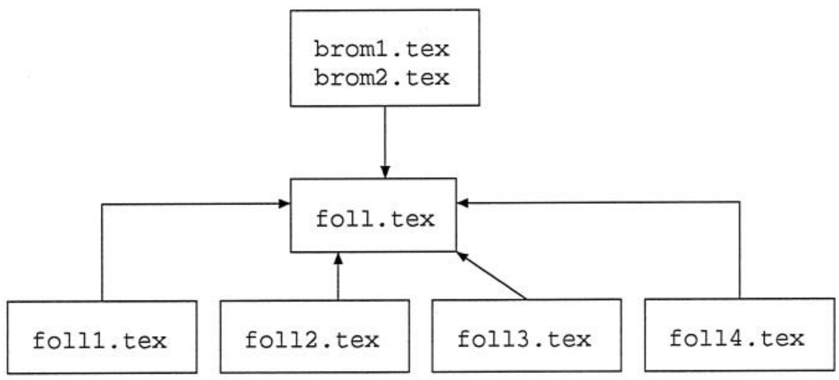
The second disk

brom1.tex	in eepic format
brom2.tex	in eepic format
eepic.sty	
epic.sty	
foll.tex	main TEXfile
foll1.tex	
foll2.tex	
foll3.tex	
foll4.tex	
foll.dvi	
subeqn.sty	
subeqnar.sty	

FIGURE FILE(in eepic format)



FM 9801

To be held in the Editorial files of JFM

Asymptotic solutions for two-dimensional low Reynolds number flow around an impulsively started circular cylinder

Masato NAKANISHI, Teruhiko KIDA, Tomoya NAKAJIMA

Detailed Derivation of Equations in Text

1 Introduction

2 Basic Relation

2.1 Derivation of Eqs.(3) ~ (5) ← (This section is added.)

In the absolute coordinate systems, the Navier-Stokes equation is given by

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\mathrm{grad} P + \vec{K} + \nu \mathbf{\Delta} \vec{u}$$

The cylinder moves unit velocity parallel to the positive x direction. The boundary conditions are therefore given by

$$\begin{aligned} \vec{u} &= (UH(t), 0) \quad \text{on S} \\ \vec{u} &\to 0 \quad \text{as } |\vec{x}| \to \infty \end{aligned}$$

In the relative coordinate systems with the fixed cylinder, the Navier-Stokes equation becomes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\mathrm{grad} P + \vec{K} + \nu \mathbf{\Delta} \vec{u}$$

The boundary conditions are given by

$$\begin{aligned} \vec{u} &= (0, 0) \quad \text{on S} \\ \vec{u} &\to -UH(t)\vec{e}_x \quad \text{as } |\vec{x}| \to \infty \end{aligned}$$

where $\vec{u}$ is the velocity vector in the relative coordinate systems and $\vec{e}_x$ is the unit vector of x-axis. Here, we normalize velocities and lengths with U and d. Then, from this equation we can easily derive the dimensionless vorticity equation (3). Here, we define the perturbation dimensionless velocity $\vec{u}_p = (u, v)$ and the perturbation stream function ψ:

$$\begin{aligned} \vec{u} &= -H(t)\vec{e}_x + \vec{u}_p \\ \vec{u}_p &\equiv (u, v) = (\partial\psi/\partial y, -\partial\psi/\partial x) \end{aligned}$$

Then, the Navier-Stokes equations are given by

$$\begin{aligned}\frac{\partial \vec{u}_p}{\partial t} + (u - H(t))\frac{\partial \vec{u}_p}{\partial x} + v\frac{\partial \vec{u}_p}{\partial y} &= -\mathrm{grad}P + \vec{K} + \frac{1}{Re}\boldsymbol{\Delta}\vec{u}_p \\ \mathrm{div}\vec{u}_p &= 0\end{aligned}$$

where P and $\vec{K}$ are the dimensionless pressure and the mass force respectively. The boundary conditions (5) are given by

$$\begin{aligned}\vec{u}_p &= (H(t), 0) \quad \text{on S} \\ \vec{u}_p &\to 0 \quad \text{as } |\vec{x}| \to \infty\end{aligned}$$

The relation between the vorticity ζ and ψ is given by

$$\zeta = \mathrm{rot}\vec{u} = \mathrm{rot}\vec{u}_p = -\boldsymbol{\Delta}\psi$$

Thus we can derive Eq.(4).

The vorticity equation with respect to the perturbation velocity field is given by

$$\frac{\partial \zeta}{\partial t} + (u - H(t))\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \frac{1}{R_e}\boldsymbol{\Delta}\zeta$$

From this equation, we easily arrive at Eqs.(6) and (7).

2.2 Derivation of Eq.(13)

From Eq.(12), $\tilde{\zeta} = \exp(-\varepsilon x)\hat{\zeta}$, so that we have

$$\begin{aligned}\frac{\partial \tilde{\zeta}}{\partial x} &= \left(-\varepsilon\hat{\zeta} + \frac{\partial \hat{\zeta}}{\partial x}\right)\exp(-\varepsilon x) \\ \frac{\partial^2 \tilde{\zeta}}{\partial x^2} &= \left(\varepsilon^2\hat{\zeta} - 2\varepsilon\frac{\partial \hat{\zeta}}{\partial x} + \frac{\partial^2 \hat{\zeta}}{\partial x^2}\right)\exp(-\varepsilon x)\end{aligned}$$

These relations are substituted into Eq.(10). Taking into account Eq.(11), we arrive at Eq.(13).

2.3 Derivation of fundamental solution G

The fundamental solution G is governed by the following equation;

$$\boldsymbol{\Delta}G - \varepsilon(\varepsilon + 2p)G = -2\pi\delta(\vec{x} - \vec{x}_0) \tag{2.1}$$

where $\delta(\vec{x})$ is the two-dimensional Dirac delta function. The delta function is expressed as

$$\begin{aligned}\delta(\vec{x} - \vec{x}_o) &= \delta(x - x_o)\cdot\delta(y - y_o) \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big\{i\Big[k_x(x - x_o) \\ &\qquad + k_y(y - y_o)\Big]\Big\}\mathrm{d}k_x\mathrm{d}k_y\end{aligned} \tag{2.2}$$

where $(-\infty < x, y, x_o, y_o < \infty)$. We define the Fourier transform of G as

$$G \equiv \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(k_x, k_y)\exp\{i[k_x(x-x_o)+k_y(y-y_o)]\}\,\mathrm{d}k_x\mathrm{d}k_y \quad (2.3)$$

Substituting Eqs.(2.2) and (2.3) into Eq.(2.1), then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(k_x, k_y)\exp\{i[k_x(x-x_o)+k_y(y-y_o)]\} \\
&\times\left\{-(k_x^2+k_y^2)-\varepsilon(\varepsilon+2p)\right\}\mathrm{d}k_x\mathrm{d}k_y \\
&= -2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i[k_x(x-x_o)+k_y(y-y_o)]\}\,\mathrm{d}k_x\mathrm{d}k_y
\end{aligned}$$

Therefore, H is given by

$$H(k_x, k_y) = \frac{2\pi}{k_x^2+k_y^2+\varepsilon(\varepsilon+2p)} \quad (2.4)$$

Substituting Eq.(2.4) into Eq.(2.3), we have

$$\begin{aligned}
G &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2\pi}{k_x^2+k_y^2+\varepsilon(\varepsilon+2p)}\exp\Bigg\{i\Bigg[k_x(x-x_o) \\
&\quad +k_y(y-y_o)\Bigg]\Bigg\}\mathrm{d}k_x\mathrm{d}k_y
\end{aligned} \quad (2.5)$$

In order to obtain G as a simple expression, we define

$$\left\{\begin{array}{lllllll} k_x & \equiv & k\cos(\alpha+\beta), & k_y & \equiv & k\sin(\alpha+\beta) \\ x-x_o & \equiv & \rho_r\cos\beta, & y-y_o & \equiv & \rho_r\sin\beta \end{array}\right.$$

Then, we have the following relations;

$$\begin{aligned}
k_x(x-x_o)+k_y(y-y_o) &= k\rho_r\cos\alpha \\
\mathrm{d}k_x\mathrm{d}k_y &\equiv k\mathrm{d}k\mathrm{d}\alpha
\end{aligned}$$

Substituting these relations into Eq.(2.5), we have

$$\begin{aligned}
G &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2\pi}{k_x^2+k_y^2+\varepsilon(\varepsilon+2p)} \\
&\quad \times\exp\{i[k_x(x-x_o)+k_y(y-y_o)]\}\,\mathrm{d}k_x\mathrm{d}k_y \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}\frac{2\pi k}{k^2+\varepsilon(\varepsilon+2p)}\exp\{[i(k\rho_r\cos\alpha)]\}\,\mathrm{d}k\mathrm{d}\alpha \\
&= \int_0^{\infty}\frac{k}{k^2+\varepsilon(\varepsilon+2p)}\left[\frac{1}{\pi}\int_0^{\pi}\exp\{[ik\rho_r\cos\alpha]\}\,\mathrm{d}\alpha\right]\mathrm{d}k \\
&= \int_0^{\infty}\frac{k}{k^2+\varepsilon(\varepsilon+2p)}J_0(k\rho_r)\mathrm{d}k = K_0\left((\varepsilon(\varepsilon+2p))^{\frac{1}{2}}\rho_r\right)
\end{aligned}$$

Thus, the solution G is given by

$$G(\vec{x}, \vec{x}_o) \equiv G(\vec{x}-\vec{x}_0) = K_0(a\rho_r)$$

where $\rho_r = |\vec{x}-\vec{x}_0|$, K_0 is the zeroth order modified Bessel function of the second kind and $a^2 = \varepsilon(\varepsilon+2p)$.

2.4 Derivation of Eq.(21)

Let us consider the additional term of Eq.(21);

$$\Psi = \frac{1}{2\pi}\oint_S \left(-\log\rho_r \frac{\partial\tilde{\psi}}{\partial n} + \tilde{\psi}\frac{\partial \log\rho_r}{\partial n}\right) \mathrm{d}s \tag{2.6}$$

This term is potential, that is, $\nabla^2\Psi = 0$. We here define the following function $\tilde{\psi}_-$:

$$\begin{aligned} \tilde{\psi}_- &= \frac{y}{p} \qquad \text{inside the body } S \\ &= 0 \qquad \text{outside the body } S \end{aligned}$$

We note that this function satisfies $\nabla^2\tilde{\psi}_- = 0$. Then, we have

$$0 = \frac{1}{2\pi}\oint_S \left(-\log\rho_r \frac{\partial\tilde{\psi}_-}{\partial n} + \tilde{\psi}_-\frac{\partial \log\rho_r}{\partial n}\right) \mathrm{d}s \qquad \text{for } \vec{x}_o \text{ outside } S \tag{2.7}$$

Therefore, we have, by taking into account of the above relation (2.7) ;

$$\Psi = \frac{1}{2\pi}\oint_S \left(-\log\rho_r \frac{\partial(\tilde{\psi}-\tilde{\psi}_-)}{\partial n} + (\tilde{\psi}-\tilde{\psi}_-)\frac{\partial \log\rho_r}{\partial n}\right) \mathrm{d}s$$

Thus, we see $\Psi = 0$ by taking into account of the boundary conditions:

$$\begin{aligned} \tilde{\psi} &= \tilde{\psi}_- = \frac{y}{p} \qquad \text{on } S \\ \frac{\partial\tilde{\psi}}{\partial n} &= \frac{\partial\tilde{\psi}_-}{\partial n} = \frac{1}{p}\frac{\partial y}{\partial n} \qquad \text{on } S \end{aligned}$$

3 Asymptotic Analysis

3.1 Derivation of $\tilde{\psi}$ and $\hat{\zeta}$ in Region (I)

3.1.1 Derivation of Eq.(39)

Substituting Eq.(38) into Eq.(34), we have as follows:

$$\begin{aligned} \hat{\zeta} \approx\ & \frac{1}{2\pi}\int_0^{2\pi}\left[K_0(\hat{p}r)\left(-I_0(\hat{p}r_1)\frac{\partial\hat{\zeta}}{\partial r_1} + \hat{\zeta}\frac{\partial I_0(\hat{p}r_1)}{\partial r_1}\right)\right. \\ & \left. +2\sum_{m=1}^{\infty} K_m(\hat{p}r)\left(-I_m(\hat{p}r_1)\frac{\partial\hat{\zeta}}{\partial r_1} + \hat{\zeta}\frac{\partial I_m(\hat{p}r_1)}{\partial r_1}\right)\cos m(\theta-\varphi)\right]_{r_1=1} \mathrm{d}\theta \end{aligned}$$

In this flow, $\hat{\zeta}$ is odd function of θ. Therefore, the terms of $\cos m\theta$ in the integrand becomes zero. Thus, we arrive at Eq.(39).

3.1.2 Derivation of Eq.(41)

$\hat{\zeta}$ is given by Eq.(39):

$$\hat{\zeta} = \varepsilon \sum_{m=1}^{\infty} c_m K_m(\hat{p}r)\sin(m\varphi) \tag{3.1}$$

This equation is substituted into Eq.(35). Using Eq.(42), we have, by taking into account of the flow symmetry, that is, $\hat{\zeta}$ is odd function of θ:

$$\begin{aligned}
\tilde{\psi} &\approx \frac{1}{2\pi}\int_0^{2\pi}\int_{r_s}^{\infty}\hat{\zeta}\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta \\
&\approx \frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_1^{\infty}\sum_{m=1}^{\infty}c_m K_m(\hat{p}r_1)\sin(m\theta)\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta \\
&\approx \sum_{m=1}^{\infty}c_m\frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_1^{\infty}K_m(\hat{p}r_1)\sin(m\theta)\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta \\
&\approx \sum_{k=1}^{\infty}c_k\frac{\varepsilon}{2\pi}\int_0^{2\pi}\left[\int_1^{r}K_k(\hat{p}r_1)\sin(k\theta)\right. \\
&\quad \times\left\{-\log r+\sum_{m=1}^{\infty}\left(\frac{r_1}{r}\right)^m\frac{\cos\{m(\theta-\varphi)\}}{m}\right\}r_1\mathrm{d}r_1 \\
&\quad +\int_r^{\infty}K_k(\hat{p}r_1)\sin(k\theta)\left\{-\log r_1\right. \\
&\quad \left.\left.+\sum_{m=1}^{\infty}\left(\frac{r}{r_1}\right)^m\frac{\cos\{m(\theta-\varphi)\}}{m}\right\}r_1\mathrm{d}r_1\right]\mathrm{d}\theta \\
&\approx \sum_{k=1}^{\infty}c_k\frac{\varepsilon}{2\pi}\int_0^{2\pi}\sin(k\theta)\left[-\int_1^{r}K_k(\hat{p}r_1)\log(r)r_1\mathrm{d}r_1\right. \\
&\quad -\int_r^{\infty}K_k(\hat{p}r_1)\log(r_1)r_1\mathrm{d}r_1 \\
&\quad +\int_1^{r}K_k(\hat{p}r_1)\sum_{m=1}^{\infty}\left(\frac{r_1}{r}\right)^m\frac{\cos\{m(\theta-\varphi)\}}{m}r_1\mathrm{d}r_1 \\
&\quad \left.+\int_r^{\infty}K_k(\hat{p}r_1)\sum_{m=1}^{\infty}\left(\frac{r}{r_1}\right)^m\frac{\cos\{m(\theta-\varphi)\}}{m}r_1\mathrm{d}r_1\right]\mathrm{d}\theta \\
&\approx \frac{\varepsilon}{2}\sum_{m=1}^{\infty}c_m\frac{1}{m}\left[\int_1^{r}K_m(\hat{p}r_1)\left(\frac{r_1}{r}\right)^m r_1\mathrm{d}r_1\right. \\
&\quad \left.+\int_r^{\infty}K_m(\hat{p}r_1)\left(\frac{r}{r_1}\right)^m r_1\mathrm{d}r_1\right]\sin(m\varphi) \\
&\approx \frac{\varepsilon}{2}\sum_{m=1}^{\infty}c_m\frac{1}{m}\left[\frac{1}{\hat{p}}\frac{1}{r^m}\left\{-r^{m+1}K_{m+1}(\hat{p}r)+K_{m+1}(\hat{p})\right\}\right. \\
&\quad \left.+\frac{1}{\hat{p}}r^m\left\{r^{-m+1}K_{m-1}(\hat{p}r)\right\}\right]\sin(m\varphi) \\
&\approx \frac{\varepsilon}{2}\sum_{m=1}^{\infty}c_m\frac{1}{m}\left[\frac{1}{\hat{p}}\left\{-\frac{2m}{\hat{p}}K_m(\hat{p}r)+\left(\frac{1}{r}\right)^m K_{m+1}(\hat{p})\right\}\right]\sin(m\varphi)
\end{aligned}$$

Here, we have used the relations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-m}K_m(z)\right) &= -z^{-m}K_{m+1}(z) \\
\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{m}K_m(z)\right) &= -z^{m}K_{m-1}(z)
\end{aligned}$$

Thus, we arrive at Eq.(41).

3.1.3 Derivation of Eqs.(43) and (44)

From the boundary condition Eq.(36), we have

$$\begin{aligned}
\tilde{\psi} &= \frac{2\varepsilon y}{\hat{p}^2} \\
&= \frac{2\varepsilon \sin\varphi}{\hat{p}^2} \qquad \text{on} \quad r = r_s(\varphi) = 1
\end{aligned} \tag{3.2}$$

We substitute $r = 1$ into Eq.(41) and compare with Eq.(3.2). Then we easily see that $c_m = 0 \quad \text{for } m \geq 2$. For $m = 1$, $\tilde{\psi}$ becomes from Eq.(41)

$$\tilde{\psi} \approx \frac{\varepsilon}{2}c_1\left[\frac{1}{\hat{p}}\left\{-\frac{2}{\hat{p}}K_1(\hat{p}) + K_2(\hat{p})\right\}\right]\sin\varphi = \frac{\varepsilon}{2}c_1\left[\frac{1}{\hat{p}}K_0(\hat{p})\right]\sin\varphi$$

Therefore, we have

$$c_1 \approx \frac{4}{\hat{p}K_0(\hat{p})}$$

Thus, Eq.(44) is obtained from Eq.(39).

Furthermore, $\tilde{\psi}$ is finally obtained as

$$\tilde{\psi} \approx \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1(\hat{p}r) + \frac{K_2(\hat{p})}{r}\right\}\sin\varphi$$

The boundary condition $(\partial\tilde{\psi}/\partial n) = (1/p)(\partial y/\partial n)$ on S, that is, $(\partial\tilde{\psi}/\partial r) = 1/p\sin\varphi$ for $r = 1$, is automatically satisfied as follows;

$$\frac{\partial\tilde{\psi}}{\partial r} \approx \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left(-2K_1'(\hat{p}) - K_2(\hat{p})\right)\sin\varphi = \frac{2\varepsilon}{\hat{p}^2}\sin\varphi$$

where prime $'$ denotes the differentiation.

3.2 Derivation of Basic Relations in Region (II) and (III)

3.2.1 Derivation of Eq.(52)

We try to derive Eq.(52). F_0 is defined by Eq.(22). We use Eq.(38). Since $r = r^*/\varepsilon^{1/2}$, that is, $r \gg r_s$, F_0 is obtained as follows;

$$\begin{aligned}
F_0 &= \frac{1}{2\pi}\oint_S\left[G\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial G}{\partial n}\right]\mathrm{d}s \\
&\approx \frac{1}{2\pi}\oint_S\left[K_0\left((\varepsilon p_0)^{\frac{1}{2}}\rho_r\right)\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial K_0\left((\varepsilon p_0)^{\frac{1}{2}}\rho_r\right)}{\partial n}\right]\mathrm{d}s
\end{aligned}$$

Since $\varepsilon^{1/2}\rho_r = (r^{*2} + \varepsilon r_s^2 - 2\varepsilon^{1/2} r^* r_s \cos(\theta_s - \varphi))^{1/2}$, we have

$$\begin{aligned}
F_0 \approx\; & \frac{1}{2\pi}\oint_S \Bigg[\Bigg(K_0\left(p_0^{\frac{1}{2}}r^*\right) I_0\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right) \\
& +2\sum_{m=1}^{\infty} K_m\left(p_0^{\frac{1}{2}}r^*\right) I_m\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right)\cos m(\theta_s - \varphi)\Bigg)\frac{\partial\hat{\zeta}}{\partial n} \\
& -\hat{\zeta}\frac{\partial}{\partial n}\Bigg(K_0\left(p_0^{\frac{1}{2}}r^*\right) I_0\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right) \\
& +2\sum_{m=1}^{\infty} K_m\left(p_0^{\frac{1}{2}}r^*\right) I_m\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right)\cos m(\theta_s - \varphi)\Bigg)\Bigg]\mathrm{d}s \\
\approx\; & \frac{1}{2\pi}\oint_S \Bigg[2K_1\left(p_0^{\frac{1}{2}}r^*\right) I_1\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right)\cos(\theta_s - \varphi)\frac{\partial\hat{\zeta}}{\partial n} \\
& -\hat{\zeta}\frac{\partial}{\partial n}\left(2K_1\left(p_0^{\frac{1}{2}}r^*\right) I_1\left((\varepsilon p_0)^{\frac{1}{2}} r_s\right)\cos(\theta_s - \varphi)\right)\Bigg]\mathrm{d}s \\
\approx\; & \frac{1}{2\pi}\oint_S \Bigg[K_1\left(p_0^{\frac{1}{2}}r^*\right)(\varepsilon p_0)^{\frac{1}{2}} r_s \sin\theta_s \sin\varphi \frac{\partial\hat{\zeta}}{\partial n} \\
& -\hat{\zeta}\frac{\partial}{\partial n}\left(K_1\left(p_0^{\frac{1}{2}}r^*\right)(\varepsilon p_0)^{\frac{1}{2}} r_s \sin\theta_s \sin\varphi\right)\Bigg]\mathrm{d}s \\
\approx\; & (\varepsilon p_0)^{\frac{1}{2}} C_0 K_1\left(p_0^{\frac{1}{2}}r^*\right)\sin\varphi
\end{aligned}$$

where

$$C_0 \approx \frac{1}{2\pi}\oint_S \left\{\frac{\partial\hat{\zeta}}{\partial n} r_s \sin\theta_s - \hat{\zeta}\frac{\partial}{\partial n}(r_s\sin\theta_s)\right\}\mathrm{d}s$$

3.2.2 Derivation of Eqs.(53) and (54)

Let us consider the integral I_f for the variable r^* defined by Eq.(48). In this region, the variable r^* is fixed and we define $F^*(r^*, \varphi)$ as $\varepsilon^{-1}F$, as shown in Eq.(55). The concept proposed by Kida and Miyai (1973) is applied to the estimation of I_f, as similar as deriving Eq.(49): it is reasonable that F^* for $\varepsilon^{1/2} r_s \le r^* \le \delta_o$, where δ_o is an order function of ε with $\lim_{\varepsilon\to 0}\varepsilon^{1/2}/\delta_o = 0$, is assumed to be indeterminate.

We first consider the following integral;

$$I_{f_1} = \int_0^{2\pi}\int_{\varepsilon^{1/2}r_s}^{\delta_o} F^* K_0\left(p_0^{\frac{1}{2}}\rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta$$

In this integral, the function F^* is indeterminate. Further, since $r^* > r_1^*$, $K_0(p_0^{1/2}\rho_r^*)$ is expressed as

$$K_0\left(p_0^{\frac{1}{2}}\rho_r^*\right) = K_0\left(p_0^{\frac{1}{2}}r^*\right) I_0\left(p_0^{\frac{1}{2}}r_1^*\right) + 2\sum_{m=1}^{\infty} K_m\left(p_0^{\frac{1}{2}}r^*\right) I_m\left(p_0^{\frac{1}{2}}r_1^*\right)\cos m(\theta - \varphi) \tag{3.3}$$

Since the velocity field is symmetry and the vorticity field is anti-symmetry with respect to the x-axis, that is, the velocity field is even and the vorticity field is odd with respect to θ, we easily arrive at, by using Eq.(3.3);

$$I_{f_1} = \sum_{m=1}^{\infty} \tilde{D}_m K_m \left(p_0^{\frac{1}{2}} r^*\right) \sin(m\varphi)$$

where $\tilde{D}_m$ is indeterminate constant, whose order is given by

$$\begin{aligned} \tilde{D}_m &= O\left(\int_{\varepsilon^{1/2} r_s}^{\delta_o} F^* I_m \left(p_0^{\frac{1}{2}} r_1^*\right) r_1^* \mathrm{d}r_1^*\right) \\ &\approx O\left(\int_{\varepsilon^{1/2} r_s}^{\delta_o} F^* {r_1}^{*m+1} \mathrm{d}r_1^*\right) \\ &\approx O(\varepsilon^{\frac{m}{2}+1} F^*) \end{aligned}$$

Second, we consider another part of I_f, I_{f2}, defined by

$$\begin{aligned} I_{f_2} &\equiv \int_0^{2\pi} \int_{\delta_o}^{\infty} F^* K_0 \left(p_0^{\frac{1}{2}} \rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta \\ &= \int_0^{2\pi} \mathrm{Pf} \int_0^{\infty} F^* K_0 \left(p_0^{\frac{1}{2}} \rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta \\ &\quad - \int_0^{2\pi} \mathrm{Pf} \int_0^{\delta_o} F^* K_0 \left(p_0^{\frac{1}{2}} \rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta \end{aligned}$$

For the second integral of the right hand side of the above relation, we can use Eq.(3.3) :

$$\begin{aligned} (\text{second term}) &= \int_0^{2\pi} \mathrm{Pf} \int_0^{\delta_o} F^* \left[K_0 \left(p_0^{\frac{1}{2}} r^*\right) I_0 \left(p_0^{\frac{1}{2}} r_1^*\right) \right. \\ &\quad \left. + 2 \sum_{m=1}^{\infty} K_m \left(p_0^{\frac{1}{2}} r^*\right) I_m \left(p_0^{\frac{1}{2}} r_1^*\right) \cos m(\theta - \varphi) \right] r_1^* \mathrm{d}r_1^* \mathrm{d}\theta \\ &= \sum_{m=1}^{\infty} \hat{D}_m K_m \left(p_0^{\frac{1}{2}} r^*\right) \sin(m\varphi) \end{aligned}$$

where $\hat{D}_m$ is not indeterminate, but it is at most of the same order of $\tilde{D}_m$. Since $I_f = I_{f1} + I_{f2}$, we finally arrive at

$$I_f = \int_0^{2\pi} \mathrm{Pf} \int_0^{\infty} F^* K_0 \left(p_o^{\frac{1}{2}} \rho_r^*\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta + \sum_{m=1}^{\infty} D_m K_m \left(p_0^{\frac{1}{2}} r^*\right) \sin(m\varphi)$$

where D_m is indeterminate constant and its order is $O(\varepsilon^{m/2+1} F^*)$.

Here, we note that F_o is of $O(\varepsilon^{1/2})$, that is, $\hat{\zeta}^* \sim O(\varepsilon^{1/2})$, and f is defined by Eq.(7). Then, we see

$$\begin{aligned} F &\sim O\left(u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y}\right) \\ &\sim O\left(\frac{\partial \zeta}{\partial r}\right) \sim O\left(\varepsilon \frac{\partial \hat{\zeta}^*}{\partial r^*}\right) \end{aligned}$$

Therefore, we may assume that F^* is of $O(\varepsilon)$. Thus, we arrive at Eq.(53) and it is appropriate to define $\hat{\zeta}^*$, $\tilde{\psi}^*$ and F^* as Eq.(55).

We apply the above mentioned procedure to the following integral in order to obtain $\tilde{\psi}^*$:

$$J_f \equiv \int_0^{2\pi} \int_{\varepsilon^{1/2} r_s}^{\infty} \hat{\zeta}^* \exp\left(-\varepsilon^{\frac{1}{2}} r_1^* \cos\theta\right) \log\left(\frac{1}{\rho^*}\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta$$

We assume that $\hat{\zeta}^*$ for $\varepsilon^{1/2} r_s \le r^* \le \delta_o$ is indeterminate. Then we have

$$\int_0^{2\pi} \int_{\varepsilon^{1/2} r_s}^{\delta_o} \hat{\zeta}^* \log\left(\frac{1}{\rho^*}\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta = \sum_{m=1}^{\infty} \tilde{H}_m \frac{1}{r^{*m}} \sin(m\varphi)$$

where $\tilde{H}_m \sim O\left(\int_{\varepsilon^{1/2} r_s}^{\delta_o} \hat{\zeta}^* {r_1}^{*m+1} \mathrm{d}r_1^*\right) \sim O(\varepsilon^{m/2+1} \hat{\zeta}^*)$. Further, we rewrite J_f as

$$J_f = \int_0^{2\pi} \left(\mathrm{Pf}\int_0^{\infty} - \mathrm{Pf}\int_0^{\delta_o} + \int_{\varepsilon^{1/2} r_s}^{\delta_o}\right) \hat{\zeta}^* \exp\left(-\varepsilon^{\frac{1}{2}} r_1^* \cos\theta\right) \log\left(\frac{1}{\rho^*}\right) r_1^* \mathrm{d}r_1^* \mathrm{d}\theta$$

The last integral of the right hand side was already obtained. The second integral is also expressed as the same form of the last integral and its coefficient is at most of the same order of that of the last integral. Thus, we arrive at Eq.(54).

3.2.3 Derivation of Eq.(56)

In this region, the basic relations are given by Eqs.(49) and (50). Let us consider the case of the circular cylinder. The first approximation of $\hat{\zeta}$ is given from Eq.(49), by using Eq.(42) and $(\partial/\partial n) = -(\partial/\partial r_1)$:

$$\begin{aligned}\hat{\zeta} &\approx \frac{1}{2\pi} \int_0^{2\pi} \Bigg[-\frac{\partial \hat{\zeta}}{\partial r_1} \left(-\log r + \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{1}{r}\right)^m \cos\{m(\theta - \varphi)\} \right) \\ &\quad + \hat{\zeta} \sum_{m=1}^{\infty} \left(\frac{1}{r}\right)^m \cos\{m(\theta-\varphi)\} \Bigg] \mathrm{d}\theta\end{aligned}$$

From the symmetry of flow, we see that $\hat{\zeta}$ is odd function with respect to θ. Thus, we arrive at

$$\hat{\zeta} \approx \sum_{m=1}^{\infty} a_m \sin(m\varphi) \left(\frac{1}{r}\right)^m \qquad (3.4)$$

where

$$a_m = -\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{m} \frac{\partial \hat{\zeta}}{\partial r} - \hat{\zeta} \right)_{r=1} \sin(m\theta) \mathrm{d}\theta$$

Thus, we arrive at Eq.(56).

3.2.4 Derivation of Eq.(57)

Substituting Eq.(3.4) into $\tilde{\psi}$ given by Eq.(50), we have

$$
\begin{aligned}
\tilde{\psi} &\approx \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r_s}^{\infty} \hat{\zeta}\log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad -\frac{\varepsilon}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r_s}^{\infty} \hat{\zeta}\log\left(\frac{1}{\rho_r}\right) r_1^2 \cos\theta \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad -\frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi) \\
&\approx \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r_s}^{\infty} \hat{\zeta}\log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad -\frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi) \\
&\approx \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{1}^{\infty} \sum_{m=1}^{\infty} a_m \sin(m\theta)\left(\frac{1}{r_1}\right)^m \log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad -\frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi)
\end{aligned}
$$

We consider the following integral by taking into account of Eq.(42),

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{1}^{\infty} \sum_{k=1}^{\infty} a_k \sin(k\theta)\left(\frac{1}{r_1}\right)^k \log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{1}^{r} \sum_{k=1}^{\infty} a_k \sin(k\theta)\left(\frac{1}{r_1}\right)^k \\
&\quad \times\left\{-\log r + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_1}{r}\right)^m \cos\{m(\theta-\varphi)\}\right\} r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad +\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r}^{\infty} \sum_{k=1}^{\infty} a_k \sin(k\theta)\left(\frac{1}{r_1}\right)^k \\
&\quad \times\left\{-\log r_1 + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{r_1}\right)^m \cos\{m(\theta-\varphi)\}\right\} r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{1}^{r} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k \sin(k\theta)\left(\frac{1}{r_1}\right)^k \frac{1}{m}\left(\frac{r_1}{r}\right)^m \\
&\quad \times \sin(m\theta)\sin(m\varphi) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad +\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r}^{\infty} \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k \sin(k\theta)\left(\frac{1}{r_1}\right)^k \frac{1}{m}\left(\frac{r}{r_1}\right)^m \\
&\quad \times \sin(m\theta)\sin(m\varphi) r_1 \mathrm{d}r_1 \mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_1^r \sum_{k=1}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\left(\frac{1}{r_1}\right)^k \left(\frac{r_1}{r}\right)^k \sin(k\varphi) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_r^{\infty} \sum_{k=1}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\left(\frac{1}{r_1}\right)^k \left(\frac{r}{r_1}\right)^k \sin(k\varphi) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{k=1}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\left(\frac{1}{r}\right)^k \frac{r^2-1}{2}\sin(k\varphi) \\
&\quad + \sum_{k=2}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\frac{r^k}{2k-2}\frac{r^2}{r^{2k}}\sin(k\varphi) \\
&\quad - a_1 r\log(r)\sin^2\theta\sin\varphi\Bigg]\mathrm{d}\theta
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\tilde{\psi} &\approx \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{k=1}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\left(\frac{1}{r}\right)^k \frac{r^2-1}{2}\sin(k\varphi) \\
&\quad + \sum_{k=2}^{\infty} a_k \frac{\sin^2(k\theta)}{k}\frac{r^k}{2k-2}\frac{r^2}{r^{2k}}\sin(k\varphi) \\
&\quad - a_1 r\log(r)\sin^2\theta\sin\varphi\Bigg]\mathrm{d}\theta \\
&\quad - \frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi) \\
&\approx \frac{1}{2}a_1\sin\varphi\left(\frac{1}{r}\frac{r^2-1}{2} - r\log r\right) \\
&\quad + \frac{1}{2}\sum_{m=2}^{\infty} a_m \frac{\sin(m\varphi)}{m}\left[\left(\frac{1}{r}\right)^m \frac{r^2-1}{2} + \frac{r^m}{2m-2}\frac{r^2}{r^{2m}}\right] \\
&\quad - \frac{1}{2}\sum_{m=1}^{\infty} E_m^s r^m \sin(m\varphi)
\end{aligned}
$$

Thus, we arrive at Eq.(57).

3.2.5 Derivation of Eqs.(58) $\sim$ (61)

From the boundary condition, we have

$$
\tilde{\psi} = \frac{2}{p_0}\sin\varphi \qquad \text{on } r = 1
$$

Therefore, we have

$$
\tilde{\psi}_{r=1} \approx -\frac{1}{2}E_1^s\sin\varphi + \frac{1}{2}\sum_{m=2}^{\infty}\left(a_m\frac{1}{m(2m-2)} - E_m^s\right)\sin(m\varphi) = \frac{2}{p_0}\sin\varphi
$$

From this equation, we arrive at Eqs.(58) and (59):

$$E_1^s = -\frac{4}{p_0}$$

$$a_m = m(2m-2)E_m^s \approx O(\varepsilon) \quad (m \geq 2)$$

Thus, we arrive at Eqs.(60) and (61) :

$$\hat{\zeta} \approx \frac{a_1}{r}\sin\varphi$$

$$\tilde{\psi} \approx \left[\frac{1}{2}a_1\left\{\frac{1}{r}\frac{r^2-1}{2} - r\log r\right\} + \frac{1}{p}r\right]\sin\varphi$$

From this equation, we have

$$\left.\frac{\partial\tilde{\psi}}{\partial r}\right|_{r=1} \approx \frac{1}{p}\sin\varphi$$

Thus, we see that the boundary condition $\partial\tilde{\psi}/\partial n = (1/p)\partial y/\partial n$ on S is automatically satisfied.

3.2.6 Derivation of Eq.(62)

Let us consider the matching of $\hat{\zeta}$ between the region (II) and (III) in the case of the circular cylinder. From Eq.(53), we have, since $K_1(z) \approx 1/z$ as $z \to 0$;

$$\hat{\zeta}^*_{III\to II} \approx \frac{C_0}{r^*}\sin\varphi$$

where suffix 'III $\to$ II' denotes the asymptotic expansion of functions in the region (III) with respect to r. From the solution of the region (II), Eq.(61), we have

$$\hat{\zeta}_{II\to III} \sim a_1\frac{\varepsilon^{\frac{1}{2}}}{r^*}\sin\varphi$$

where suffix 'II $\to$ III' denotes the asymptotic expansion of functions in the region (II) with respect to r^*. From the matching requirement of $\hat{\zeta}_{II\to III} = \varepsilon^{\frac{1}{2}}\hat{\zeta}^*_{III\to II}$, we have

$$a_1 = C_0$$

Thus, the estimation of D_m, which is of $O(\varepsilon^{m/2+1}F^*)$ mentioned in the preceding section, is reasonable, so that we have

$$D_m \approx 0$$

We substitute $\hat{\zeta}^*$ into Eq.(54), by taking into account of the above result, then we have $\tilde{\psi}^*$ as

$$\tilde{\psi}^* \approx \frac{C_0}{2\pi}p_0^{\frac{1}{2}}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}K_1\left(p_0^{\frac{1}{2}}r_1^*\right)\sin\theta\log\left(\frac{1}{\rho_r^*}\right)r_1^*\mathrm{d}r_1^*\mathrm{d}\theta + \frac{H_1}{2r^*}\sin\varphi$$

Let us consider the following integral:

$$
\begin{aligned}
&\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} K_1\left(p_0^{\frac{1}{2}}r_1^*\right)\sin\theta\log\left(\frac{1}{\rho_r^*}\right)r_1^*\mathrm{d}r_1^*\mathrm{d}\theta \\
&= \int_0^{2\pi}\Bigg[\mathrm{Pf}\int_0^{r^*} K_1\left(p_0^{\frac{1}{2}}r_1^*\right)\sin\theta\Bigg\{-\log r^* \\
&\quad + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_1^*}{r^*}\right)^m\cos\{m(\theta-\varphi)\}\Bigg\}r_1^*\mathrm{d}r_1^* \\
&\quad + \mathrm{Pf}\int_{r^*}^{\infty} K_1\left(p_0^{\frac{1}{2}}r_1^*\right)\sin\theta\Bigg\{-\log r_1^* \\
&\quad + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r^*}{r_1^*}\right)^m\cos\{m(\theta-\varphi)\}\Bigg\}r_1^*\mathrm{d}r_1^*\Bigg]\mathrm{d}\theta \\
&= \int_0^{2\pi}\Bigg[\mathrm{Pf}\int_0^{r^*} K_1\left(p_0^{\frac{1}{2}}r_1^*\right)\frac{r_1^{*2}}{r^*}\mathrm{d}r_1^* \\
&\quad + \mathrm{Pf}\int_{r^*}^{\infty} K_1\left(p_0^{\frac{1}{2}}r_1^*\right)r^*\mathrm{d}r_1^*\Bigg]\sin^2\theta\mathrm{d}\theta\sin\varphi \\
&= \pi\left[\lim_{\delta_0\to 0}\left[-\frac{r_1^{*2}}{p_0^{\frac{1}{2}}r^*}K_2\left(p_0^{\frac{1}{2}}r_1^*\right)\right]_{\delta_0}^{r^*} + r^*\frac{K_0\left(p_0^{\frac{1}{2}}r^*\right)}{p_0^{\frac{1}{2}}}\right]\sin\varphi \\
&= \pi\left[-\frac{r^*}{p_0^{\frac{1}{2}}}K_2\left(p_0^{\frac{1}{2}}r^*\right) + \frac{2}{p_0^{\frac{3}{2}}}\frac{1}{r^*} + r^*\frac{K_0\left(p_0^{\frac{1}{2}}r^*\right)}{p_0^{\frac{1}{2}}}\right]\sin\varphi
\end{aligned}
$$

Using this result, we arrive at Eq.(62):

$$
\tilde{\psi}^* \approx \frac{C_0}{2}p_0^{\frac{1}{2}}\left[-\frac{r^*}{p_0^{\frac{1}{2}}}K_2\left(p_0^{\frac{1}{2}}r^*\right) + \frac{2}{p_0^{\frac{3}{2}}}\frac{1}{r^*} + r^*\frac{K_0\left(p_0^{\frac{1}{2}}r^*\right)}{p_0^{\frac{1}{2}}}\right]\sin\varphi + \frac{H_1}{2r^*}\sin\varphi
$$

3.2.7 Derivation of Eqs.(63) and (64)

From Eq.(62), we have

$$
\tilde{\psi}^*_{III\to II} \approx \left[\frac{C_0}{2}r^*\left\{-\gamma - \log\left(\frac{p_0^{\frac{1}{2}}}{2}\right) - \log r^*\right\} + \frac{C_0}{2}\frac{r^*}{2}\right]\sin\varphi + \frac{H_1}{2r^*}\sin\varphi
$$

From Eq.(61), we have

$$
\begin{aligned}
\tilde{\psi}_{II\to III} &\approx \left[\frac{1}{2}a_1\left\{\frac{1}{r}\frac{r^2-1}{2} - r\log r\right\} + \frac{r}{p}\right]\sin\varphi \\
&\approx \left[\frac{1}{2}a_1\left\{\frac{\varepsilon^{\frac{1}{2}}}{r^*}\frac{\left(\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)^2 - 1}{2} - \frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\left(\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\right\} + \frac{1}{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right]\sin\varphi
\end{aligned}
$$

$$\approx \left[\frac{1}{2}a_1\left\{\frac{1}{2}\frac{r^*}{\varepsilon^{\frac{1}{2}}} - \frac{\varepsilon^{\frac{1}{2}}}{2r^*} - \frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\left(\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\right\} + \frac{1}{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right]\sin\varphi$$

$$\approx \frac{1}{\varepsilon^{\frac{1}{2}}}\left[\frac{1}{2}a_1\left(\frac{r^*}{2} - r^*\log\left(\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\right) + \frac{r^*}{p}\right]\sin\varphi - \frac{1}{2}a_1\frac{\varepsilon^{\frac{1}{2}}}{2r^*}\sin\varphi$$

$$\varepsilon^{\frac{1}{2}}\tilde{\psi}_{II\to III} \approx \left[\frac{1}{2}a_1\left(\frac{r^*}{2} - r^*\log\left(\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\right) + \frac{r^*}{p}\right]\sin\varphi - \frac{1}{2}a_1\frac{\varepsilon}{2r^*}\sin\varphi$$

From the matching requirement, $\tilde{\psi}^*_{III\to II} = \varepsilon^{1/2}\tilde{\psi}_{II\to III}$, we have

$$\begin{aligned}
\frac{1}{2}a_1\left(\frac{1}{2} + \log\varepsilon^{\frac{1}{2}}\right) + \frac{1}{p} &= \frac{C_0}{2}\left(-\gamma - \log\left(\frac{(2p)^{\frac{1}{2}}}{2}\right)\right) + \frac{C_0}{4} \\
&\qquad \text{(from the term of } r^*\sin\varphi\text{)} \\
-\frac{1}{2}a_1 &= -\frac{C_0}{2} \quad \text{(from the term of } r^*\log r^*\sin\varphi\text{)} \\
-\frac{1}{2}a_1\frac{\varepsilon}{2} &= \frac{H_1}{2} \quad \text{(from the term of } r^{*-1}\sin\varphi\text{)}
\end{aligned}$$

Thus, we arrive at Eqs.(63) and (64).

3.3 Derivation of Basic Relations in Region (IV) and (V)

3.3.1 Derivation of Eq.(72)

Let us consider F_o given in Eq.(22) in the region (IV). Since $R = \varepsilon r$, G is given by $G = K_0((1+\tilde{p})^{1/2}\rho^*_R)$. Therefore, we have

$$\begin{aligned}
F_0 &= \frac{1}{2\pi}\oint_S\left[G\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial G}{\partial n}\right]\mathrm{d}s \\
&= \frac{1}{2\pi}\oint_S\left[K_0\left((1+\tilde{p})^{\frac{1}{2}}\rho^*_R\right)\frac{\partial\hat{\zeta}}{\partial n} - \hat{\zeta}\frac{\partial K_0\left((1+\tilde{p})^{\frac{1}{2}}\rho^*_R\right)}{\partial n}\right]\mathrm{d}s
\end{aligned}$$

Here, we apply Eq.(38) to this equation and take into account the symmetry of flow. Then, we easily arrive at Eq.(72) :

$$\begin{aligned}
F_0 &\approx \frac{1}{2\pi}\oint_S\Big[2K_1\left((1+\tilde{p})^{\frac{1}{2}}R\right)I_1\left(\varepsilon(1+\tilde{p})^{\frac{1}{2}}r_s\right)\cos(\theta_s-\varphi)\frac{\partial\hat{\zeta}}{\partial n} \\
&\qquad -\hat{\zeta}\frac{\partial}{\partial n}\left(2K_1\left((1+\tilde{p})^{\frac{1}{2}}R\right)I_1\left(\varepsilon(1+\tilde{p})^{\frac{1}{2}}r_s\right)\cos(\theta_s-\varphi)\right)\Big]\mathrm{d}s + O(\varepsilon^2) \\
&\approx \frac{1}{2\pi}\oint_S\Big[K_1\left((1+\tilde{p})^{\frac{1}{2}}R\right)\varepsilon(1+\tilde{p})^{\frac{1}{2}}r_s\sin\theta_s\sin\varphi\frac{\partial\hat{\zeta}}{\partial n} \\
&\qquad -\hat{\zeta}\frac{\partial}{\partial n}\left(K_1\left((1+\tilde{p})^{\frac{1}{2}}R\right)\varepsilon(1+\tilde{p})^{\frac{1}{2}}r_s\sin\theta_s\sin\varphi\right)\Big] \\
&\approx \varepsilon(1+\tilde{p})^{\frac{1}{2}}\hat{C}_0K_1\left((1+\tilde{p})^{\frac{1}{2}}R\right)\sin\varphi
\end{aligned}$$

where

$$\hat{C}_0 \approx \frac{1}{2\pi} \oint_S \left\{ \frac{\partial \hat{\zeta}}{\partial n} r_s \sin\theta_s - \hat{\zeta} \frac{\partial}{\partial n}(r_s \sin\theta_s) \right\} \mathrm{d}s$$

3.3.2 Derivation of Eqs.(73) and (74)

We first consider the integral I_f defined by Eq.(48).

$$\begin{aligned} I_f &= \frac{1}{\varepsilon^2} \int_0^{2\pi} \int_{\varepsilon r_s}^{\infty} F K_0 \left((1+\tilde{p})^{\frac{1}{2}} \rho_R^* \right) R_1 \mathrm{d}R_1 \mathrm{d}\theta \\ &= \frac{1}{\varepsilon^2} \int_0^{2\pi} \int_{\delta_o}^{\infty} F K_0 \left((1+\tilde{p})^{\frac{1}{2}} \rho_R^* \right) R_1 \mathrm{d}R_1 \mathrm{d}\theta \\ &\quad + \frac{1}{\varepsilon^2} \int_0^{2\pi} \int_{\varepsilon r_s}^{\delta_o} F K_0 \left((1+\tilde{p})^{\frac{1}{2}} \rho_R^* \right) R_1 \mathrm{d}R_1 \mathrm{d}\theta \end{aligned}$$

where δ_o is of $O(\varepsilon^{1-s})$ (for $1 > s > 0$). Since F for $\varepsilon r_s \le R_1 \le \delta_o$ is indeterminate, the second term of r.h.s. of the above equation becomes as

$$\begin{aligned} &\text{(second term of r.h.s)} \\ &= \int_0^{2\pi} \int_{\varepsilon r_s}^{\delta_o} F \Bigg[K_0 \left((1+\tilde{p})^{\frac{1}{2}} R \right) I_0 \left((1+\tilde{p})^{\frac{1}{2}} R_1 \right) \\ &\quad + 2 \sum_{m=1}^{\infty} K_m \left((1+\tilde{p})^{\frac{1}{2}} R \right) I_m \left((1+\tilde{p})^{\frac{1}{2}} R_1 \right) \cos m(\theta - \varphi) \Bigg] R_1 \mathrm{d}R_1 \mathrm{d}\theta \\ &= \sum_{m=1}^{\infty} \hat{E}_m K_m \left((1+\tilde{p})^{\frac{1}{2}} R \right) \sin(m\varphi) \end{aligned}$$

where $\hat{E}_m$ is indeterminate constant and its order is

$$\begin{aligned} \hat{E}_m &\sim O\left[\int_{\varepsilon r_s}^{\delta_o} F I_m((1+\tilde{p})^{\frac{1}{2}} R_1) R_1 \mathrm{d}R_1 \right] \\ &\sim O\left[\int_{\varepsilon r_s}^{\delta_o} F R_1^{m+1} \mathrm{d}R_1 \right] \\ &\sim O(\varepsilon^{m+2} F) \end{aligned}$$

The first term is divided as $(\int_0^{\infty} - \int_0^{\delta_o})[\cdots] R_1 \mathrm{d}R_1$ and we estimate the integral due to $\int_0^{\delta_o}[\cdots] R_1 \mathrm{d}R_1$. We may easily see that this term is included in the second term of r.h.s. obtained above. This result is similar to that mentioned in section 3.3. Thus, we finally arrive at

$$\begin{aligned} I_f &= \frac{1}{\varepsilon^2} \int_0^{2\pi} \mathrm{Pf} \int_0^{\infty} F K_0 \left((1+\tilde{p})^{\frac{1}{2}} \rho_R^* \right) R_1 \mathrm{d}R_1 \mathrm{d}\theta \\ &\quad + \sum_{m=1}^{\infty} E_m K_m \left((1+\tilde{p})^{\frac{1}{2}} R \right) \sin(m\varphi) \end{aligned}$$

Here, $\hat{\zeta}$ is of $O(\varepsilon)$, so that $E_m \sim \hat{E}_m/\varepsilon^2 \sim O(\varepsilon^m F)$. Using this equation, we have in this region;

$$\hat{\zeta} \approx \varepsilon (1+\tilde{p})^{\frac{1}{2}} \hat{C}_0 K_1 \left((1+\tilde{p})^{\frac{1}{2}} R \right) \sin\varphi$$

$$
\begin{aligned}
&-\frac{1}{2\pi}\frac{1}{\varepsilon}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty FK_0\left((1+\tilde{p})^{\frac{1}{2}}\rho_R^*\right)R_1\mathrm{d}R_1\mathrm{d}\theta \\
&+\varepsilon\sum_{m=1}^{\infty}E_mK_m\left((1+\tilde{p})^{\frac{1}{2}}R\right)\sin(m\varphi)
\end{aligned}
$$

We see from Eq.(77) that $\hat{\zeta}\sim 1/r$ for $(r\to\infty)$, therefore, from the requirement of the matching, $\hat{\zeta}$ for the variable R is of $O(\varepsilon)$. Further, we suppose that u is of $O(\varepsilon^s)$ for $(s>0)$ for R. Therefore, f defined by Eq.(7) is of $O(\varepsilon^s(\partial\zeta/\partial x))\sim O(\varepsilon^{1+s}(\partial\hat{\zeta}/\partial R))\sim O(\varepsilon^{2+s})$. Thus, it is appropriate to define F^* as $F^*=F/\varepsilon^2$, and since $\hat{\zeta}^*=\hat{\zeta}/\varepsilon$, we arrive at Eq.(73).

We note however that $\hat{C}_0$ is of order of $1/\log\varepsilon$ and u becomes also of order of $1/\log\varepsilon$, as will be shown in next section 3.3.8. Thus, the first approximation of this region (IV) is given by the first term of r.h.s. of Eq.(73).

Let us consider the stream function $\tilde{\psi}$. From Eq.(24), we have

$$
\begin{aligned}
\tilde{\psi} &= \frac{1}{\varepsilon^2}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\infty}\exp(-R_1\cos\theta)\hat{\zeta}\log\left(\frac{\varepsilon}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta \\
&= \frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\infty}\exp(-R_1\cos\theta)\hat{\zeta}^*\log\left(\frac{\varepsilon}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta \\
&= \frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\delta_o}\exp(-R_1\cos\theta)\hat{\zeta}^*\log\left(\frac{\varepsilon}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta \\
&\quad+\frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\delta_o}^{\infty}\exp(-R_1\cos\theta)\hat{\zeta}^*\log\left(\frac{\varepsilon}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta
\end{aligned}
$$

Here, we apply Kida and Miyai (1973) to the above equation also. We consider the first term of r.h.s. of the above last equation. Taking into account of flow symmetry, we arrive at

$$
\begin{aligned}
&\text{(first term)} \\
&= \frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\delta_o}\exp(-R_1\cos\theta)\hat{\zeta}^*\log\left(\frac{1}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta \\
&= \frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\delta_o}\exp(-R_1\cos\theta)\hat{\zeta}^*\Bigg[-\log R \\
&\quad+\sum_{m=1}^{\infty}\left(\frac{R_1}{R}\right)^m\frac{\cos m(\theta-\varphi)}{m}\Bigg]R_1\mathrm{d}R_1\mathrm{d}\theta \\
&= \sum_{m=1}^{\infty}\frac{1}{R^m}\frac{1}{\varepsilon}\frac{1}{2\pi}\int_0^{2\pi}\int_{\varepsilon r_s}^{\delta_o}\exp(-R_1\cos\theta)\hat{\zeta}^*R_1^m\frac{\sin(m\theta)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta\sin(m\varphi) \\
&= \sum_{m=1}^{\infty}\frac{H_m^s}{R^m}\sin(m\varphi)
\end{aligned}
$$

where H_m^s is indeterminate constant and its order is

$$
H_m^s\sim O\left(\frac{1}{\varepsilon}\int_{\varepsilon r_s}^{\delta_o}\hat{\zeta}^*R_1^{m+1}\mathrm{d}R_1\right)\sim O(\varepsilon^{m+1}\hat{\zeta}^*)
$$

The second term is obtained by dividing the integral region into $(0,\infty)$ and $(0,\delta_o)$. The functional form due to $(0,\delta_o)$ is the same as the above equation. Thus, we arrive at Eq.(74).

3.3.3 Derivation of Eq.(75) ← (This section is added.)

In the region (V), we see that p_0 in section 3.2.1 is corresponding to $(1+\tilde{p})^{1/2}$. Thus, we easily arrive at

$$\begin{aligned}\hat{\zeta} &\approx \frac{1}{2\pi}\oint_S\left\{\frac{\partial\hat{\zeta}}{\partial n}\log\frac{1}{\rho_r}-\hat{\zeta}\frac{\partial}{\partial n}\log\frac{1}{\rho_r}\right\}\mathrm{d}s \\ &\quad+\frac{\varepsilon}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}F\log(\rho_r)r_1\mathrm{d}r_1\mathrm{d}\theta-\frac{\varepsilon^{1/2}}{2}\sum_{m=1}^{\infty}C_m^V r^m\sin(m\varphi)\end{aligned}$$

where C_m^V is an indeterminate constant being at most of O(1). Thus, we can arrive at Eq.(75).

3.3.4 Derivation of Eq.(76)

Eq.(76) is derived by using the result of the matching with the solution in the region (IV). In the region (V), the stream function is given by following the similar procedure as Eq.(50) :

$$\tilde{\psi} \approx \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}\hat{\zeta}\log\left(\frac{1}{\rho_r}\right)r_1\mathrm{d}r_1\mathrm{d}\theta+E_1^V r\sin\varphi$$

where E_1^V is indeterminate constant.

Here, we consider the matching between the region (IV) and (V). From the above relation, we have for $r\to\infty$;

$$\begin{aligned}\tilde{\psi}_{V\to IV} &\approx E_1^V r\sin\varphi+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{r}\hat{\zeta}\Bigg[-\log r \\ &\quad+\sum_{m=1}^{\infty}\left(\frac{r_1}{r}\right)^m\frac{\cos m(\theta-\varphi)}{m}\Bigg]r_1\mathrm{d}r_1\mathrm{d}\theta \\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r}^{\infty}\hat{\zeta}\Bigg[-\log r_1 \\ &\quad+\sum_{m=1}^{\infty}\left(\frac{r}{r_1}\right)^m\frac{\cos m(\theta-\varphi)}{m}\Bigg]r_1\mathrm{d}r_1\mathrm{d}\theta \\ &\approx E_1^V r\sin\varphi \\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r_s}^{\infty}\hat{\zeta}\sum_{m=1}^{\infty}\left(\frac{r_1}{r}\right)^m\frac{\sin(m\theta)}{m}r_1\mathrm{d}r_1\mathrm{d}\theta\sin(m\varphi) \\ &\quad+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_{r}^{\infty}\hat{\zeta}\sum_{m=1}^{\infty}\Bigg(-\left(\frac{r_1}{r}\right)^m \\ &\quad+\left(\frac{r}{r_1}\right)^m\Bigg)\frac{\sin(m\theta)}{m}r_1\mathrm{d}r_1\mathrm{d}\theta\sin(m\varphi)\end{aligned}$$

From Eq.(75), we have for $r \to \infty$;

$$\hat{\zeta} \approx \frac{1}{2\pi}\oint(-\sigma_0+\mu_0 r_s)\cos\theta_s \mathrm{d}s\sin\varphi\frac{1}{r}$$

Therefore, we have $\hat{\zeta} \approx (\tilde{C}_1^V/r)\sin\varphi$ for $r \to \infty$. Using this result, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_r^\infty \hat{\zeta}\sum_{m=1}^\infty\left(-\left(\frac{r_1}{r}\right)^m\right. \\
&\qquad \left.+\left(\frac{r}{r_1}\right)^m\right)\frac{\sin(m\theta)}{m}r_1\mathrm{d}r_1\mathrm{d}\theta\sin(m\varphi) \\
&\approx \frac{1}{2}\mathrm{Pf}\int_r^\infty \tilde{C}_1^V\frac{1}{r_1}\left(\frac{r}{r_1}-\frac{r_1}{r}\right)r_1\mathrm{d}r_1\sin\varphi \\
&\approx \frac{1}{2}\tilde{C}_1^V\left(-r\log r+\frac{r}{2}\right)\sin\varphi
\end{aligned}$$

On the other hand, we have from Eq.(74)

$$\begin{aligned}
\hat{\psi}^*_{IV\to V} &\approx \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^R \exp(-R_1\cos\theta)\hat{\zeta}^* \\
&\quad\times\left[-\log R+\sum_{m=1}^\infty\left(\frac{R_1}{R}\right)^m\frac{\cos m(\theta-\varphi)}{m}\right]R_1\mathrm{d}R_1\mathrm{d}\theta \\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_R^\infty \exp(-R_1\cos\theta)\hat{\zeta}^* \\
&\quad\times\left[-\log R_1+\sum_{m=1}^\infty\left(\frac{R}{R_1}\right)^m\frac{\cos m(\theta-\varphi)}{m}\right]R_1\mathrm{d}R_1\mathrm{d}\theta \\
&\approx \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^R \exp(-R_1\cos\theta) \\
&\quad\times\hat{\zeta}^*\sum_{m=1}^\infty\left[\left(\frac{R_1}{R}\right)^m-\left(\frac{R}{R_1}\right)^m\right]\frac{\sin(m\theta)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta\sin(m\varphi) \\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \exp(-R_1\cos\theta) \\
&\quad\times\hat{\zeta}^*\sum_{m=1}^\infty\left(\frac{R}{R_1}\right)^m\frac{\cos m(\theta-\varphi)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta
\end{aligned}$$

Here, we use the relation from Eq.(73) ; $\hat{\zeta}^* \approx (C_0^{IV}/R)\sin\varphi$ for $R \to 0$. Then, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^R \exp(-R_1\cos\theta)\hat{\zeta}^*\sum_{m=1}^\infty\left[\left(\frac{R_1}{R}\right)^m\right. \\
&\left.-\left(\frac{R}{R_1}\right)^m\right]\frac{\sin(m\theta)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta\sin(m\varphi) \\
&\approx \frac{1}{2}C_0^{IV}\left(\frac{R}{2}-R\log R\right)\sin\varphi
\end{aligned}$$

Therefore, the matching requirement gives us

$$\begin{aligned}
\tilde{C}_1^V &= C_0^{IV} = \frac{\hat{C}_0}{2} \\
E_1^V &\approx -\frac{C_0^{IV}}{2}\log\varepsilon + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} \exp(-R_1\cos\theta)\hat{\zeta}^* \sin\theta \mathrm{d}R_1 \mathrm{d}\theta
\end{aligned}$$

Thus, we arrive at Eq.(76).

3.3.5 Derivation of Eqs.(77) and (78)

Let us consider the solution in the region (V) for the circular cylinder. Basic relations are given by Eqs.(75) and (76).

$$\begin{aligned}
\hat{\zeta} &\approx \oint_S \left[\sigma_0 \frac{\partial}{\partial n}\log\rho_r - \mu_0 \log\rho_r\right] \mathrm{d}s \\
\tilde{\psi} &\approx -r\sin\varphi\left[\frac{\alpha_1}{2(1+\tilde{p})^{\frac{1}{2}}}\log\varepsilon \right.\\
&\quad \left. -\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_0^{\infty} \hat{\zeta}^* \exp\left(-R_1\cos\theta\right)\sin\theta \mathrm{d}R_1 \mathrm{d}\theta\right] \\
&\quad +\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r_s}^{\infty} \hat{\zeta}\log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta
\end{aligned}$$

For the circular cylinder, we easily arrive at Eq.(77), from Eq.(75) by taking into account of Eq.(42);

$$\hat{\zeta} \approx \sum_{m=1}^{\infty} \frac{\tilde{C}_m}{r^m}\sin\left(m\varphi\right)$$

Then, we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Pf}\int_{r_s}^{\infty} \hat{\zeta}\log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\approx \frac{1}{2\pi}\mathrm{Pf}\int_1^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{C}_m}{r_1^m}\sin\left(m\theta\right)\log\left(\frac{1}{\rho_r}\right) r_1 \mathrm{d}r_1 \mathrm{d}\theta \\
&\approx \frac{1}{2\pi}\int_0^{2\pi}\left[\mathrm{Pf}\int_1^{r} \sum_{k=1}^{\infty} \frac{\tilde{C}_k}{r_1^k}\sin\left(k\theta\right)\left\{-\log r \right.\right.\\
&\quad \left. +\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r_1}{r}\right)^m \cos\left\{m(\theta-\varphi)\right\}\right\} r_1 \mathrm{d}r_1 \\
&\quad +\mathrm{Pf}\int_r^{\infty} \sum_{k=1}^{\infty} \frac{\tilde{C}_k}{r_1^k}\sin\left(k\theta\right)\left\{-\log r_1 \right.\\
&\quad \left.\left. +\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{r}{r_1}\right)^m \cos\left\{m(\theta-\varphi)\right\}\right\} r_1 \mathrm{d}r_1\right]\mathrm{d}\theta \\
&\approx \frac{1}{2\pi}\int_0^{2\pi}\left[\mathrm{Pf}\int_1^{r} \sum_{k=1}^{\infty} \frac{\tilde{C}_k}{r_1^k}\frac{\sin^2\left(k\theta\right)}{k}\left(\frac{r_1}{r}\right)^k r_1 \mathrm{d}r_1\right.
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Pf}\int_r^\infty \sum_{k=1}^\infty \frac{\tilde{C}_k}{r_1^k}\frac{\sin^2(k\theta)}{k}\left(\frac{r}{r_1}\right)^k r_1 \mathrm{d}r_1\Bigg]\mathrm{d}\theta\sin(k\varphi) \\
\approx\ & \frac{1}{2}\left[\sum_{k=1}^\infty \tilde{C}_k \frac{1}{k}\frac{1}{2r^k}(r^2-1)\sin(k\varphi) + \sum_{k=2}^\infty \tilde{C}_k r^k \frac{1}{k}\frac{1}{2k-2}\frac{1}{r^{2k-2}}\sin(k\varphi)\right. \\
&\left. -\tilde{C}_1 r\log(r)\sin\varphi\right]
\end{aligned}
$$

Therefore, we arrive at Eq.(78) :

$$
\begin{aligned}
\tilde{\psi} \approx\ & -r\sin\varphi\left[\frac{\alpha_1}{2(1+\tilde{p})^{\frac{1}{2}}}\log\varepsilon\right. \\
&\left. -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \hat{\zeta}^* \exp(-R_1\cos\theta)\sin\theta \mathrm{d}R_1\mathrm{d}\theta\right] \\
&+\frac{1}{2}\left[\sum_{k=1}^\infty \tilde{C}_k \frac{1}{k}\frac{1}{2r^k}(r^2-1)\sin(k\varphi) + \sum_{k=2}^\infty \tilde{C}_k r^k \frac{1}{k}\frac{1}{2k-2}\frac{1}{r^{2k-2}}\sin(k\varphi)\right. \\
&\left. -\tilde{C}_1 r\log(r)\sin\varphi\right]
\end{aligned}
$$

3.3.6 Derivation of Eqs.(79) and (80)

The boundary condition is given by $\tilde{\psi} = \dfrac{y}{p} = \dfrac{\sin\varphi}{p}$ for $r = r_s = 1$. Therefore, we have Eq.(79):

$$\tilde{C}_m = 0 : \quad m \geq 2$$

and we have

$$\frac{\alpha_1}{2(1+\tilde{p})^{\frac{1}{2}}}\log\varepsilon - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \hat{\zeta}^* \exp(-R_1\cos\theta)\sin\theta\mathrm{d}R_1\mathrm{d}\theta = -\frac{1}{p}$$

The above integral is expressed as

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty \hat{\zeta}^* \exp(-R_1\cos\theta)\sin\theta\mathrm{d}R_1\mathrm{d}\theta \\
&\quad\approx\ \frac{\alpha_1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\exp(-R_1\cos\theta)\sin^2\theta\mathrm{d}R_1\mathrm{d}\theta \\
&\quad\approx\ \alpha_1\,\mathrm{Pf}\int_0^\infty \frac{K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)I_1(R_1)}{R_1}\mathrm{d}R_1
\end{aligned}
$$

Thus, we arrive at Eq.(80).

3.3.7 Derivative of Eqs.(85) and (86)

Here, we use the following relation:

$$
\begin{aligned}
\mathbf{I} \ \equiv\ & \mathrm{Pf}\int_0^\infty \frac{K_1(ax)I_1(x)}{x}\mathrm{d}x \\
=\ & -\mathrm{Pf}\int_0^\infty \frac{K_0'(ax)I_1(x)}{x}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= -\lim_{\delta_0\to 0}\frac{K_0(ax)}{a}\frac{I_1(x)}{x}\bigg|_{\delta_0}^{\infty}+\frac{1}{a}\mathrm{Pf}\int_0^\infty K_0(ax)\frac{I_1'(x)x-I_1(x)}{x^2}\mathrm{d}x\\
&= \frac{1}{2a}\left(-\gamma-\log\frac{a}{2}\right)+\frac{1}{a}\mathrm{Pf}\int_0^\infty K_0(ax)\frac{I_2(x)}{x}\mathrm{d}x
\end{aligned}
$$

Further, we use the following relation:

$$
\begin{aligned}
\mathrm{Pf}\int_0^\infty K_0(ax)\frac{I_2(x)}{x}\mathrm{d}x &= \mathrm{Pf}\int_0^\infty K_0(ax)\frac{x}{4}\sum_{n=0}^{\infty}\frac{1}{n!(n+2)!}\left(\frac{x}{2}\right)^{2n}\mathrm{d}x\\
&= \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n!(n+2)!}\frac{1}{4^n}\frac{1}{a^{2n+2}}\mathrm{Pf}\int_0^\infty x^{2n+1}K_0(x)\mathrm{d}x\\
&= \frac{1}{4}\sum_{n=1}\frac{1}{n(n+1)}\frac{1}{a^{2n}}\\
&= \frac{1}{4}\left((a^2-1)\log\frac{a^2-1}{a^2}+1\right)
\end{aligned}
$$

Substituting these relations into Eq.(80) and taking into account of $a=(1+\tilde{p})^{1/2}$, we can arrive at Eqs.(85) and (86).

3.3.8 Note of Order of F^*

Let us consider Eq.(74) in detail. Substituting the first term of r.h.s. of Eq.(73) into Eq.(74), we arrive at

$$
\begin{aligned}
\tilde{\psi}^* &\approx \frac{\alpha_1}{2\pi}\int_0^{2\pi}\mathrm{Pf}\int_0^\infty K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\exp(-R_1\cos\theta)\\
&\quad\times\sin\theta\log\left(\frac{1}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta\\
&\approx \frac{\alpha_1}{2\pi}\sum_{m=1}^{\infty}\int_0^{2\pi}\mathrm{Pf}\int_0^R K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\exp(-R_1\cos\theta)\sin\theta\\
&\quad\times\left(\frac{R_1}{R}\right)^m\frac{\sin(m\theta)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta\sin(m\varphi)\\
&\quad+\frac{\alpha_1}{2\pi}\sum_{m=1}^{\infty}\int_0^{2\pi}\mathrm{Pf}\int_R^\infty K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\exp(-R_1\cos\theta)\sin\theta\\
&\quad\times\left(\frac{R}{R_1}\right)^m\frac{\sin(m\theta)}{m}R_1\mathrm{d}R_1\mathrm{d}\theta\sin(m\varphi)\\
&\approx -\alpha_1\sum_{m=1}^{\infty}\mathrm{Pf}\int_0^R K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\left(\frac{R_1}{R}\right)^m I_m(-R_1)\mathrm{d}R_1\sin(m\varphi)\\
&\quad-\alpha_1\sum_{m=1}^{\infty}\mathrm{Pf}\int_R^\infty K_1((1+\tilde{p})^{\frac{1}{2}}R_1)\left(\frac{R}{R_1}\right)^m I_m(-R_1)\mathrm{d}R_1\sin(m\varphi)
\end{aligned}
$$

In the above equation, we consider the term of $m=1$:

$$
\begin{aligned}
\psi_1^* &= \alpha_1\mathrm{Pf}\int_0^R K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\left(\frac{R_1}{R}\right)I_1(R_1)\mathrm{d}R_1\sin\varphi\\
&\quad+\alpha_1\mathrm{Pf}\int_R^\infty K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\left(\frac{R}{R_1}\right)I_1(R_1)\mathrm{d}R_1\sin\varphi
\end{aligned}
$$

$$= \alpha_1 \, \mathrm{Pf}\int_0^R K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\left(\frac{R_1}{R}-\frac{R}{R_1}\right) I_1(R_1)\mathrm{d}R_1 \sin\varphi$$

$$+\alpha_1 \, \mathrm{Pf}\int_0^\infty K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\left(\frac{R}{R_1}\right) I_1(R_1)\mathrm{d}R_1 \sin\varphi \qquad (3.5)$$

(This part is change.

Here, we consider the term of u due to ψ_1^*. Let (X, Y) be the stretched coordinates corresponding to R.

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi^*}{\partial Y} = \frac{\partial \psi^*}{\partial R}\cos\varphi + \frac{\partial \psi^*}{\partial \varphi}\frac{\cos\varphi}{R}$$

From this equation and Eq.(3.5), we have

$$u \sim \alpha_1 \, \mathrm{Pf}\int_0^R K_1\left((1+\tilde{p})^{1/2}\right)\frac{R_1}{R^2} I_1(R_1)\mathrm{d}R_1 \cos(2\varphi) + \alpha_1 \, \mathrm{Pf}\int_R^\infty K_1\left((1+\tilde{p})^{1/2}R_1\right)\frac{1}{R_1} I_1(R_1)\mathrm{d}R_1$$

Therefore, we have

$$\begin{aligned} u &\sim \alpha_1 \, \mathrm{Pf}\int_R^\infty K_1\left((1+\tilde{p})^{1/2}R_1\right)\frac{1}{R_1} I_1(R_1)\mathrm{d}R_1 \\ &\sim \alpha_1 \, \mathrm{Pf}\int_R^\infty \left(K_1\left((1+\tilde{p})^{1/2}R_1\right) I_1(R_1) - \frac{1}{2(1+\tilde{p})^{1/2}}\right)\mathrm{d}R_1 \\ &\quad +\alpha_1 \lim_{\delta_0\to 0}\frac{1}{2(1+\tilde{p})^{1/2}}\int_R^{1/\delta_0}\frac{1}{R_1}\mathrm{d}R_1 \\ &\sim \alpha_1 \, \mathrm{Pf}\int_R^\infty \left(K_1\left((1+\tilde{p})^{1/2}R_1\right) I_1(R_1) - \frac{1}{2(1+\tilde{p})^{1/2}}\right)\mathrm{d}R_1 - \frac{\alpha_1}{2(1+\tilde{p})^{1/2}}\log R \\ &\sim O(\alpha_1) \end{aligned}$$

Thus, we arrive at

$$u \sim O(\alpha_1) \qquad \text{for fixed } R$$

Therefore, we see from this relation that $u\partial\zeta/\partial x$ is of order of α_1^2 for fixed R, that is, $O(1/|\log\varepsilon|^2)$. Hence, the assumption $F^* \sim o(1)$ is reasonable.

4 Composite Solution

4.1 $\tilde{\psi}^c$ with respect to r

4.1.1 Derivation of $\tilde{\psi}_{I\to II} = \tilde{\psi}_{II}$

The functions $\tilde{\psi}_I$ and $\tilde{\psi}_{II}$ are given by Eq.(43) and Eq.(66) respectively:

$$\tilde{\psi}_I = \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1(\hat{p}r) + \frac{K_2(\hat{p})}{r}\right\}\sin\varphi \qquad (4.1\mathrm{a})$$

$$\tilde{\psi}_{II} = \frac{C_0}{2}\left(\frac{r^2-1}{2r} - r\log r\right)\sin\varphi + \frac{r}{p}\sin\varphi \qquad (4.1\mathrm{b})$$

$$\begin{cases} C_0 = -\dfrac{4}{p}\dfrac{1}{2\left(\gamma+\frac{1}{2}\log\varepsilon\right)+\log p-\log 2} \\ \hat{p} = (2\varepsilon p)^{\frac{1}{2}} \end{cases}$$

We here use the following relations of Bessel functions for $z \to 0$:

$$\begin{cases} K_0(z) \sim -\left(\gamma+\log\left(\dfrac{z}{2}\right)\right) \\ K_1(z) \sim \dfrac{1}{z}+\dfrac{z}{2}\left(\gamma+\log\left(\dfrac{z}{2}\right)-\dfrac{1}{2}\right) \\ K_2(z) \sim \dfrac{2}{z^2}-\dfrac{1}{2} \end{cases} \tag{4.2}$$

We substitute Eq.(4.2) into Eq.(4.1a), then we have

$$\begin{aligned} \tilde{\psi}_{I\to II} &\approx \frac{2\varepsilon}{\hat{p}^2}\frac{1}{-\left(\gamma+\log\left(\frac{\hat{p}}{2}\right)\right)}\left[-\frac{2}{\hat{p}}\left\{\frac{1}{\hat{p}r}+\frac{\hat{p}r}{2}\left(\gamma+\log\left(\frac{\hat{p}r}{2}\right)-\frac{1}{2}\right)\right\}\right. \\ &\quad\left.+\frac{1}{r}\left(\frac{2}{\hat{p}^2}-\frac{1}{2}\right)\right]\sin\varphi \\ &\approx \frac{2\varepsilon}{\hat{p}^2}\frac{1}{-\left(\gamma+\log\left(\frac{\hat{p}}{2}\right)\right)}\left\{-r\left(\gamma+\log\left(\frac{\hat{p}r}{2}\right)-\frac{1}{2}\right)-\frac{1}{2r}\right\}\sin\varphi \\ &\approx \frac{1}{p}\frac{1}{\gamma+\log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\left[r\left\{\gamma+\log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}+\log r-\frac{1}{2}\right\}+\frac{1}{2r}\right]\sin\varphi \\ &\approx \frac{r}{p}\sin\varphi+\frac{1}{p}\frac{1}{\gamma+\log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\left[r\log r-\frac{r}{2}+\frac{1}{2r}\right]\sin\varphi \end{aligned}$$

Thus, we have

$$\tilde{\psi}_{I\to II} = \tilde{\psi}_{II}$$

Let us consider $\tilde{\psi}_{II\to V}$. Since $C_{0II\to V} = C_0$ in a sense of generalized expansion, we have

$$\tilde{\psi}_{II\to V} = \tilde{\psi}_{II}$$

On the other hand, $\tilde{\psi}_V$ is given by Eq.(84). The constant $\tilde{C}_{1V\to II}$ becomes as

$$\tilde{C}_{1V\to II} \approx -\frac{1}{p}\frac{2}{\gamma+\log\frac{\varepsilon}{2}+A_{V\to II}}$$

Since $A_{V\to II} \approx (1/2)\log(2p/\varepsilon)$, we have

$$\tilde{\psi}_{V\to II} = \tilde{\psi}_{II}$$

Therefore, for the variable r we arrive at

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_{II} + \tilde{\psi}_V - \tilde{\psi}_{I\to II} - \tilde{\psi}_{II\to V} = \tilde{\psi}_I + \tilde{\psi}_V - \tilde{\psi}_{II} \\
&\approx \frac{1}{p}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left\{-\frac{2}{(2\varepsilon p)^{\frac{1}{2}}}K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right) + \frac{1}{r}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)\right\}\sin\varphi \\
&\quad -\frac{1}{p\hat{A}}\left(\frac{r}{2} - \frac{1}{2r} - r\log r\right)\sin\varphi \\
&\quad +\frac{1}{p}\frac{1}{\gamma + \log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\left(\frac{r}{2} - \frac{1}{2r} - r\log r\right)\sin\varphi
\end{aligned}$$

where

$$\hat{A} = \log\varepsilon + \frac{1}{2}\log\frac{\varepsilon + 2p}{\varepsilon} - \log 2 + \gamma - \frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{\varepsilon + 2p} + 1\right\}$$

Therefore, we arrive at

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_V - \tilde{\psi}_{II} \\
&= \frac{1}{p}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left(-\frac{2}{(2\varepsilon p)^{\frac{1}{2}}}K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right) + \frac{1}{r}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)\right)\sin\varphi \\
&\quad + \left(\frac{\tilde{C}_1}{2} - \frac{C_0}{2}\right)\left(\frac{r}{2} - \frac{1}{2r} - r\log r\right)\sin\varphi
\end{aligned}$$

4.2 $\tilde{\psi}^c$ with respect to r^*

4.2.1 Deriving $\tilde{\psi}_{I\to III}$ and $\tilde{\psi}_{III\to I}$

The stream function $\tilde{\psi}_I$ is given by Eq.(43). Therefore, we have for $r^* = r\varepsilon^{1/2}$;

$$\begin{aligned}
\tilde{\psi}_I &= \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1(\hat{p}r) + \frac{K_2(\hat{p})}{r}\right\}\sin\varphi \\
&= \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1\left(\hat{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right) + \frac{\varepsilon^{\frac{1}{2}}}{r^*}K_2(\hat{p})\right\}\sin\varphi \\
&\approx \frac{2\varepsilon^{\frac{3}{2}}}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\frac{1}{r^*}K_2(\hat{p})\sin\varphi
\end{aligned}$$

From this equation, we have

$$\begin{aligned}
\tilde{\psi}_{I\to III} &\approx \frac{2\varepsilon^{\frac{3}{2}}}{2\varepsilon p}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{1}{r^*}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)\sin\varphi \\
&\approx \frac{\varepsilon^{\frac{1}{2}}}{p}\frac{1}{-\gamma - \log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\frac{1}{r^*}\left(\frac{2}{2\varepsilon p} - 1\right)\sin\varphi \\
&\approx -\frac{1}{p}\frac{1}{\gamma + \frac{1}{2}\log\frac{\varepsilon p}{2}}\left(\frac{1}{\varepsilon^{\frac{1}{2}}p}\right)\frac{1}{r^*}\sin\varphi
\end{aligned}$$

The stream function $\tilde{\psi}_{III}$ is given by Eq.(68).

$$\begin{aligned}\tilde{\psi}_{III} &= \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{r^* K_0\left((2p)^{\frac{1}{2}}r^*\right) - r^* K_2\left((2p)^{\frac{1}{2}}r^*\right) + \frac{1}{pr^*}\right\}\sin\varphi \\ &\quad -\frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi \\ &= \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{-\left(\frac{2}{p}\right)^{\frac{1}{2}} K_1\left((2p)^{\frac{1}{2}}r^*\right) + \frac{1}{p}\frac{1}{r^*}\right\}\sin\varphi - \frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi\end{aligned}$$

From the first line of this equation, we have

$$\begin{aligned}\tilde{\psi}_{III\to I} &\approx -\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{1}{p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\left[r^* K_0\left(\frac{\hat{p}}{\varepsilon^{\frac{1}{2}}}r^*\right) - r^* K_2\left(\frac{\hat{p}}{\varepsilon^{\frac{1}{2}}}r^*\right) + \frac{1}{pr^*}\right]\sin\varphi \\ &\approx -\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{1}{p^2}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\frac{1}{r^*}\sin\varphi\end{aligned}$$

Thus, we have

$$\tilde{\psi}_{I\to III} = \tilde{\psi}_{III\to I}$$

4.2.2 Deriving $\tilde{\psi}_{III\to V}$ and $\tilde{\psi}_{V\to III}$

The stream function $\tilde{\psi}_{III}$ is given by Eq.(68). From this equation, we have for $\tilde{p} = 2p/\varepsilon$;

$$\begin{aligned}\tilde{\psi}_{III\to V} &\approx \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{-\frac{2}{(\varepsilon\tilde{p})^{\frac{1}{2}}}K_1\left((\varepsilon\tilde{p})^{\frac{1}{2}}r^*\right) + \frac{2}{\varepsilon\tilde{p}}\frac{1}{r^*}\right\}\sin\varphi - \frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi \\ &\approx \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left[-\frac{2}{(\varepsilon\tilde{p})^{\frac{1}{2}}}\left\{\frac{1}{(\varepsilon\tilde{p})^{\frac{1}{2}}}\frac{1}{r^*} + \frac{(\varepsilon\tilde{p})^{\frac{1}{2}}}{2}r^*\left(\gamma + \log\frac{(\varepsilon\tilde{p})^{\frac{1}{2}}r^*}{2} - \frac{1}{2}\right)\right\}\right. \\ &\quad \left. + \frac{2}{\varepsilon\tilde{p}}\frac{1}{r^*}\right]\sin\varphi - \frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi \\ &\approx \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{-r^*\left(\gamma + \log\frac{(\varepsilon\tilde{p})^{\frac{1}{2}}r^*}{2} - \frac{1}{2}\right)\right\} - \frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi \\ &\approx \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{-r^*\left(\gamma + \log\left(\left(\frac{p}{2}\right)^{\frac{1}{2}}r^*\right) - \frac{1}{2}\right)\right\} - \frac{\varepsilon^{\frac{1}{2}}}{4}\frac{C_0}{r^*}\sin\varphi \\ &\approx \frac{1}{p}\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{1}{\gamma+\log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}r^*\left(\gamma + \log\left(\left(\frac{p}{2}\right)^{\frac{1}{2}}r^*\right) - \frac{1}{2}\right) \\ &\quad + \frac{\varepsilon^{\frac{1}{2}}}{2}\frac{1}{p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\frac{1}{r^*}\sin\varphi\end{aligned}$$

On the other hand, $\tilde{\psi}_V$ is given by Eq.(84). From this equation, we have

$$\tilde{\psi}_{V\to III} \approx \frac{1}{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\sin\varphi + \frac{\tilde{C}_{1V\to III}}{2\varepsilon^{\frac{1}{2}}}\left[\frac{r^*}{2} - r^*\log\frac{r^*}{\varepsilon^{\frac{1}{2}}} - \frac{\varepsilon}{2}\frac{1}{r^*}\right]\sin\varphi$$

where

$$\tilde{C}_{1V\to III} \approx -\frac{1}{p}\frac{2}{\gamma + \log\frac{\varepsilon}{2} + A_{V\to III}}$$

Here, $A_{V\to III} \approx (1/2)\log(2p/\varepsilon)$. Therefore, we have

$$\tilde{\psi}_{III\to V} = \tilde{\psi}_{V\to III}$$

From these results, we arrive at for the variable r^*

$$\begin{aligned}
\tilde{\psi}^c &= \tilde{\psi}_I + \tilde{\psi}_{III} + \tilde{\psi}_V - \tilde{\psi}_{I\to III} - \tilde{\psi}_{III\to V} \\
&\approx \frac{\varepsilon^{\frac{1}{2}}}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{1}{r^*}\sin\varphi \\
&\quad + \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left\{-\left(\frac{2}{p}\right)^{\frac{1}{2}} K_1\left((2p)^{\frac{1}{2}}r^*\right) + \frac{1}{p}\frac{1}{r^*}\right\}\sin\varphi \\
&\quad + \frac{1}{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\sin\varphi + \frac{\tilde{C}_1}{2}\left(\frac{1}{2}\frac{r^*}{\varepsilon^{\frac{1}{2}}} - \frac{1}{2}\frac{\varepsilon^{\frac{1}{2}}}{r^*} - \frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\sin\varphi \\
&\quad + \frac{1}{p^2}\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{1}{\gamma + \log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\frac{1}{r^*}\sin\varphi - \frac{1}{p}\frac{r^*}{\varepsilon^{\frac{1}{2}}}\sin\varphi \\
&\quad + \frac{1}{p}\frac{1}{\gamma + \log\left(\frac{\varepsilon p}{2}\right)^{\frac{1}{2}}}\left(\frac{r^*}{2\varepsilon^{\frac{1}{2}}} - \frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right)\sin\varphi
\end{aligned}$$

4.3 $\tilde{\psi}^c$ with respect to R

4.3.1 Deriving $\tilde{\psi}_{I\to III}$ and $\tilde{\psi}_{III\to I}$

$\tilde{\psi}_I$ is given by Eq.(43). For this variable $R = \varepsilon r$, we have

$$\begin{aligned}
\tilde{\psi}_I &= \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1(\hat{p}r) + \frac{K_2(\hat{p})}{r}\right\}\sin\varphi \\
&= \frac{2\varepsilon}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\left\{-\frac{2}{\hat{p}}K_1\left(\hat{p}\frac{R}{\varepsilon}\right) + \frac{\varepsilon}{R}K_2(\hat{p})\right\}\sin\varphi \\
&\approx \frac{2\varepsilon^2}{\hat{p}^2}\frac{1}{K_0(\hat{p})}\frac{1}{R}K_2(\hat{p})\sin\varphi
\end{aligned}$$

From this equation, we have

$$\begin{aligned}
\tilde{\psi}_{I\to III} &\approx \frac{1}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{\varepsilon}{R}\sin\varphi \approx \frac{1}{p}\frac{\varepsilon}{R}\frac{\frac{2}{2\varepsilon p} - 1}{-\gamma - \frac{1}{2}\log\frac{\varepsilon p}{2}}\sin\varphi \\
&\approx -\frac{1}{p}\frac{1}{\gamma + \frac{1}{2}\log\frac{\varepsilon p}{2}}\frac{1}{\varepsilon p}\frac{\varepsilon}{R}\sin\varphi
\end{aligned}$$

On the other hand, $\tilde{\psi}_{III}$ is given by Eq.(68). From this equation, we have for R:

$$\begin{aligned}
\tilde{\psi}_{III} &= \frac{C_0}{2\varepsilon^{\frac{1}{2}}}\left[\frac{R}{\varepsilon^{\frac{1}{2}}}K_0\left((2p)^{\frac{1}{2}}\frac{R}{\varepsilon^{\frac{1}{2}}}\right) - \frac{R}{\varepsilon^{\frac{1}{2}}}K_2\left((2p)^{\frac{1}{2}}\frac{R}{\varepsilon^{\frac{1}{2}}}\right) + \frac{1}{p}\frac{\varepsilon^{\frac{1}{2}}}{R}\right]\sin\varphi \\
&\quad -\frac{\varepsilon^{\frac{1}{2}}}{4}C_0\frac{\varepsilon^{\frac{1}{2}}}{R}\sin\varphi \\
&\approx \frac{C_0}{2}\frac{1}{p}\frac{1}{R}\sin\varphi
\end{aligned}$$

Therefore, we arrive at

$$\tilde{\psi}_{I\to III} = \tilde{\psi}_{III}$$

4.3.2 Deriving $\tilde{\psi}_{III\to IV}$ and $\tilde{\psi}_{IV\to III}$

$\tilde{\psi}_{III}$ is given by Eq.(68). From this equation, we easily have

$$\tilde{\psi}_{III\to IV} \approx \frac{C_0}{2}\frac{1}{p}\frac{1}{R}\sin\varphi$$

$\tilde{\psi}_{IV}$ is given by Eq.(82). From this equation, we have

$$\begin{aligned}
\tilde{\psi}_{IV\to III} &\approx \frac{\alpha_{1IV\to III}}{2\pi\varepsilon}\int_0^{2\pi}\left[\int_0^R \exp(-R_1\cos\theta)K_1\left(\left(\frac{2p}{\varepsilon}\right)^{\frac{1}{2}}R_1\right)\sin\theta\right. \\
&\quad \times\left\{-\log R + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R_1}{R}\right)^m\cos m(\theta-\varphi)\right\}R_1\mathrm{d}R_1 \\
&\quad + \int_R^{\infty}\exp(-R_1\cos\theta)K_1\left(\left(\frac{2p}{\varepsilon}\right)^{\frac{1}{2}}R_1\right)\sin\theta \\
&\quad \left.\times\left\{-\log R_1 + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R}{R_1}\right)^m\cos m(\theta-\varphi)\right\}R_1\mathrm{d}R_1\right]\mathrm{d}\theta
\end{aligned}$$

Here, we use the following relation;

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\exp(-R_1\cos\theta)\sin\theta\sin(m\theta)\mathrm{d}\theta &= -\frac{1}{2}\left(I_{m+1}(-R_1) - I_{m-1}(-R_1)\right) \\
&= -\frac{m}{R_1}I_m(-R_1)
\end{aligned}$$

Putting $R_1 = \varepsilon^{1/2}x$, we have by taking into account of flow symmetry;

$$\begin{aligned}
\tilde{\psi}_{IV\to III} &\approx -\frac{\alpha_{1IV\to III}}{\varepsilon^{\frac{1}{2}}}\sum_{m=1}^{\infty}\left[\int_0^{R/\varepsilon^{1/2}} K_1\left((2p)^{\frac{1}{2}}x\right)\left(\frac{x}{r^*}\right)^m I_m\left(-\varepsilon^{\frac{1}{2}}x\right)\mathrm{d}x\right. \\
&\quad \left.+\int_{R/\varepsilon^{1/2}}^{\infty} K_1\left((2p)^{\frac{1}{2}}x\right)\left(\frac{r^*}{x}\right)^m I_m\left(-\varepsilon^{\frac{1}{2}}x\right)\mathrm{d}x\right]\sin(m\varphi) \\
&\approx \frac{\alpha_{1IV\to III}}{2r^*}\int_0^{\infty}x^2K_1\left((2p)^{\frac{1}{2}}x\right)\mathrm{d}x\sin\varphi \\
&\approx \frac{\alpha_{1IV\to III}}{2r^*}\frac{2}{(2p)^{\frac{3}{2}}}\sin\varphi
\end{aligned}$$

Since

$$\alpha_{1IV\to III} \approx -\frac{1}{p}\left(\frac{2p}{\varepsilon}\right)^{\frac{1}{2}} \frac{2}{\gamma + \log\frac{\varepsilon}{2} + \frac{1}{2}\log\frac{2p}{\varepsilon}}$$

we arrive at

$$\tilde{\psi}_{III\to IV} = \tilde{\psi}_{IV\to III}$$

Therefore, we have for R:

$$\begin{aligned}
\tilde{\psi}^c &\approx \tilde{\psi}_I + \tilde{\psi}_{III} + \tilde{\psi}_{IV} - \tilde{\psi}_{I\to III} - \tilde{\psi}_{III\to IV} \\
&\approx \frac{2}{p}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{\varepsilon}{R}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)\sin\varphi - \frac{C_0}{2}\frac{1}{p}\frac{1}{R}\sin\varphi \\
&\quad + \frac{\alpha_1}{2\pi\varepsilon}\int_0^{2\pi}\mathrm{Pf}\int_0^{\infty}\exp(-R_1\cos\theta) \\
&\quad \times K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\sin\theta\log\left(\frac{1}{\rho_R^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta
\end{aligned}$$

4.4 Stream function ψ in real time

4.4.1 For $r = O(1)$

$\tilde{\psi}^c$ for this variable r is given by

$$\begin{aligned}
\tilde{\psi}^c &= \frac{1}{p}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left[-\frac{2}{(2\varepsilon p)^{\frac{1}{2}}}K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right)\right. \\
&\quad \left. + \frac{1}{r}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)\right]\sin\varphi \\
&\quad + \frac{1}{2}\left[\tilde{C}_1 - C_0\right]\left[\frac{r^2-1}{2r} - r\log r\right]\sin\varphi
\end{aligned}$$

where

$$\begin{aligned}
C_0 &= -\frac{1}{p}\frac{2}{\gamma + \frac{1}{2}\log\frac{\varepsilon p}{2}} \\
\tilde{C}_1 &= -\frac{1}{p}\frac{2}{\gamma + \log\frac{\varepsilon}{2} + \frac{1}{2}\log\left(1+\frac{2p}{\varepsilon}\right) - \frac{1}{2}\left[1 + \frac{2p}{\varepsilon}\log\left(\frac{2p}{\varepsilon+2p}\right)\right]}
\end{aligned}$$

We consider the inverse Laplace transformation $\mathcal{L}^{-1}$. The inverse Laplace transformation of the first term of r.h.s. of the above equation becomes as

$$\begin{aligned}
\mathcal{L}^{-1}(\text{first term}) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p}\frac{\exp(pt)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left[\frac{1}{r}K_2\left((2\varepsilon p)^{\frac{1}{2}}\right) - \frac{2}{(2\varepsilon p)^{\frac{1}{2}}}K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right)\right]\mathrm{d}p \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{p}\left[\frac{1}{r} + \frac{2}{(2\varepsilon p)^{\frac{1}{2}}}\frac{\frac{1}{r}K_1\left((2\varepsilon p)^{\frac{1}{2}}\right) - K_1\left(r(2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\right]\mathrm{d}p \\
&= \frac{1}{r}H(t) + \psi_{11}
\end{aligned}$$

In the above equation, ψ_{11} becomes as, by taking the integral contour as Figure 1.

$$\begin{aligned}
\psi_{11} &= \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{p(2\varepsilon p)^{\frac{1}{2}}}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left[\frac{1}{r}K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)-K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right)\right]\mathrm{d}p \\
&= \frac{1}{\pi i}\int_{L}\frac{\exp(pt)}{p(2\varepsilon p)^{\frac{1}{2}}}\frac{1}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\left[\frac{1}{r}K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)-K_1\left((2\varepsilon p)^{\frac{1}{2}}r\right)\right]\mathrm{d}p \\
&= \frac{1}{\pi i}\int_{\infty}^{\delta}\frac{\exp(-st)}{(-s)(2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{\pi i}{2}\right)}\frac{1}{K_0\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{\pi i}{2}\right)\right)} \\
&\quad\times\left[\frac{1}{r}K_1\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{\pi i}{2}\right)\right)-K_1\left(r(2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{\pi i}{2}\right)\right)\right](-\mathrm{d}s) \\
&\quad+\frac{1}{\pi i}\int_{-\pi}^{\pi}\frac{1}{\delta\exp(i\theta)\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}}\frac{1}{K_0\left(\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)} \\
&\quad\times\left[\frac{1}{r}K_1\left(\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)-K_1\left(r\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)\right]\delta i\exp(i\theta)\mathrm{d}\theta \\
&\quad+\frac{1}{\pi i}\int_{\delta}^{\infty}\frac{\exp(-st)}{(-s)(2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{\pi i}{2}\right)}\frac{1}{K_0\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{\pi i}{2}\right)\right)} \\
&\quad\times\left[\frac{1}{r}K_1\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{\pi i}{2}\right)\right)-K_1\left(r(2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{\pi i}{2}\right)\right)\right](-\mathrm{d}s)
\end{aligned}$$

where δ is small enough as shown in Figure 1.

Here, the second integral of r.h.s. of the above equation becomes as

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\pi i}\int_{-\pi}^{\pi}\frac{1}{\delta\exp(i\theta)\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}}\frac{1}{K_0\left(\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)} \\
&\quad\times\left[\frac{1}{r}K_1\left(\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)-K_1\left(r\left(2\varepsilon\delta\exp(i\theta)\right)^{\frac{1}{2}}\right)\right]\delta i\exp(i\theta)\mathrm{d}\theta \\
&= r-\frac{1}{r}
\end{aligned}$$

Therefore, we have finally by using the relation between $K_m\left(z\exp\left(\pm\frac{\pi i}{2}\right)\right)$ and $J_m(z)$ and $Y_m(z)$;

$$\begin{aligned}
\mathcal{L}^{-1}(\text{first term}) &= rH(t)+\frac{2}{\pi}\int_0^{\infty}\frac{\exp(-st)}{s(2\varepsilon s)^{\frac{1}{2}}}\frac{1}{J_0^2\left((2\varepsilon s)^{\frac{1}{2}}\right)+Y_0^2\left((2\varepsilon s)^{\frac{1}{2}}\right)} \\
&\quad\times\left[J_0\left((2\varepsilon s)^{\frac{1}{2}}\right)\left\{Y_1\left(r(2\varepsilon s)^{\frac{1}{2}}\right)-\frac{1}{r}Y_1\left((2\varepsilon s)^{\frac{1}{2}}\right)\right\}\right. \\
&\quad\left.-Y_0\left((2\varepsilon s)^{\frac{1}{2}}\right)\left\{J_1\left(r(2\varepsilon s)^{\frac{1}{2}}\right)-\frac{1}{r}J_1\left((2\varepsilon s)^{\frac{1}{2}}\right)\right\}\right]\mathrm{d}s
\end{aligned}$$

Let us consider the inverse Laplace transformation of the second term of r.h.s. of $\tilde{\psi}^c$.

$$\mathcal{L}^{-1}(\tilde{C}_1-C_0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\tilde{C}_1-C_0\right)\exp(pt)\mathrm{d}p$$

$$= \frac{1}{2\pi i}\int_L \frac{2}{p}\exp(pt)\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p$$
$$-\frac{1}{2\pi i}\int_{L_1}\frac{2}{p}\frac{1}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{\varepsilon+2p}{\varepsilon}-\frac{1}{2}-\frac{p}{\varepsilon}\log\frac{2p}{\varepsilon+2p}}\mathrm{d}p$$

where the integral contour L_1 is shown in Figure 2.

We here use the following relations;

$$\frac{1}{2\pi i}\int_L \frac{2}{p}\exp(pt)\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p$$
$$= \frac{1}{\pi i}\int_0^\infty \frac{\exp(-st)}{s}\left[\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}+\frac{i\pi}{2}}-\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}-\frac{i\pi}{2}}\right]\mathrm{d}s$$
$$= -\frac{4}{\pi^2}\int_0^\infty \frac{\exp(-st)}{s}\frac{\mathrm{d}s}{1+\frac{4}{\pi^2}\left(\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}\right)^2}$$
$$\equiv 2F_{21}$$

and

$$\frac{1}{2\pi i}\int_{L_1}\frac{2}{p}\exp(pt)\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(p+\frac{\varepsilon}{2}\right)-\frac{1}{2}-\frac{p}{\varepsilon}\log\left(\frac{p}{p+\varepsilon/2}\right)}\mathrm{d}p$$
$$= \frac{1}{\pi i}\int_{\frac{\varepsilon}{2}}^\infty \frac{\exp(-st)}{s}$$
$$\times\frac{-\pi i}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(s-\frac{\varepsilon}{2}\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{s-\varepsilon/2}\right)\right]^2+\frac{\pi^2}{4}}\mathrm{d}s$$
$$+\frac{1}{\pi i}\int_0^{\varepsilon/2}\frac{\exp(-st)}{s}$$
$$\times\frac{-2\frac{s}{\varepsilon}i\pi}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(\frac{\varepsilon}{2}-s\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{\varepsilon/2-s}\right)\right]^2+\pi^2\frac{s^2}{\varepsilon^2}}\mathrm{d}s$$
$$+\frac{1}{\pi i}\int_{-\pi}^{\pi}\frac{\delta i\exp(i\theta)}{\delta\exp(i\theta)}\frac{\mathrm{d}\theta}{\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{\varepsilon}{2}-\frac{1}{2}}$$
$$= \frac{2}{\gamma+\log\frac{\varepsilon}{2}-\frac{1}{2}}-\int_{\varepsilon/2}^\infty\frac{\exp(-st)}{s}$$
$$\times\frac{1}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(s-\frac{\varepsilon}{2}\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{s-\varepsilon/2}\right)\right]^2+\frac{\pi^2}{4}}\mathrm{d}s$$
$$-\frac{2}{\varepsilon}\int_0^{\varepsilon/2}\frac{\exp(-st)}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(\frac{\varepsilon}{2}-s\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{\varepsilon/2-s}\right)\right]^2+\pi^2\frac{s^2}{\varepsilon^2}}\mathrm{d}s$$
$$\equiv 2F_{22}$$

Then, we have

$$\mathcal{L}^{-1}(\tilde{C}_1-C_0) = -\frac{4}{\pi^2}\int_0^\infty\frac{\exp(-st)}{s}\frac{\mathrm{d}s}{1+\frac{4}{\pi^2}\left(\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}\right)^2}-\frac{2}{\gamma+\log\frac{\varepsilon}{2}-\frac{1}{2}}$$
$$+\int_{\varepsilon/2}^\infty\frac{\exp(-st)}{s}$$

$$\times \frac{1}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(s-\frac{\varepsilon}{2}\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{s-\varepsilon/2}\right)\right]^2+\frac{\pi^2}{4}}ds$$
$$+\frac{2}{\varepsilon}\int_0^{\varepsilon/2}\frac{\exp(-st)}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(\frac{\varepsilon}{2}-s\right)-\frac{1}{2}+\frac{s}{\varepsilon}\log\left(\frac{s}{\varepsilon/2-s}\right)\right]^2+\pi^2\frac{s^2}{\varepsilon^2}}ds$$

Therefore, we can obtain the stream function in the real time Eq.(88).

4.4.2 For $r^* = O(1)$

For this variable r^*, $\tilde{\psi}^c$ is given by

$$\begin{aligned}
\tilde{\psi}^c &\approx \frac{\varepsilon^{\frac{1}{2}}}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{1}{r^*}\sin\varphi-\frac{C_0}{(2\varepsilon p)^{\frac{1}{2}}}K_1\left((2p)^{\frac{1}{2}}r^*\right)\sin\varphi \\
&+\frac{1}{2}\left[\tilde{C}_1-C_0\right]\left[\frac{r^*}{2\varepsilon^{\frac{1}{2}}}-\frac{\varepsilon^{\frac{1}{2}}}{2r^*}-\frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right]\sin\varphi \\
C_0 &= -\frac{1}{p}\frac{2}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}} \\
\tilde{C}_1 &= -\frac{1}{p}\frac{2}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\left(1+\frac{2p}{\varepsilon}\right)-\frac{1}{2}\left\{1+\frac{2p}{\varepsilon}\log\left(\frac{2p}{\varepsilon+2p}\right)\right\}}
\end{aligned}$$

Here, we use the following relations

$$\begin{aligned}
&\frac{\varepsilon^{\frac{1}{2}}}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\frac{1}{r^*} \\
&= \frac{1}{r^*}\frac{\varepsilon^{\frac{1}{2}}}{p}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right]-\frac{1}{r^*}\frac{\varepsilon^{\frac{1}{2}}}{p}\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}
\end{aligned}$$

and

$$\frac{C_0}{(2\varepsilon)^{\frac{1}{2}}}K_1\left((2p)^{\frac{1}{2}}r^*\right)=\frac{C_0}{(2\varepsilon p)^{\frac{1}{2}}}\left[K_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{1}{(2p)^{\frac{1}{2}}r^*}\right]+\frac{1}{\varepsilon^{\frac{1}{2}}}\frac{C_0}{2p}\frac{1}{r^*}$$

Then we have

$$\begin{aligned}
\tilde{\psi}^c &\approx \frac{\varepsilon^{\frac{1}{2}}}{p}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right]\frac{1}{r^*}\sin\varphi \\
&-\frac{C_0}{(2\varepsilon p)^{\frac{1}{2}}}\left[K_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{1}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right]\sin\varphi \\
&+\frac{1}{2}\left[\tilde{C}_1-C_0\right]\left[\frac{r^*}{2\varepsilon^{\frac{1}{2}}}-\frac{\varepsilon^{\frac{1}{2}}}{2r^*}-\frac{r^*}{\varepsilon^{\frac{1}{2}}}\log\frac{r^*}{\varepsilon^{\frac{1}{2}}}\right]\sin\varphi
\end{aligned}$$

We first consider the inverse Laplace transformation $\mathcal{L}^{-1}$. Changing the integral contour to Figure 1 and taking $\delta \to 0$, we have

$$\mathcal{L}^{-1}\left[\frac{1}{p}\left\{\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right\}\right]$$

$$= \frac{1}{2\pi i}\left(\int_{\infty\exp(-i\pi)}^{\delta\exp(-i\pi)}+\int_{\delta\exp(-i\pi)}^{\delta\exp(i\pi)}+\int_{\delta\exp(i\pi)}^{\infty\exp(i\pi)}\right)\frac{\exp(pt)}{p}$$

$$\times\left\{\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right\}\mathrm{d}p$$

$$= \frac{1}{2\pi i}\int_{\infty}^{\delta}\frac{\exp(-pt)}{p}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(-\frac{i}{2}\pi\right)\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(-\frac{i}{2}\pi\right)\right)}-\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}-\frac{i\pi}{2}}\right]\mathrm{d}p$$

$$+\frac{1}{2\pi i}\int_{\delta}^{\infty}\frac{\exp(-pt)}{p}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(\frac{i}{2}\pi\right)\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(\frac{i}{2}\pi\right)\right)}-\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}+\frac{i\pi}{2}}\right]\mathrm{d}p$$

$$+\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\exp(\delta\exp(i\theta)t)}{\delta\exp(i\theta)}$$

$$\times\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right]_{p=\delta\exp(i\theta)}\delta i\exp(i\theta)\mathrm{d}\theta$$

$$= \frac{1}{2\pi i}\int_{\delta}^{\infty}\frac{\exp(-pt)}{p}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(\frac{i}{2}\pi\right)\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(\frac{i}{2}\pi\right)\right)}\right.$$

$$\left.-\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(-\frac{i}{2}\pi\right)\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\exp\left(-\frac{i}{2}\pi\right)\right)}+\frac{1}{\varepsilon p}\frac{\pi i}{\left\{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right\}^2+\frac{\pi^2}{4}}\right]\mathrm{d}p$$

$$+\frac{i}{2\pi i}\int_{-\pi}^{\pi}\left[\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right]_{p=\delta\exp(i\theta)}\mathrm{d}\theta$$

Here, we use the relation ; $K_2(z)=K_0(z)+\frac{2}{z}K_1(z)$. Then we have

$$\mathcal{L}^{-1}\left[\frac{1}{p}\left\{\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right\}\right]$$

$$= \frac{1}{2\pi}\int_{\delta}^{\infty}\frac{\exp(-pt)}{p}\left[\frac{4\left(J_0(X)Y_1(X)-J_1(X)Y_0(X)\right)}{J_0^2(X)+Y_0^2(X)}\right.$$

$$\left.+\frac{1}{\varepsilon p}\frac{\pi}{\left[\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right]^2+\frac{\pi^2}{4}}\right]\mathrm{d}p$$

$$= \frac{1}{2\pi}\int_{\delta}^{\infty}\frac{\exp(-pt)}{p}\left[-\frac{8}{\pi}\frac{1}{2\varepsilon p}\frac{1}{J_0^2(X)+Y_0^2(X)}\right.$$

$$\left.+\frac{1}{\varepsilon p}\frac{\pi}{\frac{\pi^2}{4}+\left[\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right]^2}\right]\mathrm{d}p$$

Thus, we have

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{1}{p}\left\{\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\frac{1}{\varepsilon p}\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\right\}\right] \\
&= \frac{1}{2\pi\varepsilon}\int_0^\infty \frac{\exp(-pt)}{p^2}\left[\frac{\pi}{\frac{\pi^2}{4}+\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right. \\
&\quad \left. -\frac{4}{\pi}\frac{1}{J_0^2(X)+Y_0^2(X)}\right]\mathrm{d}p
\end{aligned}
$$

where $X=(2\varepsilon p)^{1/2}$.

We next consider the following relation;

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{C_0}{(2\varepsilon p)^{\frac{1}{2}}}\left\{K_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{1}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right\}\right] \\
&= -\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p(2\varepsilon p)^{\frac{1}{2}}}\frac{\exp(pt)}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\left\{K_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{1}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right\}\mathrm{d}p
\end{aligned}
$$

We take into account of the following asymptotic behavior as $p\to 0$;

$$
K_1\left((2p)^{\frac{1}{2}}r^*\right) \approx \frac{1}{(2p)^{\frac{1}{2}}r^*}+\frac{(2p)^{\frac{1}{2}}}{2}r^*\left(\gamma+\log\left(\frac{p}{2}\right)^{\frac{1}{2}}+\log r^*-\frac{1}{2}\right)
$$

Then, we have by changing the integral contour to L ;

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{C_0}{(2\varepsilon p)^{\frac{1}{2}}}\left\{K_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{1}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right\}\right] \\
&= -\frac{1}{\varepsilon^{\frac{1}{2}}}r^*-\frac{1}{\pi i}\left(\int_0^\infty\frac{\exp(-pt)}{p(2\varepsilon p)^{\frac{1}{2}}}\exp\left(\frac{\pi i}{2}\right)\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}-\frac{\pi i}{2}}\right. \\
&\quad\times\left[K_1\left((2p)^{\frac{1}{2}}\exp\left(-\frac{\pi i}{2}\right)r^*\right)-\frac{\exp\left(\frac{\pi i}{2}\right)}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right]\mathrm{d}p \\
&\quad+\int_0^\infty\frac{\exp(-pt)}{p(2\varepsilon p)^{\frac{1}{2}}}\exp\left(-\frac{\pi i}{2}\right)\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}+\frac{\pi i}{2}} \\
&\quad\left.\times\left[K_1\left((2p)^{\frac{1}{2}}\exp\left(\frac{\pi i}{2}\right)r^*\right)-\frac{\exp\left(-\frac{\pi i}{2}\right)}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right]\mathrm{d}p\right) \\
&= -\frac{r^*}{\varepsilon^{\frac{1}{2}}}-\int_0^\infty\frac{\exp(-pt)}{p(2\varepsilon p)^{\frac{1}{2}}}\left[\left(\gamma+\frac{1}{2}\log\left(\frac{\varepsilon p}{2}\right)\right)J_1\left((2p)^{\frac{1}{2}}r^*\right)-\frac{\pi}{2}Y_1\left((2p)^{\frac{1}{2}}r^*\right)\right. \\
&\quad\left.-\frac{1}{(2p)^{\frac{1}{2}}}\frac{1}{r^*}\right]\frac{1}{\left(\gamma+\frac{1}{2}\log\left(\frac{\varepsilon p}{2}\right)\right)^2+\frac{\pi^2}{4}}\mathrm{d}p
\end{aligned}
$$

Thus, we arrive at Eq.(89).

4.4.3 For R

For this variable, $\tilde{\psi}^c$ is given by

$$
\begin{aligned}
\tilde{\psi}^c &= \left[\frac{\varepsilon}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)} - \frac{C_0}{2}\frac{1}{p}\right]\frac{1}{R}\sin\varphi \\
&\quad + \frac{1}{2\pi\varepsilon}\int_0^{2\pi}\int_0^{\infty}\exp(-R_1\cos\theta) \\
&\quad \times \alpha_1 K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\sin\theta\log\left(\frac{1}{\rho^*}\right)R_1\mathrm{d}R_1\mathrm{d}\theta
\end{aligned}
$$

The inverse transformation of the first term of r.h.s. of the above equation is given from section 4.4.2 by

$$
\mathcal{L}^{-1}\left[\frac{\varepsilon}{p}\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)} - \frac{C_0}{2}\frac{1}{p}\right]
$$

$$
= \frac{1}{2\pi}\int_0^{\infty}\frac{\exp(-pt)}{p^2}\left[\frac{\pi}{\frac{\pi^2}{4}+\left(\gamma+\frac{1}{2}\log\left(\frac{\varepsilon p}{2}\right)\right)^2}\right.
$$

$$
\left. - \frac{4}{\pi}\frac{1}{J_0^2\left((2\varepsilon p)^{\frac{1}{2}}\right)+Y_0^2\left((2\varepsilon p)^{\frac{1}{2}}\right)}\right]\mathrm{d}p
$$

Since α_1 is given by Eq.(85), $F_{22}(t)$ is given from section 4.4.1 :

$$
F_{22}(t) = -\frac{1}{2}\mathcal{L}^{-1}(\tilde{C}_1) = -\frac{1}{2}\mathcal{L}^{-1}\left(\frac{\alpha_1}{(1+\tilde{p})^{1/2}}\right)
$$

Further, we use the following relation;

$$
\mathcal{L}^{-1}\left[K_1\left(a(p-\alpha)^{\frac{1}{2}}\right)(p-\alpha)^{\frac{1}{2}}\right] = \frac{a}{4}\exp(\alpha t)\cdot\frac{1}{t^2}\exp\left(-\frac{a^2}{4t}\right)
$$

Then, we have

$$
\mathcal{L}^{-1}\left((1+\tilde{p})^{1/2}K_1\left((1+\tilde{p})^{1/2}R_1\right)\right) = \frac{R_1}{2\varepsilon}\frac{1}{t^2}\exp\left(-\frac{\varepsilon t}{2}-\frac{R_1^2}{2t\varepsilon}\right)
$$

Therefore, we have

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\alpha_1 K_1\left((1+\tilde{p})^{\frac{1}{2}}R_1\right)\right] &= \mathcal{L}^{-1}\left(\frac{\alpha_1}{(1+\tilde{p})^{1/2}}(1+\tilde{p})^{1/2}K_1\left((1+\tilde{p})^{1/2}R_1\right)\right) \\
&= -\frac{R_1}{\varepsilon}\int_0^t F_{22}(t-s)\exp\left(-\frac{\varepsilon}{2}s-\frac{R_1^2}{2\varepsilon s}\right)\frac{1}{s^2}\mathrm{d}s
\end{aligned}
$$

Therefore, we can arrive at Eq.(90) by using the convolution.

5 Aerodynamic Force

5.1 Drag coefficient of circular cylinder

The drag coefficient is obtained from Eq.(98). This equation is derived by substituting Eq.(96) into Eq.(95).

$$\tilde{C_D} \approx \frac{4\pi}{Re}\left[\varepsilon\left\{1+\frac{K_2\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\right\}+2\left(\frac{\varepsilon}{2p}\right)^{\frac{1}{2}}\frac{K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}+\tilde{C}\right]$$

Putting $x = (2\varepsilon p)^{1/2}$, we have

$$\begin{aligned}
\tilde{C_D} &= \frac{4\pi}{Re}\left[\varepsilon+\tilde{C}+\varepsilon\frac{xK_2(x)+2K_1(x)}{xK_0(x)}\right] \\
&= \frac{4\pi}{Re}\left[\varepsilon+\tilde{C}+\varepsilon\frac{xK_0(x)+2K_1(x)+2K_1(x)}{xK_0(x)}\right] \\
&= \frac{4\pi}{Re}\left[\varepsilon+\tilde{C}+\varepsilon\left(1+\frac{4}{x}\frac{K_1(x)}{K_0(x)}\right)\right] = 2\pi\left[2+\frac{\tilde{C}}{\varepsilon}+\frac{4}{x}\frac{K_1(x)}{K_0(x)}\right]
\end{aligned}$$

where we used the relation; $K_2(x) = K_0(x) + (2/x)K_1(x)$. Here $\tilde{C}$ is given by

$$\tilde{C} = \frac{2}{p}\left[\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}-\frac{1}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{\varepsilon+2p}+1\right\}}\right]$$

Thus, we have

$$\begin{aligned}
C_D &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp(pt)\tilde{C_D}\mathrm{d}p \\
&\approx \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}2\pi\exp(pt)\left[2+\frac{\tilde{C}}{\varepsilon}+\frac{4}{x}\frac{K_1(x)}{K_0(x)}\right]\mathrm{d}p \\
&= 2\pi\left[2\delta(t)+\frac{2}{\varepsilon}\left(\hat{I}_2-\hat{I}_3\right)+4\hat{I}_1\right]
\end{aligned}$$

where

$$\left\{\begin{aligned}
\hat{I}_1 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{(2\varepsilon p)^{\frac{1}{2}}}\frac{K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)}\mathrm{d}p \\
\hat{I}_2 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p}\frac{\exp(pt)}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p \\
\hat{I}_3 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p}\frac{\exp(pt)}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{\varepsilon+2p}+1\right\}}\mathrm{d}p
\end{aligned}\right.$$

5.1.1 $\hat{I}_1$

In the actual numerical calculation of these terms, $\hat{I}_i$ $(i = 1, 2, 3)$, we have to take into account of the asymptotic behavior of integrand carefully.

$$\hat{I}_1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(pt)}{(2\varepsilon p)^{\frac{1}{2}}} \frac{K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)} \mathrm{d}p$$

We can change the integral contour to L as shown in Figure 1. Then, we have

$$\begin{aligned}
\hat{I}_1 &= \frac{1}{2\pi i}\int_L \frac{\exp(pt)}{(2\varepsilon p)^{\frac{1}{2}}} \frac{K_1\left((2\varepsilon p)^{\frac{1}{2}}\right)}{K_0\left((2\varepsilon p)^{\frac{1}{2}}\right)} \mathrm{d}p \\
&= -\frac{1}{2\pi i}\int_0^\infty \left[\frac{\exp(-st)}{(2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{i\pi}{2}\right)} \frac{K_1\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{i\pi}{2}\right)\right)}{K_0\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(\frac{i\pi}{2}\right)\right)} \right. \\
&\quad \left. - \frac{\exp(-st)}{(2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{i\pi}{2}\right)} \frac{K_1\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{i\pi}{2}\right)\right)}{K_0\left((2\varepsilon s)^{\frac{1}{2}}\exp\left(-\frac{i\pi}{2}\right)\right)} \right] \mathrm{d}s
\end{aligned}$$

Here, we change the integral variable as $x^2 = 2\varepsilon s$. Then we have

$$\begin{aligned}
\hat{I}_1 &= -\frac{1}{2\pi i}\int_0^\infty \left[\frac{\exp\left(-\frac{x^2}{Re}t\right)}{ix} \frac{K_1\left(x\exp\left(\frac{i\pi}{2}\right)\right)}{K_0\left(x\exp\left(\frac{i\pi}{2}\right)\right)} \right. \\
&\quad \left. + \frac{\exp\left(-\frac{x^2}{Re}t\right)}{ix} \frac{K_1\left(x\exp\left(-\frac{i\pi}{2}\right)\right)}{K_0\left(x\exp\left(-\frac{i\pi}{2}\right)\right)} \right] \frac{x}{\varepsilon} \mathrm{d}x \\
&= \frac{1}{2\pi\varepsilon}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right) \left[\frac{K_1\left(x\exp\left(\frac{i\pi}{2}\right)\right)}{K_0\left(x\exp\left(\frac{i\pi}{2}\right)\right)} + \frac{K_1\left(x\exp\left(-\frac{i\pi}{2}\right)\right)}{K_0\left(x\exp\left(-\frac{i\pi}{2}\right)\right)} \right] \mathrm{d}x \\
&= -\frac{1}{\pi\varepsilon}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right) \frac{J_0(x)Y_1(x) - J_1(x)Y_0(x)}{J_0^2(x) + Y_0^2(x)} \mathrm{d}x
\end{aligned}$$

We here use the relation: $J_0(x)Y_1(x) - J_1(x)Y_0(x) = -2/(\pi x)$. Then we have

$$\hat{I}_1 = \frac{1}{\pi\varepsilon}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right) \frac{2}{\pi x\left(J_0^2(x) + Y_0^2(x)\right)} \mathrm{d}x$$

We consider the asymptotic behavior of integrand for $x \to 0$:

$$(\text{integrand}) \approx \frac{2}{\pi}\exp\left(-\frac{x^2}{Re}t\right) \frac{1}{x} \frac{\mathrm{d}x}{1 + \frac{4}{\pi^2}\left(\log\frac{x}{2} + \gamma\right)^2}$$

For $x \to \infty$, we have

$$\frac{1}{x}\left[\frac{1}{J_0^2(x) + Y_0^2(x)} - \frac{1}{1 + \frac{4}{\pi^2}\left(\log\frac{x}{2} + \gamma\right)^2} \right]$$

$$\approx \frac{1}{x}\left[\frac{\pi x}{2}\left(1+\frac{1}{8x^2}\right)-\frac{1}{1+\frac{4}{\pi^2}\left(\log\frac{x}{2}+\gamma\right)^2}\right]$$
$$\approx \frac{\pi}{2}+O\left(\frac{1}{x}\right)$$

Hence, we arrive at

$$\begin{aligned}\hat{I}_1 &= \frac{2}{\pi^2}\frac{2}{Re}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right)\\ &\times\left[\frac{1}{x}\left(\frac{1}{J_0^2(x)+Y_0^2(x)}-\frac{1}{1+\frac{4}{\pi^2}\left(\log\frac{x}{2}+\gamma\right)^2}\right)-\frac{\pi}{2}\right]\mathrm{d}x\\ &+\frac{2}{\pi^2}\frac{2}{Re}\int_0^\infty \exp\left(-\frac{x^2}{Re}t\right)\frac{1}{x}\frac{1}{1+\frac{4}{\pi^2}\left(\log\frac{x}{2}+\gamma\right)^2}\mathrm{d}x+\left(\frac{1}{\pi t Re}\right)^{\frac{1}{2}}\end{aligned}$$

In the actual calculation of this equation, the integral region is divided by small regions and integration formula of Gaussian type is used for these small integral regions.

5.1.2 $\hat{I}_2$

The integral contour of $\hat{I}_2$ given by the following equation is also changed into L as shown in Figure 1.

$$\hat{I}_2=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p}\frac{\exp(pt)}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p$$

We first consider the following integral I_{20};

$$I_{20}\equiv\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p$$

Then, we have

$$\begin{aligned}I_{20} &= \frac{1}{2\pi i}\int_L\frac{\exp(pt)}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}\mathrm{d}p\\ &= -\frac{1}{2\pi i}\int_0^\infty\left[\frac{\exp(-st)}{\gamma+\frac{1}{2}\log\frac{s\varepsilon}{2}+\frac{1}{2}i\pi}-\frac{\exp(-st)}{\gamma+\frac{1}{2}\log\frac{s\varepsilon}{2}-\frac{1}{2}i\pi}\right]\mathrm{d}s\\ &= \frac{1}{2}\int_0^\infty\frac{\exp(-st)}{\left(\gamma+\frac{1}{2}\log\frac{s\varepsilon}{2}\right)^2+\frac{\pi^2}{4}}\mathrm{d}s\\ &= \frac{1}{2}\frac{4}{\pi^2}\int_0^\infty\frac{\exp(-st)}{1+\frac{4}{\pi^2}\left(\gamma+\frac{1}{2}\log\frac{s\varepsilon}{2}\right)^2}\mathrm{d}s\end{aligned}$$

Here, we change the integral variable to x by $x^2=2\varepsilon s$:

$$I_{20} = \frac{4}{\pi^2 Re}\int_0^\infty\frac{x\exp\left(-\frac{x^2}{Re}t\right)}{1+\frac{4}{\pi^2}\left(\gamma+\log\frac{x}{2}\right)^2}\mathrm{d}x$$

From this equation, we have

$$\begin{aligned}\hat{I}_2 &= \int_0^t I_{20}(t)\mathrm{d}t \\ &= \frac{4}{\pi^2 Re}\int_0^\infty \frac{x}{1+\frac{4}{\pi^2}\left(\gamma+\log\frac{x}{2}\right)^2}\int_0^t \exp\left(-\frac{x^2}{Re}t\right)\mathrm{d}t\mathrm{d}x \\ &= \frac{4}{\pi^2}\int_0^\infty \frac{1-\exp\left(-\frac{x^2}{Re}t\right)}{x\left[1+\frac{4}{\pi^2}\left(\gamma+\log\frac{x}{2}\right)^2\right]}\mathrm{d}x\end{aligned}$$

5.1.3 $\hat{I}_3$

For $\hat{I}_3$, the similar procedure is used:

$$\hat{I}_3 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p}\frac{\exp(pt)}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{2p+\varepsilon}+1\right\}}\mathrm{d}p$$

Here, we define I_{30} as follows

$$I_{30} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{2p+\varepsilon}+1\right\}}\mathrm{d}p$$

Then, changing the integral contour, as shown in Figure 2, we arrive at

$$\begin{aligned}I_{30} &= \frac{1}{2\pi i}\int_{L_1}\frac{\exp(pt)}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{2p+\varepsilon}+1\right\}}\mathrm{d}p \\ &= \frac{1}{2}\int_{\varepsilon/2}^\infty\frac{\exp(-st)}{\left[\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2s-\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{-2s}{\varepsilon}\log\frac{2s}{2s-\varepsilon}+1\right\}\right]^2+\frac{\pi^2}{4}}\mathrm{d}s \\ &\quad +\int_0^{\varepsilon/2}\frac{\frac{s}{\varepsilon}\exp(-st)}{\left[\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{\varepsilon-2s}{\varepsilon}-\frac{1}{2}\left\{-\frac{2s}{\varepsilon}\log\frac{2s}{\varepsilon-2s}+1\right\}\right]^2+\left(\frac{s}{\varepsilon}\right)^2\pi^2}\mathrm{d}s\end{aligned}$$

Using the following relation;

$$\hat{I}_3 = \int_0^t I_{30}(t)\mathrm{d}t$$

we finally arrive at, by putting $s = x^2/(2\varepsilon)$;

$$\begin{aligned}\hat{I}_3 &= \frac{1}{2}\int_{\varepsilon/2}^\infty\frac{\exp(-st)-1}{-s} \\ &\quad\times\frac{\mathrm{d}s}{\left[\gamma+\frac{1}{2}\log\varepsilon-\log 2+\frac{1}{2}\log(2s-\varepsilon)-\frac{1}{2}\left\{\frac{-2s}{\varepsilon}\log\frac{2s}{2s-\varepsilon}+1\right\}\right]^2+\frac{\pi^2}{4}} \\ &\quad+\int_0^{\varepsilon/2}\frac{-s(\exp(-st)-1)}{s\varepsilon} \\ &\quad\times\frac{\mathrm{d}s}{\left[\gamma+\log\frac{1}{2}+\log\varepsilon+\frac{1}{2}\log\frac{\varepsilon-2s}{\varepsilon}-\frac{1}{2}\left\{\frac{-2s}{\varepsilon}\log\frac{2s}{\varepsilon-2s}+1\right\}\right]^2+\left(\frac{s}{\varepsilon}\right)^2\pi^2}\end{aligned}$$

$$
\begin{aligned}
= & \frac{4}{\pi^2}\int_{\varepsilon}^{\infty}\frac{1}{x}\left(1-\exp\left(-\frac{x^2}{Re}t\right)\right) \\
& \times\frac{\mathrm{d}x}{1+\frac{4}{\pi^2}\left[\gamma+\frac{1}{2}\log\frac{x^2-\varepsilon^2}{4}+\frac{x^2}{2\varepsilon^2}\log\frac{x^2}{x^2-\varepsilon^2}-\frac{1}{2}\right]^2} \\
& +\frac{4}{\pi^2}\int_0^{\varepsilon}\frac{\varepsilon^2}{x^3}\frac{1-\exp\left(-\frac{x^2}{Re}t\right)}{1+\frac{4\varepsilon^4}{x^4\pi^2}\left[\gamma+\frac{1}{2}\log\frac{\varepsilon^2-x^2}{4}+\frac{x^2}{2\varepsilon^2}\log\frac{x^2}{\varepsilon^2-x^2}-\frac{1}{2}\right]^2}\mathrm{d}x
\end{aligned}
$$

5.2 C_D^*

To obtain the asymptotic solution of C_D for $t \to 0$, we consider the asymptotic behavior of $\tilde{C}_D$ for $p \to \infty$. $\tilde{C}_D$ is given by

$$
\tilde{C}_D=\frac{4\pi}{Re}\left[\varepsilon\left\{1+\frac{K_2(x)}{K_0(x)}\right\}+2\frac{\varepsilon}{x}\frac{K_1(x)}{K_0(x)}+\tilde{C}\right] \tag{5.1}
$$

where

$$
\begin{aligned}
x &= (2\varepsilon p)^{\frac{1}{2}} \\
\tilde{C} &= \frac{2}{p}\left[\frac{1}{\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}}-\frac{1}{\gamma+\log\frac{\varepsilon}{2}+\frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon}-\frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{\varepsilon+2p}+1\right\}}\right]
\end{aligned}
$$

We use the following relations by using the relation $K_0'(x)=(\mathrm{d}K_0(x))/(\mathrm{d}x)$:

$$
\begin{aligned}
& 1+\frac{K_2(x)}{K_0(x)}+\frac{2}{x}\frac{K_1(x)}{K_0(x)} \\
= & \frac{x\left[K_0(x)+K_2(x)\right]+2K_1(x)}{xK_0(x)}=\frac{1}{xK_0(x)}\left[x\left(2K_0(x)+\frac{2}{x}K_1(x)\right)\right. \\
& \left.+2K_1(x)\right] \\
= & 2+\frac{4}{xK_0(x)}K_1(x) \\
= & 2-\frac{4}{x}\frac{K_0'(x)}{K_0(x)}
\end{aligned}
$$

and

$$
\frac{K_0'(x)}{K_0(x)}\approx-1-\frac{1}{2x}+\frac{1}{8}\frac{1}{x^2}\cdots \quad \text{as } x\to\infty
$$

Then, we arrive at for $x \to \infty$

$$
\begin{aligned}
& 1+\frac{K_2(x)}{K_0(x)}+\frac{2}{x}\frac{K_1(x)}{K_0(x)} \\
& \approx 2+\frac{4}{x}+\frac{2}{x^2}-\frac{1}{2}\frac{1}{x^3}+\cdots
\end{aligned}
$$

$\tilde{C}^*$ is defined as $\tilde{C}_D$ for $p \to \infty$, Then, from the above relation we have

$$
\tilde{C}_D^*=\frac{4\pi}{Re}\left[\varepsilon\left(2+\frac{4}{(2\varepsilon p)^{\frac{1}{2}}}+\frac{2}{2\varepsilon p}-\frac{1}{2}\frac{1}{(2\varepsilon p)^{\frac{3}{2}}}\right)+\tilde{C}\right]
$$

By the inverse transformation of this relation to real time space, we have

$$\begin{aligned}
C_D^* &= \mathcal{L}^{-1}\left(\tilde{C}_D^*\right) \\
&= \frac{4\pi}{Re}\varepsilon\left[2\delta(t) + \frac{4}{(2\varepsilon)^{\frac{1}{2}}}\frac{1}{(\pi t)^{\frac{1}{2}}} + \frac{2}{2\varepsilon}H(t)\right. \\
&\quad \left. -\frac{1}{2}\frac{1}{(2\varepsilon)^{\frac{3}{2}}}\frac{2}{\pi^{\frac{1}{2}}}t^{\frac{1}{2}} + \mathcal{L}^{-1}\left(\tilde{C}\right)\right]
\end{aligned}$$

where $H(t)$ is Heaviside function.

5.2.1 $\mathcal{L}^{-1}\left(\tilde{C}\right)$

$\tilde{C}$ is given by

$$\begin{aligned}
\tilde{C} &= \frac{2}{p}\left[\frac{1}{\gamma + \frac{1}{2}\log\frac{\varepsilon p}{2}}\right. \\
&\quad \left. -\frac{1}{\gamma + \log\frac{\varepsilon}{2} + \frac{1}{2}\log\frac{2p+\varepsilon}{\varepsilon} - \frac{1}{2}\left\{\frac{2p}{\varepsilon}\log\frac{2p}{\varepsilon+2p} + 1\right\}}\right] \\
&= \frac{2}{p}\left[\frac{1}{\gamma + \frac{1}{2}\log\frac{p\varepsilon}{2}}\right. \\
&\quad \left. -\frac{1}{\gamma + \log\frac{\varepsilon}{2} + \frac{1}{2}\log\frac{2p}{\varepsilon} + \frac{1}{2}\log\left(1+\frac{\varepsilon}{2p}\right) + \frac{1}{2}\frac{2p}{\varepsilon}\log\left(1+\frac{\varepsilon}{2p}\right) - \frac{1}{2}}\right]
\end{aligned}$$

We use the following relations:

$$\begin{aligned}
&\frac{1}{2}\log\left(1+\frac{\varepsilon}{2p}\right) + \frac{1}{2}\frac{2p}{\varepsilon}\log\left(1+\frac{\varepsilon}{2p}\right) - \frac{1}{2} \\
&= \frac{1}{2}\log(1+x) + \frac{1}{2}\frac{1}{x}\log(1+x) - \frac{1}{2} \\
&= \frac{1}{2}\left[\log(1+x) + \frac{1}{x}\log(1+x) - 1\right] \\
&\approx \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} + 1 - \frac{x}{2} + \frac{x^2}{3} - 1\right] \\
&\approx \frac{1}{2}\left[x - \frac{x}{2} + \cdots\right] \approx \frac{1}{4}x
\end{aligned}$$

where $x = \varepsilon/(2p)$. Then, we have

$$\begin{aligned}
\tilde{C} &\approx \frac{2}{p}\left[\frac{1}{\tilde{A}} - \frac{1}{\tilde{A} + \frac{1}{4}x}\right] \\
&\approx \frac{2}{p}\left[\frac{1}{\tilde{A}} - \frac{1}{\tilde{A}}\left\{1 - \frac{x}{4\tilde{A}}\right\}\right] \\
&\approx \frac{2}{p}\frac{x}{4\tilde{A}^2} = \frac{1}{2p}\frac{1}{\left(\gamma + \frac{1}{2}\log\frac{p\varepsilon}{2}\right)^2}\frac{\varepsilon}{2p} \\
&\approx \frac{\varepsilon}{4p^2}\frac{1}{\left(\gamma + \frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}
\end{aligned}$$

where $\tilde{A}$ is given by

$$\tilde{A} \equiv \gamma + \frac{1}{2}\log\frac{p\varepsilon}{2}$$

To obtain the inverse transformation of the above equation, we first consider the following inverse transformation:

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(pt)}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\mathrm{d}p \\
&= \frac{1}{2\pi i}\int_{L}\frac{\exp(pt)}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\mathrm{d}p \\
&= \int_0^\infty \exp(-st)\frac{\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}}{\left[\left(\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}\right)^2-\frac{\pi^2}{4}\right]^2+\pi^2\left[\gamma+\frac{1}{2}\log\frac{\varepsilon s}{2}\right]^2}\mathrm{d}s \\
&= \frac{1}{t}\int_0^\infty \exp(-x) \\
&\quad\times\frac{\gamma+\frac{1}{2}\log\frac{\varepsilon x}{2}-\frac{1}{2}\log t}{\left[\left(\gamma+\frac{1}{2}\log\frac{\varepsilon x}{2}-\frac{1}{2}\log t\right)^2-\frac{\pi^2}{4}\right]^2+\pi^2\left[\gamma+\frac{1}{2}\log\frac{\varepsilon x}{2}-\frac{1}{2}\log t\right]^2}\mathrm{d}x
\end{aligned}$$

Therefore, we have as $t \ll 1$:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] &\approx \frac{1}{t}\frac{-\frac{1}{2}\log t}{\left(-\frac{1}{2}\log t\right)^4}\int_0^\infty \exp(-x)\mathrm{d}x \qquad \text{when } t \ll 1 \\
&\approx \frac{8}{t}\frac{1}{(-\log t)^3}
\end{aligned}$$

Here, the integral contour L is shown in Figure 1.

Using this relation, we have

$$\mathcal{L}^{-1}\left[\frac{1}{p}\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] \approx 8\int_0^t \frac{\mathrm{d}s}{s\,(-\log s)^3}$$

We put $x = -\log s$, then we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{1}{p}\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] \\
&\approx 8\int_\infty^{-\log t}\frac{-\exp(-x)}{\exp(-x)x^3}\mathrm{d}x
\end{aligned}$$

Putting $x = -\log s$, we have

$$\mathcal{L}^{-1}\left[\frac{1}{p}\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] \approx 8\int_{-\log t}^\infty \frac{1}{x^3}\mathrm{d}x = \left[-4\frac{1}{x^2}\right]_{-\log t}^\infty = \frac{4}{(-\log t)^2}$$

Therefore, we have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{p^2}\frac{1}{\left(\gamma+\frac{1}{2}\log\frac{\varepsilon p}{2}\right)^2}\right] &\approx 4\int_0^t \frac{\mathrm{d}s}{(-\log s)^2}\\ &\approx 4\int_{\infty}^{-\log t}\frac{-\exp(-x)}{x^2}\mathrm{d}x\\ &\approx 4\int_{-\log t}^{\infty}\frac{\exp(-x)}{x^2}\mathrm{d}x\end{aligned}$$

From these relations, we finally arrive at Eq.(100):

$$\begin{aligned}C_D^* &= \mathcal{L}^{-1}\left[\tilde{C}_D^*\right]\\ &\approx 2\pi\left[2\delta(t)+\frac{4}{Re^{\frac{1}{2}}}\frac{1}{(\pi t)^{\frac{1}{2}}}+\frac{2}{Re}H(t)-\frac{1}{Re^{\frac{3}{2}}}\left(\frac{t}{\pi}\right)^{\frac{1}{2}}+\int_{-\log t}^{\infty}\frac{\exp(-x)}{x^2}\mathrm{d}x\right]\end{aligned}$$

6 Numerical results

6.1 Derivation of $C_D \sim -\frac{8\pi}{R_e}\frac{1}{\gamma+\log(R_e/4)}$ ←(This section is added.)

This equation is derived for $R_e \ll 1$ and $t \to \infty$. Let us consider the first term of r.h.s. of $\hat{I}_1$. Changing the integral variable by $x = \delta y$ where $\delta = (R_e/t)^{1/2}$, we have

$$\begin{aligned}(\text{1st term of } \hat{I}_1) &= \frac{2}{\pi^2}\frac{2}{R_e}\int_0^{\infty}\exp(-y^2)\Big[\frac{1}{y}\Big(\frac{1}{J_0^2(\delta y)+Y_0^2(\delta y)}\\ &\quad -\frac{1}{1+\frac{4}{\pi^2}\left(\gamma+\log\frac{\delta y}{2}\right)}\Big)-\frac{\pi}{2}\delta\Big]\mathrm{d}y\end{aligned}$$

Since $Y_0(\delta y) = \frac{2}{\pi}\left(\gamma+\log\frac{\delta y}{2}\right)J_0(\delta y)+O(\delta^2 y^2)$, we see

$$J_0^2(\delta y)+Y_0^2(\delta y) = 1+\frac{4}{\pi^2}\left(\gamma+\log\frac{\delta y}{2}\right)^2+O(\delta^2 y^2)$$

Taking into account of $R_e \ll 1$ and $t \to \infty$, that is, $\delta \ll 1$, we easily see

$$(\text{1st term of } \hat{I}_1) = O(\delta)$$

Since the second term of r.h.s. of $\hat{I}_1$ is zero for $t \to \infty$, we consider the terms of $4 \times$ (3rd term of r.h.s. of $\hat{I}_1$) $+ \frac{2}{\varepsilon}\hat{I}_2$:

$$\begin{aligned}&\left(4\times(\text{3rd term of r.h.s. of } \hat{I}_1)+\frac{2}{\varepsilon}\hat{I}_2\right)\\ &= \frac{16}{\pi^2}\frac{1}{R_e}\int_0^{\infty}\frac{1}{x}\frac{1}{1+\frac{4}{\pi^2}\left(\gamma+\log\frac{x}{2}\right)^2}\mathrm{d}x\end{aligned}$$

This integral is obtained by replacing $x = 2\exp(-\gamma + \frac{\pi}{2}y)$:

$$\left(4 \times (\text{3rd term of r.h.s. of } \hat{I}_1) + \frac{2}{\varepsilon}\hat{I}_2\right)$$
$$= \frac{8}{\pi}\frac{1}{R_e}\int_{-\infty}^{\infty}\frac{1}{1+y^2}\mathrm{d}y = \frac{8}{R_e}$$

The second term of r.h.s. of $\hat{I}_3$ is given by

$$\begin{aligned}
&(\text{2nd term of r.h.s. of } \hat{I}_3)\\
&= \int_0^1 y\left(1 - \exp\left(-\frac{R_e}{4}ty^2\right)\right)/\Big[\frac{\pi^2 y^4}{4}\\
&\quad +\left(\gamma + \log\frac{R_e}{4} - \frac{1}{2} + y^2\log y + \frac{1}{2}(1-y^2)\log(1-y^2)\right)^2\Big]\mathrm{d}y\\
&\approx \int_0^1 y\frac{1}{\left(\gamma + \log\frac{R_e}{4} - \frac{1}{2}\right)^2}\mathrm{d}y\\
&\approx \frac{1}{2}\frac{1}{\left(\gamma + \frac{R_e}{4} - \frac{1}{2}\right)^2}
\end{aligned}$$

The first term of r.h.s. of $\hat{I}_3$ is given by changing the integral variable x to $\frac{R_e}{2}y$:

$$\begin{aligned}
&(\text{1st term of r.h.s. of } \hat{I}_3)\\
&= \int_1^\infty \frac{1}{y}\left(1 - \exp\left(-\frac{R_e}{4}ty^2\right)\right)/\Big[\frac{\pi^2}{4}\\
&\quad +\left(\gamma + \log\frac{R_e}{4} - \frac{1}{2} - \frac{1}{2}\log(y^2-1) + \frac{y^2}{2}\log\frac{y^2}{y^2-1}\right)^2\Big]\mathrm{d}y
\end{aligned}$$

Here we use the following relation;

$$\log(y^2-1) + y^2\log\frac{y^2}{y^2-1} \sim 2\log y + 1 \quad \text{for } y \to \infty$$

Then, we have

$$\begin{aligned}
&(\text{1st term of r.h.s. of } \hat{I}_3)\\
&= \int_1^\infty \frac{1}{y}\frac{\mathrm{d}y}{\frac{\pi^2}{4} + \left(\gamma + \log\frac{R_e}{4} + \log y\right)^2} + O\left(\frac{1}{\left(\gamma + \log\frac{R_e}{4}\right)^2}\right)
\end{aligned}$$

Replacing y by $y = \exp(X - \gamma - \log\frac{R_e}{4})$, we have

$$\begin{aligned}
&(\text{1st term of r.h.s. of } \hat{I}_3)\\
&\approx \int_{\gamma+\log\frac{R_e}{4}}^\infty \frac{1}{X^2 + \frac{\pi^2}{4}}\mathrm{d}X \approx 2 - \frac{1}{\gamma + \log\frac{R_e}{4}}
\end{aligned}$$

We note that R_e is assumed to be very small, that is, $\gamma + \log\frac{R_e}{4}$ is large negative. Therefore, terms derived from the 3rd term of $\hat{I}_1$ plus $\hat{I}_2$ are canceled and we can obtain the present result for $t \to \infty$ and small R_e.

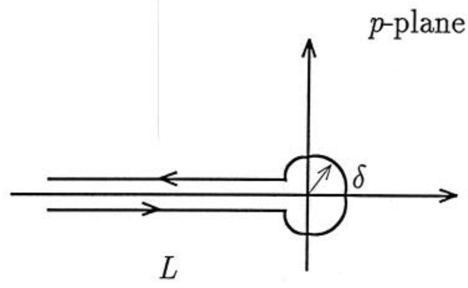


Figure 1: Integral Contour L

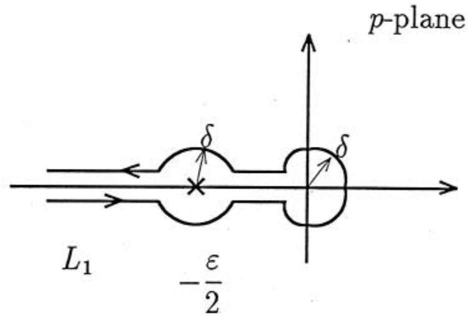


Figure 2: Integral Contour L_1