A detailed derivation of the amplitude equation that apply in the *Weakly Non-linear Oscillations of Nearly Inviscid Liquid Bridges* (J.A. Nicolás and J.M. Vega, Journal of Fluid Mechanics, vol. 328 (1996), pp. 95-128, to be referred to as NV).

A more detailed derivation than that in [NV, §§ 3.1-3.4] is given below. In order to make the derivation self-contained, some equations in [NV, §§ 2-3] are first listed:

$$u_r + r^{-1}u + w_z = 0, \tag{1}$$

$$u_t + w(u_z - w_r) = -q_r + C(u_{rr} + r^{-1}u_r - r^{-2}u + u_{zz}), \tag{2}$$

$$w_t + u(w_r - u_z) = -q_z + C(w_{rr} + r^{-1}w_r + w_{zz}), \tag{3}$$

$$u = 0, \qquad w = h'_\pm(t) \quad \text{at} \quad z = \pm\Lambda + h_\pm(t), \tag{4}$$

$$u = w_r = q_r = 0 \qquad \text{at} \quad r = 0, \tag{5}$$

$$u = f_t + f_z w \qquad \text{at} \quad r = f, \tag{6}$$

$$(w_r + u_z)(1 - f_z^2) + 2(u_r - w_z)f_z = 0 \qquad \text{at} \quad r = f, \tag{7}$$

$$q - \frac{u^2 + w^2}{2} + \frac{ff_{zz} - 1 - f_z^2}{f(1 + f_z^2)^{3/2}} = 2C\frac{u_r - (w_r + u_z)f_z + w_z f_z^2}{1 + f_z^2} \qquad \text{at} \quad r = f \tag{8}$$

$$f = 1 \qquad \text{at} \quad z = \pm\Lambda + h_\pm(t). \tag{9}$$

$$\int_{-\Lambda+h_-}^{\Lambda+h_+} f(z,t)^2 dz = 2\Lambda. \tag{10}$$

$$h_\pm(t) = \beta_\pm \mu \exp[i(\Omega + \omega_\pm \delta)t] + \text{c.c.}, \tag{11}$$

$$C \ll 1, \qquad \mu \ll 1 \quad \text{and} \quad \delta \ll 1, \tag{12}$$

$$u = \varepsilon(AU_0 e^{i\Omega t} + \text{c.c.}) + \varepsilon\sqrt{C}u_1 + \varepsilon^2 u_2 + \varepsilon^2\sqrt{C}u_3 + \varepsilon C u_4 + \varepsilon^3 u_5 + \mu u_6 + \text{HOT}$$
$$w = \cdots, \qquad q - 1 = \cdots, \qquad f - 1 = \cdots, \tag{13}$$

$$0 < \varepsilon \ll 1. \tag{14a}$$

$$\int_0^{2\pi/\Omega}\int_{-\Lambda}^{\Lambda}\int_0^1 (U_0 u_k + W_0 w_k)e^{-i\Omega t} r dr dz dt = 0 \qquad \text{for all } k \geq 1, \tag{14b}$$

$$\varepsilon dA/dt = -\left[(1+i)\alpha_1\sqrt{C} + \alpha_2 C\right]\varepsilon A + i\alpha_3\varepsilon^3 A|A|^2 + i\mu\left(\alpha_4^+\beta_+ e^{i\omega_+\delta t} - \alpha_4^-\beta_- e^{i\omega_-\delta t}\right) + \text{HOT}, \tag{15a}$$

$$\delta\varepsilon dA/d\tau = H_1\varepsilon\sqrt{C} + H_2\varepsilon^2 + H_3\varepsilon C + H_4\varepsilon^2\sqrt{C} + H_5\varepsilon^3 + H_6\mu + \text{HOT}, \tag{15b}$$

$$\tau = \delta t. \tag{16}$$

1 The solution in the bulk

We shall only need to calculate the first three terms in the expansions (13). The (linear) inviscid eigenmode (U_0, W_0, Q_0, F_0) is given by

$$U_{0r} + r^{-1}U_0 + W_{0z} = i\Omega U_0 + Q_{0r} = i\Omega W_0 + Q_{0z} = 0, \tag{17}$$

$$W_0 = 0 \quad \text{at} \quad z = \pm\Lambda, \qquad U_0 = W_{0r} = 0 \quad \text{at} \quad r = 0, \tag{18}$$

$$U_0 - i\Omega F_0 = Q_0 + F_0 + F_0'' = 0 \quad \text{at} \quad r = 1, \tag{19}$$

$$F_0(\pm\Lambda) = \int_{-\Lambda}^{\Lambda} F_0(z)dz = 0. \tag{20}$$

The terms of orders $\varepsilon\sqrt{C}$ and ε^2 in (13) are given by

$$u_{kr} + r^{-1}u_k + w_{kz} = 0, \tag{21}$$

$$u_{kt} + q_{kr} + (H_k U_0 e^{i\Omega t} + \text{c.c.}) = w_{kt} + q_{kz} + (H_k W_0 e^{i\Omega t} + \text{c.c.}) = 0, \tag{22}$$

$$w_k = G_k^{\pm} \quad \text{at} \quad z = \pm\Lambda, \qquad u_k = w_{kr} = q_{kr} = 0 \quad \text{at} \quad r = 0, \tag{23}$$

$$u_k = \phi_k, \qquad q_k = \psi_k \quad \text{at} \quad r = 1, \tag{24}$$

$$f_k = 0 \quad \text{at} \quad z = \pm\Lambda, \qquad \int_{-\Lambda}^{\Lambda} f_k dz + \gamma_k = 0 \tag{25}$$

for $k = 1$ and 2, where

$$\gamma_1 = 0, \qquad \gamma_2 = \int_{-\Lambda}^{\Lambda} (A^2 F_0^2 e^{i\Omega t} + \text{c.c.} + 2|AF_0|^2)dz/2. \tag{26}$$

Eqs. (21), (22) and (25) and the boundary conditions at $r = 0$ are obtained upon substitution of (13) and (15b) into (1)-(3), (5) and (10), when taking into account that $U_{0z} \equiv W_{0r}$. The remaining boundary conditions (and the functions $G_k^{\pm}$, ϕ_k and ψ_k) will be obtained below by applying matching conditions with the Stokes and the interface boundary layers.

2 The solution in the Stokes boundary layers

For the sake of brevity we give details only for the boundary layer near $z = \Lambda$, where we use the stretched coordinate

$$\xi = [z - \Lambda - h_+(t)]/\sqrt{C},$$

with h_+ as given in (11). We seek the expansions

$$\begin{aligned} u &= \varepsilon A\tilde{U}_0 e^{i\Omega t} + \text{c.c.} + \varepsilon\sqrt{C}\tilde{u}_1 + \varepsilon^2\tilde{u}_2 + \text{HOT}, \\ w &= h_+' + \sqrt{C}\left(\varepsilon A\widetilde{W}_0 e^{i\Omega t} + \text{c.c.} + \text{HOT}\right) \\ q - 1 &= \varepsilon A\tilde{Q}_0 e^{i\Omega t} + \text{c.c.} + \varepsilon\sqrt{C}\tilde{q}_1 + \varepsilon^2\tilde{q}_2 + \text{HOT} + O(\mu). \end{aligned} \tag{27}$$

Substitution of (27) and (15b) into (1)-(4) yields

$$\begin{aligned} \tilde{Q}_{0\xi} &= i\Omega\tilde{U}_0 - \tilde{U}_{0\xi\xi} + \tilde{Q}_{0r} = \tilde{U}_{0r} + r^{-1}\tilde{U}_0 + \widetilde{W}_{0\xi} \\ &= \tilde{u}_{1t} - \tilde{u}_{1\xi\xi} + \tilde{q}_{1r} + \left(H_1\tilde{U}_0 e^{i\Omega t} + \text{c.c.}\right) = \tilde{q}_{1\xi} = 0, \end{aligned} \tag{28}$$

$$\tilde{q}_{2\xi} - \left(A\tilde{U}_0 e^{i\Omega t} + \text{c.c.}\right)\left(A\tilde{U}_{0\xi} e^{i\Omega t} + \text{c.c.}\right) = 0, \tag{29}$$

$$\tilde{u}_{2t} - \tilde{u}_{2\xi\xi} + \tilde{q}_{2r} + \left(H_2\tilde{U}_0 e^{i\Omega t} + \text{c.c.}\right) + \left(A\widetilde{W}_0 e^{i\Omega t} + \text{c.c.}\right)\left(A\tilde{U}_{0\xi} e^{i\Omega t} + \text{c.c.}\right) = 0, \tag{30}$$

$$\tilde{U}_0 = \widetilde{W}_0 = \tilde{u}_1 = \tilde{u}_2 = 0 \qquad \text{at} \quad \xi = 0, \tag{31}$$

and integration of (28)-(31) leads to

$$\tilde{U}_0 = K_0^+(r)(1 - \Gamma), \qquad \widetilde{W}_0 = -\left(dK_0^+/dr + r^{-1}K_0^+\right)\left[\xi + (1 - i)(1 - \Gamma)/\sqrt{2\Omega}\right], \tag{32}$$

$$\tilde{u}_1 = \left[K_1^+(1 - \Gamma) - (1 - i)H_1 K_0^+ \xi\Gamma/2\sqrt{2\Omega}\right] Ae^{i\Omega t} + \text{c.c.}, \tag{33}$$

$$\begin{aligned}\tilde{u}_2 =& \tilde{U}_{22}(r,\xi,\Lambda)e^{2i\Omega t} + \left[K_2^+(1-\Gamma) - (1-i)H_2K_0^+\xi\Gamma/2\sqrt{2\Omega}\right]e^{i\Omega t} \\ &+ |A|^2\bar{K}_0^+(dK_0^+/dr + r^{-1}K_0^+)\left[i\left(|\Gamma|^2-1\right) + 2(1-2i)(\bar{\Gamma}-1) + (1+i)\sqrt{2\Omega}\xi\bar{\Gamma}\right]\Big/2\Omega \\ &+ |A|^2\bar{K}_0^+(dK_0^+/dr)\left[|\Gamma|^2 - 1 + 2i(\Gamma-\bar{\Gamma})\right]/2\Omega + \text{c.c.}\end{aligned} \tag{34}$$

where overbars and c.c. stand for the complex conjugate, $K_s^+ = K_s^+(r)$ is an arbitrary function for $s = 0$, 1 and 2, that is to be calculated, the function $\Gamma = \Gamma(\xi)$ is given by

$$\Gamma(\xi) = \exp\left[(1+i)\sqrt{\Omega/2}\xi\right], \tag{35}$$

and the function $\tilde{U}_{22}$ is not calculated (because it will be not needed in the sequel).

Now, the functions K_0^+, K_1^+ and K_2^+, and the functions G_1^+ and G_2^+ appearing in the boundary conditions (23) are obtained by applying matching conditions between the solutions in the bulk (13) and in this boundary layer (27). After applying a similar procedure to the boundary layer near $z = -\Lambda$ we obtain

$$U_0(r,\pm\Lambda) = K_0^\pm(r), \qquad u_1(r,\pm\Lambda) = AK_1^\pm(r)e^{i\Omega t} + \text{c.c.}, \tag{36}$$

$$G_1^\pm = \pm\left[(1-i)AW_{0z}(r,\pm\Lambda)e^{i\Omega t} + \text{c.c.}\right]/\sqrt{2\Omega}, \qquad G_2^\pm = 0, \tag{37}$$

$$\begin{aligned}u_2 = & K_2^\pm(r)e^{i\Omega t} + K^\pm(r)e^{2i\Omega t} + \text{c.c.} \\ & - |A|^2\left[3(1-i)\bar{U}_0U_{0r} + \text{c.c.} + 4r^{-1}|U_0|^2\right]/2\Omega, \quad \text{at} \quad z = \pm\Lambda,\end{aligned} \tag{38}$$

where the first continuity equation (17) has been taken into account and u_2 has been also obtained for convenience. Again the functions $K^\pm$ will not appear in the sequel and are not considered here.

3 The solution in the interface boundary layer

Here we use the stretched coordinate

$$\eta = [r - f(z,t,\tau)]/\sqrt{C},$$

and seek the expansions

$$\begin{aligned} u &= f_t + \delta f_\tau + f_z w + \sqrt{C}\left(\varepsilon AU_0^* e^{i\Omega t} + \text{c.c.} + \varepsilon\sqrt{C}u_1^* + \text{HOT}\right), \\ w &= \varepsilon AW_0^* e^{i\Omega t} + \text{c.c.} + \varepsilon\sqrt{C}w_1^* + \varepsilon^2 w_2^* + \varepsilon^2\sqrt{C}w_3^* + \text{HOT}, \\ q - 1 &= (u^2+w^2)/2 + \left(\varepsilon AP_0^* e^{i\Omega t} + \text{c.c.}\right) + \varepsilon\sqrt{C}p_1^* + \varepsilon^2 p_2^* + \varepsilon^2\sqrt{C}p_3^* + \text{HOT}.\end{aligned} \tag{39}$$

Substitution of (39) and (15b) into (1)-(3) and (6)-(8) yields the following equations

$$P_{0\eta}^* = i\Omega W_0^* - W_{0\eta\eta}^* + P_{0z}^* = U_{0\eta}^* + W_{0z}^* + i\Omega F_0 = 0, \tag{40}$$

$$p_{1\eta}^* = \Omega^2\left(AF_0e^{i\Omega t} + \text{c.c.}\right), \tag{41}$$

$$w_{1t}^* - w_{1\eta\eta}^* + p_{1z}^* + \left[(H_1W_0^* - AW_{0\eta}^*)e^{i\Omega t} + \text{c.c.}\right] = 0, \tag{42}$$

$$u_{1\eta}^* + w_{1z}^* + f_{1t} + \left[(H_1F_0 - i\Omega\eta AF_0 + AU_0^*)e^{i\Omega t} + \text{c.c.}\right] = p_{2\eta}^* = 0, \tag{43}$$

$$\begin{aligned} & w_{2t}^* - w_{2\eta\eta}^* + p_{2z}^* + (H_2W_0^*e^{i\Omega t} + \text{c.c.}) + (AU_0^*e^{i\Omega t} + \text{c.c.})(AW_{0\eta}^*e^{i\Omega t} + \text{c.c.}) \\ & \quad + (AW_0^*e^{i\Omega t} + \text{c.c.})(AW_{0z}^*e^{i\Omega t} + \text{c.c.}) - \Omega^2\left(AF_0'e^{i\Omega t} + \text{c.c.}\right)\left(AF_0e^{i\Omega t} + \text{c.c.}\right) = 0,\end{aligned} \tag{44}$$

$$\begin{aligned} & p_{3\eta}^* + \left(2i\Omega H_2F_0e^{i\Omega t} + \text{c.c.}\right) + f_{2tt} + 2(AW_0^*e^{i\Omega t} + \text{c.c.})(i\Omega AF_0'e^{i\Omega t} + \text{c.c.}) \\ & \quad + (AF_0'e^{i\Omega t} + \text{c.c.})\left[A(i\Omega W_0^* - W_{0\eta\eta}^*)e^{i\Omega t} + \text{c.c.}\right] = 0,\end{aligned} \tag{45}$$

$$w_{3t}^* - w_{3\eta\eta}^* + p_{3z}^* - w_{2\eta}^* + \left[H_1(\partial w_2^*/\partial A) + H_2(\partial w_1^*/\partial A) + H_4 W_0^* e^{i\Omega t} + \text{c.c.}\right]$$
$$+ u_1^*(AW_{0\eta}^* e^{i\Omega t} + \text{c.c.}) + w_{1\eta}^*(AU_0^* e^{i\Omega t} + \text{c.c.}) + w_1^*(AW_{0z}^* e^{i\Omega t} + \text{c.c.}) + w_{1z}^*(AW_0^* e^{i\Omega t} + \text{c.c.})$$
$$+ (AF_0' e^{i\Omega t} + \text{c.c.})\left[A(i\Omega U_0^* - U_{0\eta\eta}^* + 2W_{0z\eta}^*)e^{i\Omega t} + \text{c.c.}\right] + \left[A(F_0 + F_0'')e^{i\Omega t} + \text{c.c.}\right](AW_{0\eta}^* e^{i\Omega t} + \text{c.c.})$$
$$+ (AF_0' e^{i\Omega t} + \text{c.c.})\, f_{1tt} - \Omega^2 f_{1z}\,(AF_0 e^{i\Omega t} + \text{c.c.}) + (AF_0' e^{i\Omega t} + \text{c.c.})\,(2i\Omega F_0 H_1 e^{i\Omega t} + \text{c.c.}) = 0 \tag{46}$$

and the following boundary conditions at $\eta = 0$

$$P_0^* + F_0'' + F_0 = W_{0\eta}^* = U_0^* = 0 \tag{47}$$
$$p_1^* + f_1 + f_{1zz} = w_{1\eta}^* + (i\Omega AF_0' e^{i\Omega t} + \text{c.c.}) = u_1^* = 0 \tag{48}$$
$$p_2^* + f_2 + f_{2zz} - (AF_0 e^{i\Omega t} + \text{c.c.})^2 + (AF_0' e^{i\Omega t} + \text{c.c.})^2/2 = w_{2\eta}^* = 0, \tag{49}$$
$$p_3^* + f_3 + f_{3zz} = 2f_1(AF_0 e^{i\Omega t} + \text{c.c.}) - f_{1z}(AF_0' e^{i\Omega t} + \text{c.c.}), \tag{50}$$
$$w_{3\eta}^* + f_{2tz} + (H_2 F_0' e^{i\Omega t} + \text{c.c.}) + (AF_0'' e^{i\Omega t} + \text{c.c.})(AW_0^* e^{i\Omega t} + \text{c.c.})$$
$$+ (AF_0' e^{i\Omega t} + \text{c.c.})\left[A(U_{0\eta}^* - W_{0z}^*)e^{i\Omega t} + \text{c.c.}\right] = 0. \tag{51}$$

Integration of (40)-(51) yields

$$P_0^* = -(F_0 + F_0''), \qquad W_0^* = -i(F_0' + F_0''')/\Omega, \qquad U_0^* = i(F_0'' + F_0^{iv} - \Omega^2 F_0)\eta/\Omega,$$
$$p_1^* = -f_1 - f_{1zz} + \Omega^2(AF_0 e^{i\Omega t} + \text{c.c.})\eta,$$
$$w_1^* = -\sqrt{2\Omega}\left[(1+i)AF_0'\Gamma(\eta)e^{i\Omega t} + \text{c.c.}\right] + (i\Omega AF_0' e^{i\Omega t} + \text{c.c.})\,\eta + \text{POL}$$
$$u_1^* = 2[AF_0''\Gamma(\eta)e^{i\Omega t} + \text{c.c.}] + \text{POL}$$
$$p_2^* = -f_2 - f_{2zz} + (AF_0 e^{i\Omega t} + \text{c.c.})^2 - (AF_0' e^{i\Omega t} + \text{c.c.})^2/2,$$
$$w_{2\eta}^* = \eta\text{POL}, \qquad p_3^* = \text{POL}$$
$$w_{3\eta}^* = i|A|^2\left[3(2\bar{F}_0'' + 2\bar{F}_0^{iv} - \Omega^2\bar{F}_0)F_0' + (\bar{F}_0' + \bar{F}_0''')F_0''\right]/\Omega$$
$$- 2i|A|^2\left[(2\bar{F}_0'' + 2\bar{F}_0^{iv} - \Omega^2\bar{F}_0)F_0' + (\bar{F}_0' + \bar{F}_0''')F_0''\right]\Gamma(\eta)/\Omega$$
$$- (1-i)\sqrt{2/\Omega}|A|^2\left(\bar{F}_0'' + \bar{F}_0^{iv} - \Omega^2\bar{F}_0\right)F_0'\eta\Gamma(\eta) + \text{c.c.} + \eta\text{POL} + \text{OT},$$

where the function Γ is as defined in (35), POL stands for a polinomial in the η variable (whose coefficients may depend on the remaining variables) and OT stands for oscillatory terms in the short time variable, of the type OT= $W_{31\eta}^*(\eta, z, \tau)e^{i\Omega t} + W_{32\eta}^*(\eta, z, \tau)e^{2i\Omega t} + \text{c.c.}$. In fact, when taking into account the actual expressions for POL and some results below, it may be seen (after a careful, involved analysis that is omitted for the sake of brevity) that POL identically vanishes in the expressions giving $w_{2\eta}^*$ and $\tilde{w}_{3\eta}$; but this assertion is not essential to obtain the results in this Section.

Now, in order to apply matching conditions between the solution in the bulk (13) and the solution in this boundary layer (39), we take into account that the solution in the bulk satisfies

$$q(f, z; t, \tau) = q(1, z; t, \tau) + (f-1)q_r(1, z; t, \tau) + O(\varepsilon^3)$$
$$u(f, z; t, \tau) = \ldots, \qquad w_r(f, z; t, \tau) = \ldots,$$

to obtain, at $r = 1$,

$$\phi_1 \equiv u_1 = f_{1t} + (H_1 F_0 e^{i\Omega t} + \text{c.c.}), \qquad \psi_1 \equiv q_1 = -f_1 - f_{1zz}, \tag{52}$$
$$\phi_2 \equiv u_2 = f_{2t} + (H_2 F_0 e^{i\Omega t} + \text{c.c.}) - (AF_0 e^{i\Omega t} + \text{c.c.})(AU_{0r} e^{i\Omega t} + \text{c.c.})$$

$$+(AF_0'e^{i\Omega t}+\text{c.c.})(AW_0e^{i\Omega t}+\text{c.c.}), \tag{53}$$

$$\psi_2 \equiv q_2 = -f_2 - f_{2zz} - (AF_0e^{i\Omega t}+\text{c.c.})(AQ_{0r}e^{i\Omega t}+\text{c.c.}) + (AF_0e^{i\Omega t}+\text{c.c.})^2 \\ - (AF_0'e^{i\Omega t}+\text{c.c.})^2/2 + (AW_0e^{i\Omega t}+\text{c.c.})^2/2 + (i\Omega AF_0e^{i\Omega t}+\text{c.c.})^2/2, \tag{54}$$

$$w_{2r} = i|A|^2\left[3(2\bar{F}_0^{iv}+2\bar{F}_0''-\Omega^2\bar{F}_0)F_0' + (\bar{F}_0'''+\bar{F}_0')F_0''\right]/\,\Omega - |A|^2\bar{W}_{0rr}F_0 + \text{c.c.} + \text{OT}, \tag{55}$$

where w_{2r} has been also obtained for convenience and OT stands for oscillatory terms in the short time scale, of the type $k_1(z)e^{i\Omega t} + k_2(z)e^{2i\Omega t} + \text{c.c.}$, where the functions k_1 and k_2 need not being calculated.

4 Solvability conditions

Here we shall calculate the coefficients in the amplitude equation (15b) by eliminating secular terms in the short time scale, $t \sim 1$. To this end, we first obtain an *integral solvability condition* as follows. First, introduce into (1)-(8) the time scales $t \sim 1$ and $\tau = \delta t$ by replacing in (2)-(4) and (6) the time derivative by $\partial/\partial t + \delta\partial/\partial\tau$. Then multiply (2) by $rU_0e^{-i\Omega t}$, (3) by $rW_0e^{-i\Omega t}$, the second and third equations in (17) by $-rue^{-i\Omega t}$ and $-rwe^{-i\Omega t}$ respectively, add, integrate in $0 < r < f$, $-\Lambda + h_- < z < \Lambda + h_+$, integrate by parts and take into account the boundary conditions (4) and (6)-(9) to obtain

$$\frac{\partial}{\partial t}\left(e^{-i\Omega t}I_1\right) + e^{-i\Omega t}I_2 = e^{-i\Omega t}(I_3 + I_4 + I_5^+ - I_5^-), \tag{56}$$

where

$$I_1 = \int_{-\Lambda+h_-}^{\Lambda+h_+}\int_0^f (uU_0 + wW_0)r\,dr\,dz - \int_{-\Lambda+h_-}^{\Lambda+h_+} Q_0(f,z)f(f-1)dz,$$

$$I_2 = \int_{-\Lambda+h_-}^{\Lambda+h_+}\int_0^f \delta(u_\tau U_0 + w_\tau W_0)r\,dr\,dz - \int_{-\Lambda+h_-}^{\Lambda+h_+} \delta f_\tau f Q_0(f,z)dz,$$

$$I_3 = \int_{-\Lambda+h_-}^{\Lambda+h_+}\int_0^f (uW_0 - wU_0)(u_z - w_r)r\,dr\,dz \\ - C\int_{-\Lambda+h_-}^{\Lambda+h_+}\int_0^f [(U_{0r}u_r + U_{0z}u_z + W_{0r}w_r + W_{0z}w_z)r + U_0u/r]dr\,dz,$$

$$I_4 = \int_{-\Lambda+h_-}^{\Lambda+h_+} [(uU_0 + wW_0)f - f(f-1))Q_{0r} - (f-1)Q_0]_{r=f}\, f_t dz \\ + \int_{-\Lambda+h_-}^{\Lambda+h_+}\left[i\Omega(f-1)Q_0 + (U_0 - f_zW_0)\left(1 + \frac{ff_{zz}-1-f_z^2}{f(1+f_z^2)^{3/2}} - \frac{u^2+w^2}{2}\right) + CU_0(u_r - f_zu_z)\right]_{r=f} f dz \\ - C\int_{-\Lambda+h_-}^{\Lambda+h_+}\left[W_0\left(u_z + f_zw_z + 2f_z\frac{u_r - w_z}{1-f_z^2}\right) + 2(U_0 - f_zW_0)\frac{u_r - (w_r + u_z)f_z + w_zf_z^2}{1+f_z^2}\right]_{r=f} f dz,$$

$$I_5^{\pm} = \int_0^1\left[\left(\frac{\partial h_\pm}{\partial t}W_0 + Q_0\right)\frac{\partial h_\pm}{\partial t} + \delta Q_0\frac{\partial h_\pm}{\partial\tau} - W_0(q-1) + C(U_0u_z + W_0w_z)\right]_{z=\pm\Lambda+h_\pm} r\,dr.$$

Now, secular terms are eliminated by integrating (56) in the short time scale in the interval $]0,t[$, dividing by t, letting $t \to \infty$ and requiring I_1 to be bounded as $t \to \infty$. Then we obtain the following *solvability condition*

$$\lim_{t\to\infty} t^{-1}\int_0^t e^{-i\Omega t}(I_2 - I_3 - I_4 - I_5^+ + I_5^-)dt = 0. \tag{57}$$

In order to apply (57) we first take into account eqs. (6), (11) and (17)-(19), and the structure of the solution in the bulk and in the Stokes and the interface boundary layers to obtain

$$
\begin{aligned}
I_2 =&\delta \int_{-\Lambda}^{\Lambda}\int_0^1 (U_0 u_\tau + W_0 w_\tau) r dr dz \\
&- \delta \int_{-\Lambda}^{\Lambda} \left[f_\tau f Q_0 + (f-1) f_\tau Q_{0r} - (f-1)(U_0 u_\tau + W_0 w_\tau)\right]_{r=1} dz \\
&+ 2\varepsilon C \left\{ H_1 e^{i\Omega t} \int_0^1 \int_{-\infty}^0 U_0(r,\Lambda)[\tilde{U}_0(r,\xi) - U_0(r,\Lambda)] r dr d\xi + \text{c.c.} \right\} + \text{HOT}
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
I_3 =& 2\varepsilon^2 \sqrt{C} \int_0^1 \int_{-\infty}^0 \left[A \left(\xi W_{0z}(r,\Lambda) \tilde{U}_0 - U_0(r,\Lambda) \widetilde{W}_0 \right) e^{i\Omega t} + \text{c.c.} \right] \left(A \tilde{U}_{0\xi} e^{i\Omega t} + \text{c.c.} \right) r dr d\xi \\
&- 2\varepsilon C \int_0^1 \int_{-\infty}^0 U_{0z}(r,\Lambda) \left[A \tilde{U}_{0\xi}(r,\xi) e^{i\Omega t} + \text{c.c.} \right] r dr d\xi \\
&- C\varepsilon A e^{i\Omega t} \int_{-\Lambda}^{\Lambda} \int_0^1 \left[(U_{0r}^2 + U_{0z}^2 + W_{0r}^2 + W_{0z}^2) r + U_0^2/r \right] dr dz \\
&- C\varepsilon \bar{A} e^{-i\Omega t} \int_{-\Lambda}^{\Lambda} \int_0^1 \left[(|U_{0r}|^2 + |U_{0z}|^2 + |W_{0r}|^2 + |W_{0z}|^2) r + |U_0|^2/r \right] dr dz + \text{HOT},
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
I_4 =& \int_{-\Lambda}^{\Lambda} \{ \varepsilon \big(f W_0(1,z) + (f-1) W_{0r}(1,z) \big) \big(A W_0^*(0,z) e^{i\Omega t} + \text{c.c.} \big) + i\Omega (1 - f/2) F_0 f_t \\
&+ (f-1) \big[f_t (U_{0r}(1,z) + U_0(1,z)) - f Q_{0r}(1,z) + F_0 + F_0'' \big] - (f-1)^2 (F_0'' + F_0^{iv}) \} f_t dz \\
&- i\Omega \int_{-\Lambda}^{\Lambda} \left\{ (f-1)^2 (1 - \Omega^2 f) + (f-1) f_{zz} + \frac{3}{2} f_z^2 (1 + f_{zz}) + \frac{\varepsilon^2}{2} f \left(W_0^*(0,z) e^{i\Omega t} + \text{c.c.} \right)^2 \right\} F_0 dz \\
&+ \int_{-\Lambda}^{\Lambda} \big[U_{0r}(f-1) - f_z W_0 \big]_{r=1} \left\{ f - 1 + f f_{zz} + \frac{1}{2} f_z^2 - \frac{1}{2} \left[f_t^2 + \varepsilon^2 \big(A W_0^*(0,z) e^{i\Omega t} + \text{c.c.} \big)^2 \right] \right\} dz \\
&+ \frac{\Omega^2}{2} \int_{-\Lambda}^{\Lambda} (f-1)^3 U_{0r}(1,z) dz + \int_{-\Lambda}^{\Lambda} \big[U_{0rr} (f-1)^2/2 - W_{0r} f_z (f-1) \big]_{r=1} (f - 1 + f_{zz}) dz \\
&- C \int_{-\Lambda}^{\Lambda} f_{tz} W_0(1,z) dz + C \int_{-\Lambda}^{\Lambda} \left[f_t + \varepsilon \left(A W_{0z}^*(0,z) e^{i\Omega t} + \text{c.c.} \right) \right] U_0(1,z) dz + \text{HOT},
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
I_5^{\pm} =& \left(i\Omega \mu \beta_{\pm} e^{i(\Omega t + \omega_{\pm} \tau)} + \text{c.c.} \right) \int_0^1 r Q_0(r, \pm\Lambda) dr \\
&\pm \sqrt{C} \varepsilon \int_0^1 U_0(r,\Lambda) \left[A \tilde{U}_{0\xi}(r,0) e^{i\Omega t} + \text{c.c.} + \sqrt{C} \tilde{u}_{1\xi}(r,0) + \varepsilon \tilde{u}_{2\xi}(r,0) \right] r dr + \text{HOT},
\end{aligned}
\tag{61}
$$

where u and w (the velocity components in the bulk) and f are as given in (13), $\tilde{U}_0$, $\widetilde{W}_0$, $\tilde{u}_1$, $\tilde{u}_2$ and W_0^* are as given in (32)-(34) and the second equation below (51) and HOT stands for

$$
\text{HOT} = o(\mu + \varepsilon C + \varepsilon^2 \sqrt{C} + \varepsilon^3)
\tag{62}
$$

Also, in order to obtain (58) and (59) we have used the solution in the Stokes boundary layer near $z = -\Lambda$ (not given in §2), and the fact that the functions U_0^2 and $U_0 W_{0z}$ are even in the z variable; in order to obtain (60) we have taken into account the equation

$$
i\Omega \int_{-\Lambda + h_-}^{\Lambda + h_+} (f-1) f Q_0(1,z) dz = -i\Omega \int_{-\Lambda}^{\Lambda} [f(f-1) + 2 f_z^2 + (2f-1) f_{zz}] F_0 dz + o(\varepsilon\mu)
$$

that is obtained upon substitution of (9), (11), (19) and (20), and integration by parts.

Now, H_1, H_2 and H_6 are readily calculated upon substitution of the resulting equations into (57) and setting to zero the coefficients of $\varepsilon\sqrt{C}$, ε^2 and μ, to obtain

$$
H_1 = -(1+i)\alpha_1 A, \qquad H_2 = 0, \qquad H_6 = i \left(\alpha_4^+ \beta_+ e^{i\omega_+ \tau} - \alpha_4^- \beta_- e^{i\omega_- \tau} \right),
\tag{63}
$$

where

$$\alpha_1 = \int_0^1 Q_{0r}(r,\Lambda)^2 r dr \left[\sqrt{2\Omega^3}\int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1,z) dz\right]^{-1}, \tag{64}$$

$$\alpha_4^{\pm} = -\Omega \int_0^1 Q_0(r,\Lambda) r dr \left[2\int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1,z) dz\right]^{-1} \tag{65}$$

and we have taken into account (17) and the following equations

$$\int_{-\Lambda}^{\Lambda}\int_0^1 (U_0^2 + W_0^2) r dr = -\int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1,z) dz, \tag{66}$$
$$\int_0^1 U_0(r,\Lambda)\tilde{U}_{0\xi}(r,0) r dr = -(1+i)\sqrt{\Omega/2}\int_0^1 U_0(r,\Lambda)^2 dr$$

Eq. (66) is obtained when multiplying the second and third equations (17) by rU_0 and rW_0 respectively, adding, integrating, integrating by parts and taking into account the first equation (17) and (18)-(19); the second equation is readily obtained when taking into account (32) and (36).

In order to calculate α_1 and $\alpha_4^{\pm}$ we need the nontrivial eigenfunctions of (17)-(20) that were first calculated in a semianalytical form by Sanz [1985]; we simply collect his results here. The eigenfrequencies Ω are exactly the (real) solutions of one of the following equations

$$\Lambda \tan\Lambda = \sum_{n \text{ odd}} a_n r_n \qquad \text{or} \qquad \Lambda\cot\Lambda + \sum_{n \text{ even}} a_n r_n = 0 \tag{67}$$

where

$$a_0 = 1, \qquad a_n = 2\Omega^2/(\Omega^2 q_n - s_n) \quad \text{if } n \geq 1, \tag{68}$$

$$q_n = I_0(l_n), \qquad r_n = q_n/(l_n^2-1), \qquad s_n = l_n(l_n^2-1)I_1(l_n), \qquad l_n = n\pi/2\Lambda \quad \text{if } n \geq 0, \tag{69}$$

and I_0 and I_1 are the first two modified Bessel functions. If the first equation (67) holds (*odd modes*), Q_0 and F_0 are defined (up to a constant factor) by

$$Q_0 = \sum_{n \text{ odd}} a_n I_0(l_n r)\cos[l_n(z+\Lambda)], \qquad F_0 = \Lambda\sin z/\cos\Lambda + \sum_{n \text{ odd}} a_n r_n \cos[l_n(z+\Lambda)], \tag{70}$$

while if the second equation holds (*even modes*), then

$$Q_0 = \sum_{n \text{ even}} a_n I_0(l_n r)\cos[l_n(z+\Lambda)], \qquad F_0 = \Lambda\cos z/\sin\Lambda + \sum_{n \text{ even}} a_n r_n \cos[l_n(z+\Lambda)]. \tag{71}$$

U_0 and W_0 are readily calculated by means of the second and third equations (16).

Now, α_1 and $\alpha_4^{\pm}$ are readily calculated. Notice that these constants are real and that $\alpha_4^- = \alpha_4^+$ and $\alpha_4^- = -\alpha_4^+$ for even and odd modes respectively, i.e., for the m-th mode we have

$$\alpha_4^- = (-1)^m \alpha_4^+. \tag{72}$$

The coefficients H_3 and H_5 of the amplitude equation (15b) will depend on the terms of orders $\varepsilon\sqrt{C}$ and ε^2 in the expansions (13), that are considered now. When taking into account (26), (37), (52)-(54) and (63), the solutions of (14b), (17)-(25) for $k=1$ and 2 are seen to be given by

$$u_1 = AU_1 e^{i\Omega t} + \text{c.c.}, \quad w_1 = AW_1 e^{i\Omega t} + \text{c.c.}, \quad q_1 = AQ_1 e^{i\Omega t} + \text{c.c.}, \quad f_1 = AF_1 e^{i\Omega t} + \text{c.c.}, \tag{73}$$

$$u_2 = A^2 U_{22} e^{2i\Omega t} + \text{c.c.} + u_{20}, \quad w_2 = A^2 W_{22} e^{2i\Omega t} + \text{c.c.} + w_{20}, \tag{74}$$

$$q_2 = A^2 Q_{22} e^{2i\Omega t} + \text{c.c.} + |A|^2 Q_{20}, \quad f_2 = A^2 F_{22} e^{2i\Omega t} + \text{c.c.} + |A|^2 F_{20}, \tag{75}$$

where the nonoscillatory (in the short time scale) components of the velocity field, u_{20} and w_{20}, will be considered in §4, while (U_1, W_1, Q_1, F_1), $(U_{22}, W_{22}, Q_{22}, F_{22})$ and (Q_{20}, F_{20}) are given by

$$U_{1r} + r^{-1}U_1 + W_{1z} = U_{22r} + r^{-1}U_{22} + W_{22z} = 0, \tag{76}$$

$$i\Omega U_1 + Q_{1r} - (1+i)\alpha_1 U_0 = 2i\Omega U_{22} + Q_{22r} = Q_{20r} = 0, \tag{77}$$

$$i\Omega W_1 + Q_{1z} - (1+i)\alpha_1 W_0 = 2i\Omega W_{22} + Q_{22z} = Q_{20z} = 0, \tag{78}$$

$$W_1 = \pm(1-i)W_{0z}/\sqrt{2\Omega}, \qquad W_{22} = 0 \quad \text{at} \quad z = \pm\Lambda, \tag{79}$$

$$U_1 - i\Omega F_1 + (1+i)\alpha_1 F_0 = U_{22} - 2i\Omega F_{22} + F_0 U_{0r} - F_0' W_0 = 0 \quad \text{at} \quad r = 1, \tag{80}$$

$$Q_1 + F_1 + F_1'' = Q_{22} + F_{22} + F_{22}'' + F_0 Q_{0r} - F_0^2 + (F_0'^2 - W_0^2 + \Omega^2 F_0^2)/2 = 0 \quad \text{at} \quad r = 1, \tag{81}$$

$$Q_{20} + F_{20} + F_{20}'' + (\bar{F}_0 Q_{0r} + \text{c.c.}) - 2|F_0|^2 + |F_0'|^2 - |W_0|^2 - \Omega^2 |F_0|^2 = 0 \quad \text{at} \quad r = 1, \tag{82}$$

$$F_1(\pm\Lambda) = F_{22}(\pm\Lambda) = F_{20}(\pm\Lambda) = \int_{-\Lambda}^{\Lambda} F_1 dz = \int_{-\Lambda}^{\Lambda} (F_{22} + F_0^2/2) dz = \int_{-\Lambda}^{\Lambda} (F_{20} + |F_0|^2) dz = 0, \tag{83}$$

$$\int_{-\Lambda}^{\Lambda} \int_0^1 (U_0 U_1 + W_0 W_1) r dr dz = 0. \tag{84}$$

Notice that u_2 has no oscillatory terms with frequency Ω in the short time scale; this is a consequence of condition (14b) for $k = 2$. Then $K_2^+ = 0$ in (34) (see (38)); since, in addition, $H_2 = 0$ (see (63)), we have

$$\int_0^1 U_0(r, \Lambda) \tilde{u}_{2\xi}(r, 0) r dr = \text{independent of } t. \tag{85}$$

Also, the problem posed by (76)-(81) and (83)-(84) giving (U_1, W_1, Q_1, F_1) possesses a solution if and only if the constant α_1 is as given in (64), as is readily seen (upon multiplication of (77), (78) and the second and third equations in (17) by rU_0, rW_0, $-rU_1$ and $-rW_1$ respectively, addition, integration in $-\Lambda < z < \Lambda$, $0 < r < 1$, integration by parts and substitution of (17)-(20), (76), (79)-(81) and (83)).

If (64) holds then (76)-(81) and (83)-(84) uniquely define U_1, W_1, Q_1 and F_1, that may be obtained in a semianalytical form. In particular, Q_1 and F_1 are given by

$$Q_1 = -(1-i)[bQ_0 + \partial Q_0/\partial\Lambda] + \sum_{n \text{ odd}} b_n I_0(l_n r) \cos[l_n(z+\Lambda)]/\sqrt{2\Omega} \tag{86a}$$

$$\begin{aligned} F_1 = &-(1-i)[bF_0 + \partial F_0/\partial\Lambda + 2\Lambda(\partial\Omega/\partial\Lambda - \alpha_1\sqrt{2\Omega}) \sin z/(\Omega \cos\Lambda)] \\ &+ \sum_{n \text{ odd}} b_n r_n \cos[l_n(z+\Lambda)]/\sqrt{2\Omega}, \end{aligned} \tag{87a}$$

if Ω is a solution of the first equation in (67), or

$$Q_1 = -(1-i)[bQ_0 + \partial Q_0/\partial\Lambda] + \sum_{n \text{ even}} b_n I_0(l_n r) \cos[l_n(z+\Lambda)]/\sqrt{2\Omega} \tag{86b}$$

$$\begin{aligned} F_1 = &-(1-i)[bF_0 + \partial F_0/\partial\Lambda + 2\Lambda(\partial\Omega/\partial\Lambda - \alpha_1\sqrt{2\Omega}) \cos z/(\Omega \sin\Lambda)] \\ &+ \sum_{n \text{ even}} b_n r_n \cos[l_n(z+\Lambda)]/\sqrt{2\Omega}, \end{aligned} \tag{87b}$$

if Ω is a solution of the second equation in (67); U_1 and W_1 are given by

$$U_1 = [iQ_{1r} + (1-i)\alpha_1 U_0]/\Omega, \qquad W_1 = [iQ_{1z} + (1-i)\alpha_1 W_0]/\Omega. \tag{88}$$

Here the constant b_n is given by

$$b_n = 2\Omega(\partial\Omega/\partial\Lambda - \alpha_1\sqrt{2\Omega})q_n a_n/(\Omega^2 q_n - s_n) \qquad \text{for } n = 0, 1, \ldots,$$

and the constants a_n, l_n, q_n, r_n and s_n are as given in (68)-(69). The results below do not depend on the constant b, that is uniquely determined by (84).

Similarly, if neither $\Omega_1 = 2\Omega$ nor $\Omega_1 = 0$ are solutions of the second equation (67), then (76)-(83) uniquely define $(U_{22}, W_{22}, Q_{22}, F_{22})$ and (Q_{20}, F_{20}), that may be written in a semianalytical form as

$$Q_{22} = 2\Omega^2 c_0 + 4\Omega^2 \sum_{k=1}^{\infty} c_{2k} I_0(2l_k r)\cos[2l_k(z+\Lambda)] \tag{89}$$

$$F_{22} = i[F_0'(z)W_0(1,z) - F_0(z)U_{0r}(1,z)]/2\Omega + 2\sum_{k=1}^{\infty} c_{2k} l_k I_1(2l_k)\cos[2l_k(z+\Lambda)], \tag{90}$$

$$U_{22} = iQ_{22r}/2\Omega, \qquad W_{22} = iQ_{22z}/2\Omega, \tag{91}$$

$$Q_{20} = D_3 - g_0/2, \tag{92}$$

$$F_{20} = D_4 \cos z - D_3 - \sum_{k=1}^{\infty} g_{2k}(1-4l_k^2)^{-1}\cos[2l_k(z+\Lambda)], \tag{93}$$

where

$$D_1 = \int_{-\Lambda}^{\Lambda} F_0(z)^2 dz/4\Lambda + \sum_{k=1}^{\infty}\left[s_{2k}e_{2k} + 2(1-4l_k^2)d_{2k}\Omega^2 q_{2k}\right]/\left(4\Omega^2 q_{2k} - s_{2k}\right)\left(1-4l_k^2\right),$$

$$D_2 = \Lambda\cot\Lambda - 1 + 8\Omega^2\sum_{k=0}^{\infty} r_{2k}/(4\Omega^2 q_{2k} - s_{2k}),$$

$$D_3 = \left[\tan\Lambda\sum_{k=1}^{\infty}(1-4l_k^2)^{-1}g_{2k} + \int_{-\Lambda}^{\Lambda}|F_0(z)|^2 dz/2\right]\Big/(\Lambda - \tan\Lambda),$$

$$D_4 = \left[\Lambda\sum_{k=1}^{\infty}(1-4l_k^2)^{-1}g_{2k} + \int_{-\Lambda}^{\Lambda}|F_0(z)|^2 dz/2\right]\Big/(\Lambda\cos\Lambda - \sin\Lambda),$$

and for $n \geq 0$, the constants l_n, r_n, q_n and s_n are as given in (69), while c_n, d_n, e_n and g_n are as given by

$$c_n = \left[2D_1/D_2 + (1-l_n^2)d_n/2 + e_n\right]/(4\Omega^2 q_n - s_n),$$

$$d_n = -i\int_{-\Lambda}^{\Lambda}[F_0'(z)W_0(1,z) - F_0(z)U_{0r}(1,z)]\cos[l_n(z+\Lambda)]dz/(\Lambda\Omega),$$

$$e_n = \int_{-\Lambda}^{\Lambda}[(2-3\Omega^2)F_0(z)^2 + W_0(1,z)^2 - F_0'(z)^2]\cos[l_n(z+\Lambda)]dz/2\Lambda,$$

$$g_n = \int_{-\Lambda}^{\Lambda}\left[(\Omega^2-2)|F_0(z)|^2 - |W_0(1,z)|^2 + |F_0'(z)|^2\right]\cos[l_n(z+\Lambda)]dz/\Lambda.$$

Now H_3, H_4 and H_5 are readily obtained upon substitution of (13) (with the terms of orders $\varepsilon\sqrt{C}$ and ε^2 as given by (73)-(75)) and (15b) into (58)-(61), substitution of the resulting equations into (57) and setting to zero the coefficients of εC, $\varepsilon^2\sqrt{C}$ and ε^3, to obtain (after some algebraic manipulations)

$$H_3 = -\alpha_2 A, \qquad H_4 = 0, \qquad H_5 = i\alpha_3 A|A|^2, \tag{94}$$

where the real constants α_2 and α_3 are given by

$$(\alpha_2 - 2 - 2\alpha_1^2/\Omega)\int_{-\Lambda}^{\Lambda} F_0(z)Q_0(1,z)dz = -4F_0'(\Lambda)F_0''(\Lambda) \tag{95}$$
$$+ (1+i)\int_0^1 Q_{0r}(r,\Lambda)Q_{1r}(r,\Lambda)rdr/\sqrt{2\Omega^3}$$
$$+\int_{-\Lambda}^{\Lambda}[2Q_0(1,z)^2 - \Omega^2 F_0(z)^2 - (1+i)\alpha_1 F_1(z)Q_0(1,z)]dz$$

$$2(\alpha_3/\Omega)\int_{-\Lambda}^{\Lambda} F_0(z)Q_0(1,z)dz = \int_{-\Lambda}^{\Lambda}[(2-3\Omega^2)F_0F_{22} - F_0'F_{22}' + (2-\Omega^2)F_0F_{20} - F_0'F_{20}' \tag{96}$$
$$-2W_0(1,z)W_{22}(1,z) - (F_{22}+F_{20})Q_0(1,z)]\,F_0dz + \int_{-\Lambda}^{\Lambda}(F_{22}-F_{20})W_0(1,z)^2dz$$
$$+\int_{-\Lambda}^{\Lambda}\left[(11F_0 + 17F_0'')F_0'^2 + 4(F_0+F_0'')F_0''F_0 - (6+\Omega^2)F_0^3 - F_0W_0(1,z)^2\right]F_0dz/2.$$

In order to obtain (93)-(96) we have taken into account that iU_0, iW_0, Q_0 and F_0 are real (see (68)-(70) or (71)); also we have used (63), (66), (85) and the following equations

$$\int_0^1\int_{-\infty}^0 U_0(r,\Lambda)[\tilde{U}_0(r,\xi) - U_0(r,\Lambda)]rdrd\xi = -(1-i)\int_0^1 rU_0(r,\Lambda)^2dr\Big/\sqrt{\Omega}\,, \tag{97}$$

$$U_{0z}(r,\Lambda) = W_{0r}(r,\Lambda) = 0 \tag{98}$$

$$\int_{-\Lambda}^{\Lambda}\int_0^1 (U_0U_1 + W_0W_1)rdrdz = \int_{-\Lambda}^{\Lambda}\left[-\alpha_1(1-i)F_0Q_0(1,z)/\Omega - F_1Q_0(1,z)\right]dz \tag{99}$$

$$\int_{-\Lambda}^{\Lambda}\int_0^1 \left[(U_{0r}^2 + U_{0z}^2 + W_{0r}^2 + W_{0z}^2)r + U_0^2/r\right]drdz$$
$$= 4F_0'(\Lambda)F_0''(\Lambda) + \int_{-\Lambda}^{\Lambda}\left[\Omega^2F_0^2 + 2F_0''Q_0(1,z)\right]dz, \tag{100}$$

$$\int_0^1 U_0(r,\Lambda)\tilde{u}_{1\xi}(r,0)rdr = \left[-(1+i)\sqrt{\frac{\Omega}{2}}\int_0^1 U_0(r,\Lambda)U_1(r,\Lambda)rdr\right.$$
$$\left. +\alpha_1\int_0^1 U_0(r,\Lambda)^2rdr\Big/\sqrt{2\Omega}\right]Ae^{i\Omega t} + \text{c.c.} \tag{101}$$

Eqs. (97) and (98) readily follow from (18), (19), (32) and (35). Eq. (99) is obtained by multiplying the first equation in (77) and (78) by rU_0 and rW_0 respectively, adding, integrating in $-\Lambda < z < \Lambda$, $0 < r < 1$ and integrating by parts. Similarly, (100) is obtained by multiplying the equations

$$U_{0rr} + U_{0zz} + r^{-1}U_{0r} - r^{-2}U_0 = W_{0rr} + W_{0zz} + r^{-1}W_{0r} = 0$$

(that are readily obtained from (17)) by rU_0 and rW_0 respectively, adding, integrating in $-\Lambda < z < \Lambda$, $0 < r < 1$ and integrating by parts. Equation (101) is readily obtained when taking into account (33) and (36).