The additional material to the paper "Wave pattern formation in a fluid annulus with a radially – vibrating inner cylinder" by T.S. Krasnopolskaya and G.J.F. van Heijst

Appendix B. An extended analysis of the non-axisymmetric resonant case, including secondary modes

Here we will take into account the existence of the secondary modes. Therefore, we approximate the free surface displacement by the expression

$$\zeta \approx \frac{1}{N_{nm}}\zeta_{nm}(t)\psi^c_{nm}(r,\theta)+\zeta_{00}(t)+\sum_{h=1}\frac{\zeta_{0h}(t)\psi_{0h}(r)}{N_{0h}}+\sum_{h=1}\frac{1}{N_{2n\,h}}\zeta_{2n\,h}(t)\psi^c_{2n\,h}(r,\theta). \quad \text{(B 1)}$$

Besides the resonance modes, in (B.1) account is taken of secondary modes, which are excited due to the nonlinear coupling of the vibrational modes and to the direct non-resonant excitation. So for the general case we seek the functions $\zeta^{c,s}_{nm}$, ζ_{0h} and $\zeta^{c,s}_{2n\,h}$ in the following form (see also Miles, 1984c):

$$\zeta_{nm} = \epsilon_1^{1/2}\lambda_1[p_2(\tau_1)cos\frac{(\omega t)}{2} + q_2(\tau_1)sin\frac{(\omega t)}{2}] \quad \text{(B 2)}$$

for the dominant modes and

$$\zeta_{0h} = \epsilon_1\lambda_1[A_{0h}(\tau_1)cos\omega t + B_{0h}(\tau_1)sin\omega t + C_{0h}(\tau_1)],$$

$$\zeta_{2n\,h} = \epsilon_1\lambda_1[A_{2n\,h}(\tau_1)cos\omega t + B_{2n\,h}(\tau_1)sin\omega t + C_{2n\,h}(\tau_1)], \quad \text{(B 3)}$$

for the secondary modes, where λ_1, ϵ_1 and τ_1 are as before in (3.55) - (3.56); the variables $p_2(\tau_1), q_2(\tau_1), A_{bh}(\tau_1), B_{bh}(\tau_1), C_{bh}(\tau_1)$ $(b = 0, 2n)$ are slowly varying dimensionless amplitudes of the dominant and the secondary modes.

For free-surface vibrations of the form (B.1), it can be assumed that the velocity potential ϕ_1 contains the terms

$$\phi_1 = \phi_{nm}(t)\psi^c_{nm}(r,\theta)\frac{\cosh k_{nm}(x+d)}{N_{nm}\cosh k_{nm}d} + \sum_{h=1}\phi_{0h}(t)\psi_{0h}(r)\frac{\cosh k_{0h}(x+d)}{N_{0h}\cosh k_{0h}d} +$$

$$\sum_{h=1}\phi_{2n\,h}(t)\psi^c_{2n\,h}(r,\theta)\frac{\cosh k_{2n\,h}(x+d)}{N_{2n\,h}\cosh k_{2n\,h}d}. \quad \text{(B 4)}$$

Using average over the surface $x = 0$ values of the terms of the nonlinear boundary condition (3.61) instead of the linear condition (3.11a) the potential ϕ_0 can be written in the form:

$$\phi_0 = \frac{4\epsilon_1 g R_1}{\pi\omega_{nm}(R_2^2-R_1^2)}\sin\omega t\left(\frac{r^2}{2} - R_2^2\ln r\right) + \frac{8\epsilon_1 g R_1 d}{\pi\omega_{nm}(R_2^2-R_1^2)}\sin\omega t\frac{(d+x)^2}{2d}, \quad \text{(B 5)}$$

and

$$\zeta_{00} = \frac{4\epsilon_1 d g R_1}{\pi\omega^2_{nm}(R_2^2-R_1^2)}\cos\omega t - \frac{1}{2}K_{00}(\zeta_{nm})^2 \quad \text{(B 6)}$$

where

$$K_{00} = \frac{\gamma_{10}}{N^2_{nm}(R_2^2-R_1^2)}[k^2_{nm}\int_{R_1}^{R_2}(\chi'_{nm})^2 r dr - k^2_{nm}N^2_{nm} + n^2\int_{R_1}^{R_2}\chi^2_{nm}\frac{1}{r}dr]$$

The potential ϕ_2 has the same form as in (3.59). Substitution of ϕ_0 and ϕ_2 in to the kinematic boundary condition (3.61) gives a possibility to express the amplitudes of ϕ_1 in the following way:

$$\phi_{nm}(t) = \gamma_{10}\dot{\zeta}_{nm}[1 - \gamma_{11}\zeta_{nm}^2 + \gamma_{10}\sum_{h=1} M_{0h}\zeta_{0h} + \gamma_{10}\sum_{h=1} M_{2n\,h}\zeta_{2n\,h} - k_{nm}^2\gamma_{10}\zeta_{00}]$$
$$+\gamma_{10}\zeta_{nm}[\sum_{h=1}\phi_{0h}(t)L_{0h} + \sum_{h=1}\phi_{2n\,h}(t)L_{2n\,h} - \epsilon_1 D\sin\omega t]$$
$$- \frac{\gamma_{10}}{d}\zeta_{nm}[\dot{\zeta}_{00} + K_{00}\zeta_{nm}\dot{\zeta}_{nm}] + O(\epsilon_1^2); \quad \text{(B 7)}$$
$$\phi_{0h}(t) = \gamma_{0h}\dot{\zeta}_{0h} + K_{0h}\zeta_{nm}\dot{\zeta}_{nm} + O(\epsilon_1^2);$$
$$\phi_{2n\,h}(t) = \gamma_{2n\,h}\dot{\zeta}_{2n\,h} + K_{2n\,h}\zeta_{nm}\dot{\zeta}_{nm} + O(\epsilon_1^2). \quad \text{(B 8)}$$

Where γ_{10}, γ_{11} and D are given in (3.63), and

$$M_{0h} = \frac{\pi k_{nm}}{N_{nm}^2 N_{0h}}[k_{0h}\int_{R_1}^{R_2}\chi_{0h}'\chi_{nm}'\chi_{nm}r\,dr - k_{nm}\int_{R_1}^{R_2}\chi_{0h}\chi_{nm}^2 r\,dr],$$
$$M_{2n\,h} = \frac{\pi}{2N_{nm}^2 N_{2n\,h}}[k_{2n\,h}k_{nm}\int_{R_1}^{R_2}\chi_{2n\,h}'\chi_{nm}'\chi_{nm}r\,dr - k_{nm}^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2 r\,dr$$
$$+2n^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2\frac{1}{r}dr];$$
$$L_{0h} = \frac{\pi k_{0h}}{N_{nm}^2 N_{0h}}[k_{nm}\int_{R_1}^{R_2}\chi_{0h}'\chi_{nm}'\chi_{nm}r\,dr - k_{0h}\int_{R_1}^{R_2}\chi_{0h}\chi_{nm}^2 r\,dr];$$
$$L_{2n\,h} = \frac{\pi}{2N_{nm}^2 N_{2n\,h}}[k_{2n\,h}k_{nm}\int_{R_1}^{R_2}\chi_{2n\,h}'\chi_{nm}'\chi_{nm}r\,dr - k_{2n\,h}^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2 r\,dr$$
$$+ 2n^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2\frac{1}{r}dr]; \quad \text{(B 9)}$$
$$\gamma_{0h} = [k_{0h}\tanh(k_{0h}d)]^{-1}; \qquad \gamma_{2n\,h} = [k_{2n\,h}\tanh(k_{2n\,h}d)]^{-1};$$
$$K_{0h} = \frac{\pi\gamma_{10}\gamma_{0h}}{N_{nm}^2 N_{0h}}[k_{nm}^2\int_{R_1}^{R_2}\chi_{0h}(\chi_{nm}')^2 r\,dr - k_{nm}^2\int_{R_1}^{R_2}\chi_{0h}\chi_{nm}^2 r\,dr + n^2\int_{R_1}^{R_2}\chi_{0h}\chi_{nm}^2\frac{1}{r}dr];$$
$$K_{2n\,h} = \frac{\pi\gamma_{10}\gamma_{2n\,h}}{2N_{nm}^2 N_{2n\,h}}[k_{nm}^2\int_{R_1}^{R_2}\chi_{2n\,h}(\chi_{nm}')^2 r\,dr - k_{nm}^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2 r\,dr$$
$$+ n^2\int_{R_1}^{R_2}\chi_{2n\,h}\chi_{nm}^2\frac{1}{r}dr]. \quad \text{(B 10)}$$

From the dynamic boundary condition (3.64), in the same way as before, we also get

$$A_{0h} = \frac{\lambda_1}{\gamma_{0h}(\omega_{0h}^2 - \omega^2)}\frac{\omega^2}{8}(E_{0h} + Q_{0h})(p_2^2 - q_2^2) +$$
$$\frac{16\pi g}{\gamma_{0h}\lambda_1(\omega_{0h}^2 - \omega^2)}\sum_{l=1}^{\infty}\frac{(-1)^l\eta b_{0lh}}{(\alpha_l^2 - \eta^2)\alpha_l d\hat{\chi}_{0\,l}'(\alpha_l R_1)} - \frac{16gR_1a_{0h}}{\gamma_{0h}\lambda_1(\omega_{0h}^2 - \omega^2)(R_2^2 - R_1^2)}; \quad \text{(B 11)}$$
$$B_{0h} = \frac{\lambda_1}{\gamma_{0h}(\omega_{0h}^2 - \omega^2)}\frac{\omega^2}{4}(E_{0h} + Q_{0h})(p_2q_2); \qquad C_{0h} = \frac{\lambda_1}{g}\frac{\omega^2}{8}(E_{0h} + Q_{0h})(p_2^2 + q_2^2);$$

$$A_{2n\,h} = \frac{\lambda_1}{\gamma_{2n\,h}(\omega_{2n\,h}^2 - \omega^2)}\frac{\omega^2}{8}(E_{2n\,h} + Q_{2n\,h})(p_2^2 - q_2^2);$$

$$B_{2n\,h} = \frac{\lambda_1}{\gamma_{2n\,h}(\omega_{2n\,h}^2 - \omega^2)}\frac{\omega^2}{4}(E_{2n\,h} + Q_{2n\,h})(p_2 q_2);$$

$$C_{2n\,h} = \frac{\lambda_1}{g}\frac{\omega^2}{8}(E_{2n\,h} + Q_{2n\,h})(p_2^2 + q_2^2); \tag{B 12}$$

and derive the following evolution equations:

$$\frac{dp_2}{d\tau_1} = -\hat{\alpha}p_2 - [\beta_1 - \beta_2 + \frac{A_1 + A_2}{2}(p_2^2 + q_2^2)]q_2 + (\beta_3 + \beta_4)q_2; \tag{B.13a}$$

$$\frac{dq_2}{d\tau_1} = -\hat{\alpha}q_2 + [\beta_1 - \beta_2 + \frac{A_1 + A_2}{2}(p_2^2 + q_2^2)]p_2 + (\beta_3 + \beta_4)p_2. \tag{B.13b}$$

Here $\hat{\alpha}, \beta, \beta_2, \beta_3, A_1$ are the same constant coefficients as in (3.65). Other constants are the following:

$$E_{0h} = K_{0h} + \frac{\pi}{N_{0h}N_{nm}^2}\int_{R_1}^{R_2} \chi_{0h}\chi_{nm}^2 r dr;$$

$$E_{2n\,h} = K_{2n\,h} + \frac{\pi}{2N_{2n\,h}N_{nm}^2}\int_{R_1}^{R_2} \chi_{2n\,h}\chi_{nm}^2 r dr;$$

$$Q_{0h} = K_{0h} + \frac{\pi\gamma_{10}^2}{2N_{0h}N_{nm}^2}[k_{nm}^2\int_{R_1}^{R_2} \chi_{0h}(\chi_{nm}')^2 r dr + \gamma_{10}^{-2}\int_{R_1}^{R_2} \chi_{0h}\chi_{nm}^2 r dr$$
$$+ n^2\int_{R_1}^{R_2} \chi_{0h}\chi_{nm}^2\frac{1}{r}dr];$$

$$Q_{2n\,h} = K_{2n\,h} + \frac{\pi\gamma_{10}^2}{4N_{2n\,h}N_{nm}^2}[k_{nm}^2\int_{R_1}^{R_2} \chi_{2n\,h}(\chi_{nm}')^2 r dr + \gamma_{10}^{-2}\int_{R_1}^{R_2} \chi_{2n\,h}\chi_{nm}^2 r dr$$
$$+ n^2\int_{R_1}^{R_2} \chi_{2n\,h}\chi_{nm}^2\frac{1}{r}dr];$$

$$A_2 = \lambda_1^2\sum_{h=1}[K_{0h}(L_{0h} - R_{0h} + 2\gamma_{0h}^{-1}\gamma_{10}^{-1}S_{0h}) + K_{2n\,h}(L_{2n\,h} - R_{2n\,h} + 2\gamma_{0h}^{-1}\gamma_{10}^{-1}S_{2n\,h})] +$$
$$\lambda_1\omega^2\sum_{h=1}[\frac{W_{1h}(E_{0h} + Q_{0h})}{4\gamma_{0h}(\omega_{0h}^2 - \omega^2)} + \frac{S_{3h}(E_{0h} + Q_{0h})}{2g} + \frac{W_{2h}(E_{2n\,h} + Q_{2n\,h})}{4\gamma_{2n\,h}(\omega_{2n\,h}^2 - \omega^2)} + \frac{S_{5h}(E_{2n\,h} + Q_{2n\,h})}{2g}]$$
$$+ (k_{nm}^2\gamma_{10} + 3\gamma_{10}^{-1})\frac{\lambda_1^2}{2}K_{00} \tag{B.14}$$

$$\beta_4 = \frac{4dgR_1}{\pi\omega_{nm}^2(R_2^2 - R_1^2)}\left(k_{nm}^2\gamma_{10} + \frac{5}{\gamma_{10}} - \frac{2}{d}\right)$$
$$+ \frac{8\pi g\omega}{Sd\lambda_1}\sum_{h=1}[\frac{W_{1h}}{\gamma_{0h}N_{0h}\omega_{nl}(\omega_{0h}^2 - \omega^2)}\sum_{l=1}^{\infty}\frac{(-1)^l\eta}{(\alpha_l^2 - \eta^2)\alpha_l\hat{\chi}_{0\,l}'(\alpha_l R_1)}b_{0lh}; \tag{B.15}$$

$$W_{1h} = \lambda_1(S_{3h} - 2R_{3h}); \qquad W_{2h} = \lambda_1(S_{5h} - 2R_{5h});$$

$$S_{3h} = \gamma_{10}M_{0h} + 5\gamma_{10}^{-1}S_{0h} + 4\gamma_{0h}L_{0h};$$

$$S_{5h} = \gamma_{10} M_{2n\,h} + 5\gamma_{10}^{-1} S_{02n\,h} + 4\gamma_{2n\,h} L_{02n\,h}; \qquad R_{3h} = \gamma_{10} M_{0h} + \gamma_{0h}(L_{0h} + R_{0h});$$

$$R_{5h} = \gamma_{10} M_{2n\,h} + \gamma_{2n\,h}(L_{2n\,h} + R_{2n\,h});$$

$$S_{0h} = \frac{\pi}{N_{nm}^2 N_{0h}} \int_{R_1}^{R_2} \chi_{0h} \chi_{nm}^2 r dr; \qquad S_{2n\,h} = \frac{\pi}{2N_{nm}^2 N_{2n\,h}} \int_{R_1}^{R_2} \chi_{2n\,h} \chi_{nm}^2 r dr;$$

$$R_{0h} = \frac{\pi}{N_{nm}^2 N_{0h}} [k_{nm} k_{0h} \int_{R_1}^{R_2} \chi_{0h}' \chi_{nm}' \chi_{nl} r dr + \gamma_{10}^{-1} \gamma_{0h}^{-1} \int_{R_1}^{R_2} \chi_{0h} \chi_{nm}^2 r dr];$$

$$R_{2n\,h} = \frac{\pi}{2N_{nm}^2 N_{2n\,h}} [k_{nm} k_{2n\,h} \int_{R_1}^{R_2} \chi_{2n\,h}' \chi_{nm}' \chi_{nm} r dr + \gamma_{10}^{-1} \gamma_{2n\,h}^{-1} \int_{R_1}^{R_2} \chi_{2n\,h} \chi_{nm}^2 r dr$$

$$+ 2n^2 \int_{R_1}^{R_2} \chi_{2n\,h} \chi_{nm}^2 \frac{1}{r} dr]. \tag{B.16}$$

For the single dominant mode with amplitudes p_2 and q_2 the system (B.13) has a solution corresponding to harmonic vibrations (i.e. $dp_2/d\tau_1 = 0$ and $dq_2/d\tau_1 = 0$) for which

$$p_2^2 + q_2^2 = \frac{2}{A_1 + A_2} \{-(\beta_1 - \beta_2) \pm [(\beta_3 + \beta_4)^2 - \hat{\alpha}^2]^{1/2}\}. \tag{B.17}$$

Thus, the difference between the secondary modes model and the resonant modes model consists of: (I) a change in the amplitude of parametric excitation at the value of β_4 which is conditioned by the direct excitation of the mean level oscillations and the axisymmetric waves and then by the energy transformation from them into the cross-waves; (II) a change in the coefficient of nonlinearity of the system, when A_2 depends on indirect excitation of the mean level variations and axisymmetric modes ψ_{0h} and non-symmetric modes $\psi_{2n\,h}$. This coefficient can influence to the stability of the resonant cross-wave, as was shown by Becker and Miles (1991). However, the value of β_4 may be negligibly small by the same "geometrical" reason as for the value of $\zeta_{00}(t)$.