A detailed analysis of the oscillatory boundary layers that appear in the paper *On the steady streaming flow due to high-frequency vibration in nearly inviscid liquid bridges*, J.A. Nicolás, Damián Rivas and J.M. Vega, Journal of Fluid Mechanics, vol. 354 (1998), pp. 147-174, to be referred to NRV in the sequel.

A detailed analysis of the viscous boundary layers summarized in [NRV, Appendix B] is given below. In order to make the derivation self-contained, some equations in [NRV, §§2,-3 and Appendix A] are first listed (their number as equations of NRV is indicated in brackets)

$$u_r + r^{-1}u + w_z = 0, \qquad (1) \quad [\mathbf{(2.1)}]$$

$$u_t + w(u_z - w_r) = -q_r + C(u_{rr} + r^{-1}u_r - r^{-2}u + u_{zz}), \qquad (2) \quad [\mathbf{(2.2)}]$$

$$w_t + u(w_r - u_z) = -q_z + C(w_{rr} + r^{-1}w_r + w_{zz}), \qquad (3) \quad [\mathbf{(2.3)}]$$

$$u = 0, \qquad w = h'_\pm(t), \qquad f = 1 \quad \text{at } z = \pm\Lambda + h_\pm(t), \qquad (4) \quad [\mathbf{(2.4)}]$$

$$u = w_r = q_r = 0 \quad \text{at } r = 0, \qquad (5) \quad [\mathbf{(2.5)}]$$

$$u = f_t + f_z w \; at \; r = f, \qquad (6) \quad [\mathbf{(2.6)}]$$

$$(w_r + u_z)(1 - f_z^2) + 2(u_r - w_z)f_z = 0 \quad \text{at } r = f, \qquad (7) \quad [\mathbf{(2.7)}]$$

$$q - \frac{u^2 + w^2}{2} + \frac{f f_{zz} - 1 - f_z^2}{f(1 + f_z^2)^{3/2}} = 2C\frac{u_r - (w_r + u_z)f_z + w_z f_z^2}{1 + f_z^2} \quad \text{at } r = f, \qquad (8) \quad [\mathbf{(2.8)}]$$

$$\varepsilon^2\Omega^{1/3} \ll C \quad \text{and} \quad \varepsilon^2\Omega^{17/6}C^{1/2} \ll 1, \qquad (9)[\mathbf{(2.12a)}]$$

$$\varepsilon\Omega \ll 1, \qquad (10)[\mathbf{(2.12b)}]$$

$$W = \beta_\pm \quad \text{at } z = \pm\Lambda, \qquad U = W_r = 0 \quad \text{at } r = 0, \qquad Q = 0 \quad \text{at } r = 1. \qquad (11) \quad [\mathbf{(3.3)}]$$

$$\xi = k(1 - r) \quad \text{and} \quad \eta = kz \qquad (12) \quad [\mathbf{(3.5)}]$$

$$\left.\begin{aligned} u &= \mathrm{i}\Omega\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\left[U(1 - \xi/k, z) + u_1 + k^{-1}u_2 + \mathrm{HOT}\right] + \text{c.c.},\\ w &= \mathrm{i}\Omega\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\left[W(1 - \xi/k, z) + w_1 + k^{-1}w_2 + \mathrm{HOT}\right] + \text{c.c.},\\ q - 1 &= \Omega^2\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\left[Q(1 - \xi/k, z) + k^{-1}(q_1 + k^{-1}q_2) + \mathrm{HOT}\right] + \text{c.c.}, \end{aligned}\right\} \qquad (13) \quad [\mathbf{(3.6)}]$$

$$u_1 - f_1 = q_1 + f_{1\eta\eta} = 0 \quad \text{at } \xi = 0 \qquad (14) \quad [\mathbf{(3.9)}]$$

$$u_2 - f_2 - 2\mathrm{i}\omega f_{1\eta\eta} = q_2 + f_{2\eta\eta} + 2f_{1\eta z} + 2\mathrm{i}\omega u_{1\xi} = 0 \quad \text{at } \xi = 0, \qquad (15) \quad [(\mathbf{A.7})]$$

$$f - 1 = \varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\big[U(1,z) + f_1 + k^{-1} f_2 + \mathrm{HOT}\big] + \text{c.c.}, \qquad (16) \quad [(\mathbf{3.7})]$$

$$\left.\begin{aligned} u_1 &= A\mathrm{e}^{-\xi+\mathrm{i}\eta} + B\mathrm{e}^{-\xi-\mathrm{i}\eta}, \qquad w_1 = iA\mathrm{e}^{-\xi+\mathrm{i}\eta} - \mathrm{i}B\mathrm{e}^{-\xi-\mathrm{i}\eta},\\ q_1 &= A\mathrm{e}^{-\xi+\mathrm{i}\eta} + B\mathrm{e}^{-\xi-\mathrm{i}\eta}, \qquad f_1 = A\mathrm{e}^{\mathrm{i}\eta} + B\mathrm{e}^{-\mathrm{i}\eta}, \end{aligned}\right\} \qquad (17) \quad [(\mathbf{3.10})]$$

$$\omega = \Omega C \sim 1, \qquad (18) \quad [(\mathbf{3.12})]$$

$$\left.\begin{aligned} u &= \mathrm{i}\Omega\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\tilde{u}_1 + \text{c.c} + \mathrm{HOT}, \qquad w = \mathrm{i}\Omega\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\tilde{w}_1 + \text{c.c.} + \mathrm{HOT},\\ q - 1 &= (\Omega^2/k)\varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\tilde{q}_1 + \text{c.c.} + \mathrm{HOT}, \quad f - 1 = \varepsilon \mathrm{e}^{\mathrm{i}\Omega t}\tilde{f}_1 + \text{c.c.} + \mathrm{HOT}, \end{aligned}\right\} \qquad (19) \quad [(\mathbf{3.14})]$$

$$\tilde{u}_1 - \tilde{f}_1 = \tilde{q}_1 + \tilde{f}_1'' = 0 \quad \text{at } \xi = 0, \qquad (20) \quad [(\mathbf{3.16})]$$

$$\tilde{w}_1 = \beta_+, \quad \tilde{f}_1 = 0 \quad \text{at } \tilde{\eta} = 0, \qquad (21) \quad [(\mathbf{3.16})]$$

$$\left.\begin{aligned} U &= \big[(\beta_+ + \beta_-)U^o + (\beta_+ - \beta_-)U^e\big]/2,\\ W &= \big[(\beta_+ + \beta_-)W^o + (\beta_+ - \beta_-)W^e\big]/2,\\ Q &= \big[(\beta_+ + \beta_-)Q^o + (\beta_+ - \beta_-)Q^e\big]/2, \end{aligned}\right\} \qquad (22) \quad [(\mathbf{A.1})]$$

$$U^o = Q^o_r, \qquad U^e = Q^e_r, \qquad W^o = Q^o_z, \qquad W^e = Q^e_z, \qquad (23) \quad [(\mathbf{A.2})]$$

$$Q^o = z + \frac{2}{\Lambda}\sum_{m\,odd}\frac{I_0(l_m r)}{l_m^2 I_0(l_m)}\cos(l_m(z+\Lambda)), \qquad (24) \quad [(\mathbf{A.3})]$$

$$Q^e = \frac{3 - 2\Lambda^2 - 3r^2 + 6z^2}{12\Lambda} - \frac{2}{\Lambda}\sum_{m\,even}\frac{I_0(l_m r)}{l_m^2 I_0(l_m)}\cos(l_m(z+\Lambda)), \qquad (25) \quad [(\mathbf{A.4})]$$

$$3A_z - (4\omega + \mathrm{i}/2)A = 3B_z + (4\omega + \mathrm{i}/2)B = 0, \qquad (26) \quad [(\mathbf{3.11})]$$

1. The viscous-interface boundary layer (region (d))

Here we analyse region (d), in order to obtain the appropriate boundary conditions that apply in regions (b) and (c) near the interface. The thickness of this region is of the order of $\sqrt{C/\Omega}$. The solution in this region will be found in terms of the slow variable z and the fast (stretched) variables

$$\zeta = \sqrt{\Omega/C}\,(f - r) \quad \text{and } \eta = kz, \qquad (27)$$

that define a moving coordinate system attached to the interface. In addition, we shall use the new dependent variables u^*, w^*, p^* and f^* defined as

$$\begin{aligned} u &= f_t + (kf_\eta + f_z)w + \varepsilon k\sqrt{\Omega C}\,u^*, \qquad & w &= \varepsilon\Omega w^*\\ q &= 1 + (u^2 + w^2)/2 + \varepsilon k^2 p^*, & f &= 1 + \varepsilon f^* \end{aligned} \qquad (28)$$

to rewrite eqs. (1)-(3) and (6)-(8) as follows

$$u^*_\zeta - w^*_\eta = k^{-1}(w^*_z + \Omega^{-1} f^*_t) + \mathrm{HOT}, \qquad (29)$$

$$p^*_\zeta = k\sqrt{C/\Omega}\,\Omega^{-2} f^*_{tt} + \varepsilon k^2 \sqrt{C/\Omega}\big(2\Omega^{-1} f^*_{t\eta} w^* + \Omega^{-1} f^*_\eta w^*_t - f^*_\eta w^*_{\zeta\zeta}\big) + \mathrm{HOT}, \tag{30}$$

$$\Omega^{-1} w^*_t - w^*_{\zeta\zeta} + p^*_\eta = -k^{-1} p^*_z + \varepsilon k(u^* w^*_\zeta - w^* w^*_\eta - \Omega^{-2} f^*_{tt} f^*_\eta) \tag{31}$$
$$+ \varepsilon k^2 \sqrt{C/\Omega}\, f^*_\eta (u^*_{\zeta\zeta} - \Omega^{-1} u^*_t) + \mathrm{HOT},$$

$$u^* = 0, \qquad p^* + f^*_{\eta\eta} = -2k^{-1}(f^*_{\eta z} + \omega u^*_\zeta) + \mathrm{HOT}, \quad \text{at } \zeta = 0, \tag{32}$$

$$w^*_\zeta = k\sqrt{C/\Omega}\big[\Omega^{-1} f^*_{t\eta} + \varepsilon k(w^* f^*_{\eta\eta} - w^*_\eta f^*_\eta - u^*_\zeta f^*_\eta)\big] + \mathrm{HOT} \quad \text{at } \zeta = 0, \tag{33}$$

where ω is as defined in (18) and $\mathrm{HOT} = o\big(k^{-1} + \varepsilon k^2 \sqrt{C/\Omega}\big)$. Here we are assuming that u^*, w^*, p^* and f^*, and their derivatives with respect to ζ, z, η and $\tau = \Omega t$ are of order unity, and that $\varepsilon = o(\Omega^{-1})$ (see restrictions (9)-(10)).

In order to obtain the boundary conditions near the interface that are used in regions (a), (b) and (c), we shall consider the *oscillatory, resonant* part of the solution in this region, that is written as

$$\begin{aligned} u^* &= ie^{i\Omega t}\big(U^*_0 + k\sqrt{C/\Omega}\, U^*_1 + k^{-1} U^*_2\big) + \text{c.c.} + \mathrm{HOT} + \mathrm{NRT}, \quad w^* = ie^{i\Omega t}(W^*_0 + \cdots) + \cdots , \\ p^* &= e^{i\Omega t}(P^*_0 + \cdots) + \cdots , \quad f^* = e^{i\Omega t}(F_0 + k^{-1} F_2) + \text{c.c.} + \mathrm{HOT} + \mathrm{NRT}, \end{aligned} \tag{34}$$

where HOT and NRT stand for *higher order terms* than those displayed and *non-resonant terms* (depending on time as $\exp(im\Omega t)$, with $m \neq \pm 1$) respectively, and according to (16) and (17),

$$F_0 = A(z)e^{i\eta} + B(z)e^{-i\eta} + U(1,z), \quad \text{if } \Lambda - z \gg k^{-1} \quad \text{and } \Lambda + z \gg k^{-1}. \tag{35}$$

with $U(1,z)$ as given by (22)-(25). Notice that, for the sake of brevity, we are anticipating that the term of order $O\big(k\sqrt{C/\Omega}\big)$ in the expansion of f^* identically vanishes. The functions U^*_m, W^*_m and P^*_m, for $m = 0,1,2$, appearing in (34) do not depend on time and are calculated from the problems that result when (34) is inserted into (29)-(33) (and the coefficients of the monomials corresponding to the associated asymptotic orders are set to zero); in addition, the following matching requirements with the solution in regions (a), (b) and (c) must be taken into account: $W^*_{0\zeta} = W^*_{1\zeta\zeta} = W^*_{2\zeta} = 0$ at $\zeta = \infty$. For the sake of brevity we do not write these problems. Instead we only give here that part of their solution that will be needed below

$$\begin{aligned} &U^*_0 = -\zeta F_{0\eta\eta\eta\eta}, \quad W^*_0 = -F_{0\eta\eta\eta}, \quad W^*_1 = \big[-(1-i)\Gamma(\zeta) - \zeta\big] F_{0\eta}, \\ &U^*_1 = \big[2i - \sqrt{2}\, i\Gamma(\zeta) - \zeta^2/2\big] F_{0\eta\eta}, \\ &P^*_0 = -F_{0\eta\eta}, \quad P^*_1 = -\zeta F_0, \quad P^*_2 = -F_{2\eta\eta} - 2F_{0\eta z} + 2i\omega F_{0\eta\eta\eta\eta}, \end{aligned} \tag{36}$$

where the real constant ω is as given in (18) and

$$\Gamma(\zeta) = \sqrt{2}\exp\big[-(1+i)\zeta/\sqrt{2}\big]. \tag{37}$$

Now, when taking into account (27)-(28) and (34)-(37) we readily find the *asymptotic behavior*, as $\zeta \to \infty$, of the solution in this region, that is given by

$$\begin{aligned} u &= \varepsilon\Omega i e^{i\Omega t}\Big[F_0 - k\sqrt{C/\Omega}\,\zeta F_{0\eta\eta\eta\eta} + k^{-1}(F_2 + 2i\omega F_{0\eta\eta})\Big] + \text{c.c.} + \mathrm{HOT} + \mathrm{NRT}, \\ w &= -\varepsilon\Big[i\Omega e^{i\Omega t}\big(F_{0\eta\eta\eta} + k\sqrt{C/\Omega}\,\zeta F_{0\eta}\big) + \text{c.c.}\Big] + \mathrm{HOT} + \mathrm{NRT}, \\ q - 1 &= -\varepsilon k^2\Big[e^{i\Omega t}\big(-F_{0\eta\eta} + k\sqrt{C/\Omega}\,\zeta F_0 + k^{-1}(F_{2\eta\eta} + 2F_{0\eta z} - 2i\omega F_{0\eta\eta\eta\eta})\big) + \text{c.c.}\Big] + \mathrm{HOT} + \mathrm{NRT}. \end{aligned}$$

Matching of this asymptotic behavior for u and $q-1$ with the related asymptotic behavior, as $\xi \to 0$, of the solution in regions (b) and (c) (see (13) and (19)) readily justifies the boundary conditions (11) (at $r = 1$), (14), (15) and (20) (at $\xi = 0$).

In order to obtain the *steady boundary conditions* outside this boundary layer we must consider the steady part of the solution. For convenience we consider the steady parts of u^*, w^*_ζ and p^* that are written as

$$\begin{aligned} u^* &= \varepsilon k U_0^s + \text{HOT} + \text{NST}, \qquad w^*_\zeta = \varepsilon k^2 \sqrt{C/\Omega}\, H_1 + \text{HOT} + \text{NST}, \\ p^* &= \varepsilon k \big(P_0^s + k\sqrt{C/\Omega}\, P_1^s\big) + \text{HOT} + \text{NST}, \end{aligned} \tag{38}$$

where NST stands for *non-steady terms*. When inserting (28), (34), (36) and (38) into (29)-(33), we obtain three linear problems giving H_1, P_0^s and P_1^s, (U_0^s needs not being calculated). If the solution of the problem giving P_1^s is inserted into the problem giving H_1, then after some algebraic manipulations, we obtain

$$\begin{aligned} H_{1\zeta} &= \big[(1-i)(F_{0\eta}\bar{F}_{0\eta\eta\eta})_\eta + \sqrt{2}\zeta F_{0\eta}\bar{F}_{0\eta\eta\eta\eta}\big]\Gamma(\zeta) + \text{c.c.} \quad \text{in } \ 0 < \zeta < \infty, \\ H_1 &= iF_{0\eta\eta}\bar{F}_{0\eta\eta\eta} - 2iF_{0\eta}\bar{F}_{0\eta\eta\eta\eta} + \text{c.c.} \quad \text{at } \ \zeta = 0, \end{aligned} \tag{39}$$

where $\Gamma(\zeta)$ is as defined in (37). Notice that $H_{1\zeta} \to 0$ as $\zeta \to \infty$, as required by matching conditions with the outer region, where $H_1(\zeta = \infty)$ drives the leading order part of the steady flow, as it will be seen below. Integration of (39) yields

$$H_1 = -(6iF_{0\eta}\bar{F}_{0\eta\eta\eta\eta} + iF_{0\eta\eta}\bar{F}_{0\eta\eta\eta} + \text{c.c.}) \quad \text{at } \ \zeta = \infty. \tag{40}$$

Now we apply matching conditions with the solution in region (b). To this end, we only need to take into account that the solution in region (b) satisfies

$$\begin{aligned} u(f,z,t) &= u(1,z,t) + (f-1)u_r(1,z,t) + \text{HOT}, \\ w_r(f,z,t) &= w_r(1,z,t) + (f-1)w_{rr}(1,z,t) + \text{HOT}, \end{aligned}$$

and use (12), (18), (26), (27)-(28), (38) and (40) to obtain that the steady part of the solution in region (b) must be such that

$$\begin{aligned} u &= \varepsilon^2\Omega k\big[iF_{0\eta}\bar{F}_{0\eta\eta\eta} - iF_0\bar{u}_{1\xi} + \text{c.c.}\big] + \text{HOT} + \text{NST}, \\ w_\xi &= -\varepsilon^2\Omega k(6iF_{0\eta}\bar{F}_{0\eta\eta\eta\eta} + iF_{0\eta\eta}\bar{F}_{0\eta\eta\eta} + iF_0\bar{w}_{1\xi\xi} + \text{c.c.}) + \text{HOT} + \text{NST}, \end{aligned}$$

at $\xi = 0$, where ξ, u_1, w_1 and F_0 are as given by (12), (17) and (35). Then after some algebraic manipulations we obtain

$$\begin{aligned} u &= -\varepsilon^2\Omega k\big[i\bar{U}(1,z)(Ae^{i\eta} + Be^{-i\eta}) + \text{c.c.}\big] + \text{NST} \\ w_\xi &= -\varepsilon^2\Omega k[\bar{U}(1,z)(Ae^{i\eta} - Be^{-i\eta}) + \text{c.c. }] - 8(|B|^2 - |A|^2) + o(\varepsilon^2\Omega k^2) + \text{NST}, \end{aligned} \tag{41}$$

at $\xi = 0$, where as above, $U(1,z)$ is as given by (22)-(25). These equations provide the boundary conditions for the steady part of the solution in region (b), that is considered in §3.2 (NRV).

2. The remaining oscillatory viscous regions (e) and (f)

The main object of this section is to show that these two regions have a higher order effect in the steady flow in the bulk, which needs not being considered, and to justify the boundary conditions (11) at $z = \pm\Lambda$.

The Stokes boundary layer near $z = \Lambda$ may be described in terms of the streched variable

$$\eta^* = [\Lambda - z + \varepsilon(\beta_+ e^{i\Omega t} + \text{c.c.})]\sqrt{\Omega/C} \tag{42}$$

that define a moving coordinate system attached to the disk. The solution in this region may be written as

$$\begin{aligned} u &= \varepsilon\Omega(\tilde{U}_1(r,\eta^*)e^{i\Omega t} + \text{c.c.}) + \varepsilon^2\Omega\tilde{u}_2(r,\eta^*,t) + \text{HOT}, \\ w &= \varepsilon\left[i\Omega\beta_+ + \sqrt{\Omega C}\,\widetilde{W}_1(r,\eta^*)\right]e^{i\Omega t} + \text{c.c.} + \varepsilon^2\sqrt{\Omega C}\tilde{w}_2(r,\eta^*,t) + \text{HOT}, \\ q - 1 &= \varepsilon\Omega^2[\tilde{Q}_1(r,\eta^*)e^{i\Omega t} + \text{c.c.}] + \varepsilon^2\Omega^2\tilde{q}_2(r,\eta^*,t) + \text{HOT}, \end{aligned} \tag{43}$$

where HOT stands for higher order terms than those displayed. The functions $\tilde{U}_1$, $\tilde{Q}_1$, $\widetilde{W}_1$, $\tilde{u}_2$, $\tilde{q}_2$ and $\tilde{w}_2$ may be calculated from the equations and boundary conditions that result when Eqs. (42)-(43) are replaced into (1)-(3) and (4). But none of these functions will be needed below; instead, we shall only need to take into account that these functions are $O(1)$ quantities in this region.

Now, in order to apply matching conditions with the outer solution we only need to take into account (42)-(43) and the fact that the solution in regions (a) and (c) satisfy

$$\begin{aligned} u(r,\Lambda + \varepsilon(\beta_+ e^{i\Omega t} + \text{c.c.}),t) &= u(r,\Lambda) + \varepsilon(\beta_+ e^{i\Omega t} + \text{c.c.})u_r(r,\Lambda,t) + \text{HOT}, \\ w(r,\Lambda + \varepsilon(\beta_+ e^{i\Omega t} + \text{c.c.}),t) &= \cdots, \quad q(r,\Lambda + \varepsilon(\beta_+ e^{i\Omega t} + \text{c.c.}),t) - 1 = \cdots, \end{aligned}$$

to obtain the appropriate boundary conditions at $z = \Lambda$ for the solution in regions (a) and (c). After a similar analysis of the Stokes boundary layer near $z = -\Lambda$, we obtain the following boundary conditions

$$w = \varepsilon(i\Omega\beta_\pm e^{i\Omega t} + \text{c.c.}) + \text{HOT} \qquad \text{at} \quad z = \pm\Lambda \tag{44}$$

$$u = O(\varepsilon^2\Omega) + \text{NST}, \qquad w = O(\varepsilon^2\Omega) + \text{NST} \quad \text{at} \quad z = \pm\Lambda \tag{45}$$

for the oscillatory and steady parts of the solution in regions (a) and (c) respectively. Here HOT and NST stand, respectively, for higher order terms than those displayed and non-steady terms, depending on the time variable as $\exp(im\Omega t)$, with $m \neq 0$. Eq. (44) justifies the boundary conditions (11) and (21) at $z = \pm\Lambda$ that were used in the analysis of regions (a) and (c), and the estimates in Eq. (45) are used in §3.3 (NRV) to derive the asymptotic model.

The oscillatory viscous regions (f) provide higher order terms in the analysis of region (c), that are not considered in this paper (NRV).