i\,\alpha\,Wp\,\left(We_z{}^2 + 4\,\omega_1{}^2\,We_z{}^2\,Wp^2 + We_z{}^2\,\cos(2\,\omega_1\,t_1) - 2\,\omega_1{}^2\,We_z{}^2\,Wp^2\,\cos(2\,\omega_1\,t_1)\right)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)}$$
$$-\frac{3\,\omega_1\,We_z{}^2\,Wp^2\,\sin(2\,\omega_1\,t_1)}{\left(1+4\,\omega_1{}^2\,Wp^2\right)\,\left(1+\omega_1{}^2\,Wp^2\right)(1+S)} + 2\,Wp\,\left(\frac{We_z\,\cos(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)} + \frac{\omega_1\,We_z\,Wp\,\sin(\omega_1\,t_1)}{(1+\omega_1{}^2\,Wp^2)}\right)\,D$$
$$-2\,i\,\alpha\,Wp$$

$$Wp\,A_{7,1} = D \quad Wp\,A_{7,3} = i\,\alpha \quad Wp\,A_{7,4} = -1 \quad Wp\,A_{7,7} = S\,Wp\,\left(-\alpha^2 + D^2\right) \quad Wp\,A_{7,10} = -D$$

$$Wp\,A_{8,2} = D \quad Wp\,A_{8,5} = i\,\alpha \quad Wp\,A_{8,8} = S\,Wp\,\left(-\alpha^2 + D^2\right)$$

$$Wp\,A_{9,3} = D \quad Wp\,A_{9,6} = i\,\alpha \quad Wp\,A_{9,9} = S\,Wp\,\left(-\alpha^2 + D^2\right) \quad Wp\,A_{9,10} = -\alpha$$

$$Wp\,A_{10,7} = D \quad Wp\,A_{10,9} = i\,\alpha$$

DMAC flow: Given below are the nonzero components of $Wp\,\mathbf{A(t)}$ that correspond to the constitutive equation. The components corresponding to the momentum and continuity equations,as well as the form of the matrix $\mathbf{E}$ are the same as those in the CCMAC case given above. The equations for the DAC case may be recovered by setting ω_1 to zero.

$$
\begin{aligned}
Wp\,A_{1,1} &= 1 + iWe_z\,\alpha\,(1-\rho)\,\cos(\omega_1 t_1)\\
Wp\,A_{1,7} &= \frac{2\,i\,We_z\,Wp\,\alpha\,(\cos(\omega_1 t_1) + Wp\,\omega_1\,\sin(\omega_1 t_1))}{(1+Wp^2\,\omega_1^2)} - 2\,Wp\,D\\
Wp\,A_{2,1} &= Wp\,(-1+2\rho)\\
Wp\,A_{2,2} &= 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)\\
Wp\,A_{2,7} &= -Wp^2\,\frac{2\,(1+Wp^2\,\omega_1^2)^2 + 2\,i\,We_z\,\alpha\,(-1+2\,\rho)\,\cos(\omega_1 t_1)}{(1+Wp^2\,\omega_1^2)^2}\\
&\quad -Wp^2\,\frac{i\,We_z\,Wp\,\alpha(-1+2\,\rho)\,\omega_1\,(3+Wp^2\,\omega_1^2)\,\sin(\omega_1 t_1)}{(1+Wp^2\,\omega_1^2)^2} + Wp^2\,(-1+2\rho)D\\
Wp\,A_{2,8} &= i\alpha\,We_z\,Wp\frac{\cos(\omega_1 t_1) + Wp\,\omega_1\,\sin(\omega_1 t_1)}{(1+Wp^2\,\omega_1^2)} - Wp\,D\\
Wp\,A_{3,1} &= We_z\,\cos(\omega_1 t_1)\\
Wp\,A_{3,3} &= 1 + i\,We_z\,\alpha\,(1-\rho)\,\cos(\omega_1 t_1)\\
Wp\,A_{3,7} &= -i\,\alpha\,Wp\,\frac{(1+We_z^2+Wp^2\,\omega_1^2)(1+4Wp^2\,\omega_1^2) - We_z^2\,(-1+2\,Wp^2\,\omega_1^2)\cos(2\,\omega_1 t_1)}{(1+4\,Wp^2\,\omega_1^2)(1+Wp^2\,\omega_1^2)}\\
&\quad -i\,\alpha\,Wp\,\frac{3\,We_z^2\,Wp\,\omega_1\sin(2\,\omega_1 t_1)}{(1+4\,Wp^2\,\omega_1^2)(1+\omega_1^2\,Wp^2)} + \frac{We_z\,Wp\,(\cos(\omega_1 t_1) + Wp\,\omega_1\,\sin(\omega_1 t_1))\,D}{(1+Wp^2\,\omega_1^2)}\\
Wp\,A_{3,9} &= -Wp\,D\\
Wp\,A_{4,2} &= 2\,Wp\,(-1+2\,\rho)\\
Wp\,A_{4,4} &= 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1\,t_1)\\
Wp\,A_{4,7} &= 8\,Wp^3\,(-1+2\,\rho)\\
Wp\,A_{4,8} &= -2\,i\,We_z\,Wp^2\,\alpha\,\frac{(-1+2\,\rho)(2\,cos(\omega_1 t_1) + Wp\,\omega_1\,(3+Wp^2\,\omega_1^2)\sin(\omega_1 t_1))}{(1+Wp^2\omega_1^2)} +\\
&\quad 2\,Wp^2\,(-1+2\,\rho)\,D\\
Wp\,A_{5,2} &= We_z\,\cos(\omega_1 t_1)\\
Wp\,A_{5,3} &= Wp\,(-1+2\,\rho)\\
Wp\,A_{5,5} &= 1 + i\,\alpha\,We_z\,(1-\rho)\,\cos(\omega_1 t_1)\\
Wp\,A_{5,7} &= 2\,We_z\,Wp^2\,\frac{2\,\cos(\omega_1 t_1) + Wp\,\omega_1(3+Wp^2\,\omega_1^2)\,\sin(\omega_1 t_1)}{(1+Wp^2\,\omega_1^2)}
\end{aligned}
$$

$$Wp\,A_{5,8} = i\,\alpha\,Wp\,\frac{(1+We_z^2+Wp^2\,\omega_1^2)(1+4\,Wp^2\,\omega_1^2) - We_z^2\,(-1+2\,Wp^2\,\omega_1^2)\,\cos(2\,\omega_1 t_1) + 3\,We_z^2\,Wp\,\omega_1\,\sin(2\,\omega_1\,t_1)}{(1+4\,Wp^2\,\omega_1^2)(1+Wp^2\,\omega_1^2)} + \frac{We_z\,Wp\,(\cos(\omega_1 t_1) + Wp\,\omega_1\,\sin(\omega_1 t_1))}{(1+Wp^2\,\omega_1^2)}\,D$$

$$Wp\,A_{5,9} = i\,We_z\,Wp^2\,\alpha\,(-1+2\,\rho)\frac{2\,\cos(\omega_1 t_1) + Wp\,\omega_1\,(3+Wp^2\,\omega_1^2)\,\sin(t_1\,\omega_1)}{(1+Wp^2\,\omega_1^2)^2} + Wp^2\,(-1+2\,\rho)\,D$$

$$Wp\,A_{6,3} = 2\,We_z\,\cos(\omega_1 t_1)$$

$$Wp\,A_{6,6} = 1 + i\,We_z\,\alpha(1-\rho)\,\cos(\omega_1 t_1)$$

$$Wp\,A_{6,9} = -2\,i\,Wp\,\alpha\frac{(1+We_z^2+Wp^2\,\omega_1^2)(1+4\,Wp^2\,\omega_1^2) - We_z^2\,(-1+2\,Wp^2\,\omega_1^2)\,\cos(2\,\omega_1 t_1)}{(1+4\,Wp^2\,\omega_1^2)(1+4\,Wp^2\,\omega_1^2)} - 2\,i\,Wp\,\alpha\frac{3\,We_z^2\,\omega_1\,Wp\,\sin(2\,\omega_1 t_1)}{(1+4\,Wp^2\,\omega_1^2)(1+4\,Wp^2\,\omega_1^2)} + 2\,We_z\,Wp\frac{\cos(\omega_1 t_1) + Wp\,\omega_1\,\sin(2\,\omega_1 t_1)}{(1+Wp^2\,\omega_1^2)}D$$

DAP flow: Below are given the nonzero components of the linear operator $Wp\,\mathbf{L}$ for DAP flow. Once again, the momentum and continuity equations are the same as for the CCMAC case, as are the terms in the matrix $\mathbf{E}$.

$$Wp\,L_{1,1} = 1 + We_z\;i\,\alpha\,\rho\,(1-\rho)$$

$$Wp\,L_{1,7} = 2\,i,\,We_z\,Wp\,\alpha(-1+2\,\rho) - 2\,Wp\,D$$

$$Wp\,L_{2,1} = Wp\,(-1+2\,\rho)$$

$$Wp\,L_{2,2} = Wp\,L_{1,1}$$

$$Wp\,L_{2,7} = -2\,Wp^2 - 2\,i\,We_z\,Wp^2\,\alpha\,(-1+2\,\rho)^2 + Wp^2\,(-1+2\,\rho)\,D$$

$$Wp\,L_{2,8} = i\,We_z\,Wp\,\alpha\,(-1+2\,\rho) - \,Wp\,D$$

$$Wp\,L_{3,1} = We_z\,(-1+2\,\rho)$$

$$Wp\,L_{3,3} = Wp\,L_{1,1}$$

$$Wp\,L_{3,7} = -2\,We_z Wp + 8\,i\,\rho\,We_z^2\,Wp\,\alpha(1-\rho) - i\,\alpha\,Wp\,(1+2\,We_z^2) + We_z\,Wp\,(-1+2\,\rho)\,D$$

$$Wp\,L_{3,9} = i\,We_z\,\alpha\,(-1+2\,\rho)\,D$$

$$Wp\,L_{4,2} = 2\,Wp\,(-1+2\,\rho)$$

$$Wp\,L_{4,4} = Wp\,L_{1,1}$$

$$Wp\,L_{4,7} = 8\,Wp^3(-1+2\,\rho)$$

$$Wp\,L_{4,8} = -4\,i\,We_z\,Wp^2\,\alpha(1-2\,\rho)^2 + 2\,Wp^2(-1+2\,\rho)\,D$$

$$Wp\,L_{5,2} = We_z\,(-1+2\,\rho)$$

$$Wp\,L_{5,3} = Wp\,(-1+2\,\rho)$$

$$Wp\,L_{5,5} = Wp\,L_{1,1}$$

$$Wp\,L_{5,7} = 8\,We_z\,Wp^2(-1+2\,\rho)$$

$$Wp\,L_{5,8} = -i\,\alpha\,Wp\,(1+2\,We_z^2) + 8\,i\,\rho\,We_z^2\,Wp\,(1-\rho) + We_z(-1+2\,\rho)\,D$$

$$Wp\,L_{5,9} = -2\,i\,We_z\,Wp^2\,\alpha(1-2\,\rho)^2 + Wp^2(-1+2\,\rho)\,D$$

$$Wp\,L_{6,3} = 2\,We_z\,(-1+2\,\rho)$$

$$Wp\,L_{6,6} = Wp\,L_{1,1}$$

$$Wp\,L_{6,7} = 8\,We_z^2\,Wp\,(-1+2\,\rho)$$

$$Wp\,L_{6,9} = -2\,i\,\alpha\,Wp\,(1+2\,We_z^2\,(1-2\,\rho)^2) + 2\,We_z\,Wp\,(-1+2\,\rho)\,D$$