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Appendices (Mathematical and Technical Details)

Appendix A *Derivation of* (2.21)

In this appendix we derive infinite system (2.21) from (2.19) under assumption (2.20), i.e. $S_1 = O(\epsilon)$ and $S_2 = O(\epsilon^2)$. First we expand $(1-2S_1+S_2)^{-\frac{3}{2}}$ in powers of ϵ. Let $y \equiv 2S_1 - S_2 = O(\epsilon)$, then $(1-y)^{-\frac{3}{2}}$ can be expanded in powers of y as

$$\begin{aligned}(1-y)^{-\frac{3}{2}} &= 1 + \frac{3}{2}\cdot\frac{y}{1!} + \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{y^2}{2!} + \frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{y^3}{3!} + \dots \\ &= \sum_{p=0}^{\infty}\frac{(2p+1)!!}{2^p}\frac{y^p}{p!} \\ &= \sum_{p=0}^{\infty}\frac{(2p+2)!}{2^{2p+1}(p+1)!p!}y^p. \end{aligned} \tag{A.1}$$

Substituting $y = 2S_1 - S_2$ into (A.1) yields

$$(1-2S_1+S_2)^{-\frac{3}{2}} = \sum_{p=0}^{\infty}\sum_{j=0}^{p}\frac{(-1)^j(2p+2)!}{2^{p+j+1}(p+1)!p!}\begin{pmatrix} p \\ j \end{pmatrix} S_1^{p-j}S_2^j,$$

where $\begin{pmatrix} p \\ j \end{pmatrix}$ is the Binomial coefficient. Since $S_1^{p-j}S_2^j = O(\epsilon^{p+j})$, we put $i = p+j$ and rearrange the above expression as

$$(1-2S_1+S_2)^{-\frac{3}{2}} = \sum_{i=0}^{\infty}\sum_{j=0}^{[i/2]} C(i,j){S_1}^{i-2j}{S_2}^j, \tag{A.2}$$

where

$$C(i,j) = \frac{(-1)^j\{2(i-j+1)\}!}{2^{i+1}(i-j+1)!j!(i-2j)!}, \tag{A.3}$$

and $[x]$ denotes the largest integer i which satisfies $i \le x$. Note that ${S_1}^{i-2j}{S_2}^j = O(\epsilon^i)$. Substituting

$$S_1 \equiv \frac{\alpha s_x + \beta s_y}{\alpha^2+\beta^2} \quad\text{and}\quad S_2 \equiv \frac{s_x^2+s_y^2+s_z^2}{\alpha^2+\beta^2},$$

into ${S_1}^{i-2j}$ and ${S_2}^j$ gives

$$S_1{}^{i-2j} = \frac{1}{(\alpha^2+\beta^2)^{i-2j}} \sum_{k=0}^{i-2j} \begin{pmatrix} i-2j \\ k \end{pmatrix} (\alpha s_x)^k (\beta s_y)^{i-2j-k}, \tag{A.4a}$$

$$S_2{}^j = \frac{1}{(\alpha^2+\beta^2)^j} \sum_{r=0}^{j} \sum_{s=0}^{j-r} \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} j-r \\ s \end{pmatrix} (s_x^2)^s (s_y^2)^{j-r-s} (s_z^2)^r. \tag{A.4b}$$

By using (A.2) with (A.4a,b), we may write (2.19) as

$$\sum_{n,m} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} X_{n,m} \\ Y_{n,m} \\ Z_{n,m} \end{pmatrix} \mathrm{e}^{\mathrm{i}(n\lambda_1+\delta m\lambda_2)} = -\frac{1}{4\pi} \text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha \mathrm{d}\beta}{(\alpha^2+\beta^2)^{\frac{3}{2}}}$$
$$\times \begin{pmatrix} -s_z - \beta s_{2z} + s_y s_{2z} - s_z s_{2y} \\ \alpha s_{2z} + s_z s_{2x} - s_x s_{2z} \\ -\alpha + s_x - \alpha s_{2y} + \beta s_{2x} + s_x s_{2y} - s_y s_{2x} \end{pmatrix}$$
$$\times \sum_{i=0}^{\infty} \sum_{j=0}^{[i/2]} \sum_{k=0}^{i-2j} \sum_{r=0}^{j} \sum_{s=0}^{j-r}$$
$$C(i,j) \begin{pmatrix} i-2j \\ k \end{pmatrix} \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} j-r \\ s \end{pmatrix}$$
$$\times \frac{\alpha^k \beta^{i-2j-k}}{(\alpha^2+\beta^2)^{i-j}} s_x{}^{k+2s} s_y{}^{i-(k+2s)-2r} s_z{}^{2r}. \tag{A.5}$$

Since

$$\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} \equiv \sum_{n,m} \begin{pmatrix} X_{n,m}(t) \\ Y_{n,m}(t) \\ Z_{n,m}(t) \end{pmatrix} (1 - \mathrm{e}^{\mathrm{i}(n\alpha+\delta m\beta)}) \mathrm{e}^{\mathrm{i}(n\lambda_1+\delta m\lambda_2)},$$

(see (2.17a)) we have

$$s_x{}^{k+2s} s_y{}^{i-(k+2s)-2r} s_z{}^{2r} = \sum^{(i)} I^{(i)}(\alpha,\beta) U^{(i,k,r,s)} \mathrm{e}^{\mathrm{i}(N\lambda_1+\delta M\lambda_2)}, \tag{A.6}$$

where

$$\sum^{(i)} \equiv \sum_{n_1,m_1} \cdot \sum_{n_2,m_2} \cdots \sum_{n_i,m_i}, \tag{A.7}$$

$$I^{(i)}(\alpha,\beta) \equiv (1-\mathrm{e}^{\mathrm{i}(n_1\alpha+\delta m_1\beta)})(1-\mathrm{e}^{\mathrm{i}(n_2\alpha+\delta m_2\beta)})\cdots(1-\mathrm{e}^{\mathrm{i}(n_i\alpha+\delta m_i\beta)}), \tag{A.8}$$

$$U^{(i,k,r,s)} \equiv X_{n_1,m_1}\cdots X_{n_\eta,m_\eta}\cdot Y_{n_{\eta+1},m_{\eta+1}}\cdots Y_{n_\mu,m_\mu} \cdot Z_{n_{\mu+1},m_{\mu+1}}\cdots Z_{n_i,m_i}, \tag{A.9}$$

$\eta = k+2s$, $\mu = i-2r$, $N = n_1+n_2+\cdots+n_i$ and $M = m_1+m_2+\cdots+m_i$. Note that when $i=0$, $U^{(i,k,r,s)}=1$, and that when $\eta=0$, $\eta=\mu$ and $\mu=i$, $U^{(i,k,r,s)}$ does not include X, Y and Z component, respectively.

Equating coefficients of $\exp[\mathrm{i}(n\lambda_1+\delta m\lambda_2)]$ on both-hand sides of (A.5) yields

$$\begin{aligned}\dot{X}_{n,m} = &\sum_{i=0}^{\infty}\sum_{j=0}^{[i/2]}\sum_{k=0}^{i-2j}\sum_{r=0}^{j}\sum_{s=0}^{j-r} C(i,j)\begin{pmatrix} i-2j \\ k\end{pmatrix}\begin{pmatrix} j \\ r\end{pmatrix}\begin{pmatrix} j-r \\ s\end{pmatrix}\\ &\times\Bigg\{-\sum_{\{n,m\}}^{(i+1)} G_1^{(i+1,j,k)}U^{(i,k,r,s)}Z_{n_{i+1},m_{i+1}} - \sum_{\{n,m\}}^{(i+1)} H_\beta^{(i+1,j,k)}U^{(i,k,r,s)}Z_{n_{i+1},m_{i+1}}\\ &+\sum_{\{n,m\}}^{(i+2)} H_1^{(i+2,j,k)}U^{(i,k,r,s)}Y_{n_{i+1},m_{i+1}}Z_{n_{i+2},m_{i+2}}\\ &-\sum_{\{n,m\}}^{(i+2)} H_1^{(i+2,j,k)}U^{(i,k,r,s)}Z_{n_{i+1},m_{i+1}}Y_{n_{i+2},m_{i+2}}\Bigg\}, \quad .\end{aligned} \tag{A.10a}$$

$$\begin{aligned}\dot{Y}_{n,m} = &\sum_{i=0}^{\infty}\sum_{j=0}^{[i/2]}\sum_{k=0}^{i-2j}\sum_{r=0}^{j}\sum_{s=0}^{j-r} C(i,j)\begin{pmatrix} i-2j \\ k\end{pmatrix}\begin{pmatrix} j \\ r\end{pmatrix}\begin{pmatrix} j-r \\ s\end{pmatrix}\\ &\times\Bigg\{\sum_{\{n,m\}}^{(i+1)} H_\alpha^{(i+1,j,k)}U^{(i,k,r,s)}Z_{n_{i+1},m_{i+1}}\\ &+\sum_{\{n,m\}}^{(i+2)} H_1^{(i+2,j,k)}U^{(i,k,r,s)}Z_{n_{i+1},m_{i+1}}X_{n_{i+2},m_{i+2}}\\ &-\sum_{\{n,m\}}^{(i+2)} H_1^{(i+2,j,k)}U^{(i,k,r,s)}X_{n_{i+1},m_{i+1}}Z_{n_{i+2},m_{i+2}}\Bigg\},\end{aligned} \tag{A.10b}$$

$$\begin{aligned}\dot{Z}_{n,m} = &\sum_{i=0}^{\infty}\sum_{j=0}^{[i/2]}\sum_{k=0}^{i-2j}\sum_{r=0}^{j}\sum_{s=0}^{j-r} C(i,j)\begin{pmatrix} i-2j \\ k\end{pmatrix}\begin{pmatrix} j \\ r\end{pmatrix}\begin{pmatrix} j-r \\ s\end{pmatrix}\\ &\times\Bigg\{-\sum_{\{n,m\}}^{(i)} G_\alpha^{(i,j,k)}U^{(i,k,r,s)} + \sum_{\{n,m\}}^{(i+1)} G_1^{(i+1,j,k)}U^{(i,k,r,s)}X_{n_{i+1},m_{i+1}}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\{n,m\}}^{(i+1)} H_{\alpha}^{(i+1,j,k)} U^{(i,k,r,s)} Y_{n_{i+1},m_{i+1}} + \sum_{\{n,m\}}^{(i+1)} H_{\beta}^{(i+1,j,k)} U^{(i,k,r,s)} X_{n_{i+1},m_{i+1}} \\
&+ \sum_{\{n,m\}}^{(i+2)} H_{1}^{(i+2,j,k)} U^{(i,k,r,s)} X_{n_{i+1},m_{i+1}} Y_{n_{i+2},m_{i+2}} \\
&\left. - \sum_{\{n,m\}}^{(i+2)} H_{1}^{(i+2,j,k)} U^{(i,k,r,s)} Y_{n_{i+1},m_{i+1}} X_{n_{i+2},m_{i+2}} \right\}. \qquad \text{(A.10c)}
\end{aligned}
$$

where $\sum_{\{n,m\}}^{(i)}$ means the summation $\sum^{(i)}$ defined in (A.7) under the conditions $n_1 + n_2 + \cdots + n_i = n$ and $m_1 + m_2 + \cdots + m_i = m$. The integration coefficients $G_{\alpha}^{(i,j,k)}$, $G_{1}^{(i+1,j,k)}$, $H_{\alpha}^{(i+1,j,k)}$, $H_{\beta}^{(i+1,j,k)}$ and $H_{1}^{(i+2,j,k)}$ are defined by

$$
G_{\alpha}^{(i,j,k)} \equiv -\frac{1}{4\pi}\text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha F^{(i,j,k)}(\alpha,\beta) I^{(i)}(\alpha,\beta) \mathrm{d}\alpha \mathrm{d}\beta, \qquad \text{(A.11a)}
$$

$$
G_{1}^{(i+1,j,k)} \equiv -\frac{1}{4\pi}\text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{(i,j,k)}(\alpha,\beta) I^{(i+1)}(\alpha,\beta) \mathrm{d}\alpha \mathrm{d}\beta, \qquad \text{(A.11b)}
$$

$$
\begin{aligned}
H_{\{\alpha,\beta\}}^{(i+1,j,k)} \equiv -\frac{1}{4\pi}\text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\alpha,\beta\} F^{(i,j,k)}(\alpha,\beta) \\
\times \mathrm{i}\delta m_{i+1} \mathrm{e}^{\mathrm{i}(n_{i+1}\alpha + \delta m_{i+1}\beta)} I^{(i)}(\alpha,\beta) \mathrm{d}\alpha \mathrm{d}\beta, \qquad \text{(A.11c)}
\end{aligned}
$$

$$
\begin{aligned}
H_{1}^{(i+2,j,k)} \equiv -\frac{1}{4\pi}\text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{(i,j,k)}(\alpha,\beta) \\
\times \mathrm{i}\delta m_{i+2} \mathrm{e}^{\mathrm{i}(n_{i+2}\alpha + \delta m_{i+2}\beta)} I^{(i+1)}(\alpha,\beta) \mathrm{d}\alpha \mathrm{d}\beta, \qquad \text{(A.11d)}
\end{aligned}
$$

where

$$
F^{(i,j,k)}(\alpha,\beta) \equiv \frac{\alpha^k \beta^{i-2j-k}}{(\alpha^2+\beta^2)^{i-j+\frac{3}{2}}}. \qquad \text{(A.12)}
$$

Note that $i \geq 0$, $0 \leq j \leq i/2$ and $0 \leq k \leq i - 2j$, and also that the above integrals depend not only on i, j, k but also on the vectors $\boldsymbol{n} \equiv (n_1, n_2, \cdots, n_\lambda)$ and $\boldsymbol{m} \equiv (m_1, m_2, \cdots, m_\lambda)$ whose dimension λ is equal to the superscript of G or H. In appendix B we will show how to evaluate these integrals.

Since $G_{\alpha}^{(0,0,0)} = 0$, as will be shown in appendix B, there is no term of zeroth degree in $\boldsymbol{A}_{n,m}$ on the right hand side of equations (A.10a)-(A.10c). The equation for the Fourier coefficients can therefore be written symbolically in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}_{n,m} = \boldsymbol{L}\boldsymbol{A}_{n,m} + \sum_{\{n,m\}}^{(2)} \{\boldsymbol{A}\}^2 + \cdots + \sum_{\{n,m\}}^{(K)} \{\boldsymbol{A}\}^K + \cdots \quad , \tag{A.13}$$

in which

$$\boldsymbol{L}\boldsymbol{A}_{n,m} = \begin{pmatrix} (-G_1^{(1,0,0)} - H_\beta^{(1,0,0)})Z_{n,m} \\ H_\alpha^{(1,0,0)} Z_{n,m} \\ (-3G_\alpha^{(1,0,1)} + G_1^{(1,0,0)} + H_\beta^{(1,0,0)})X_{n,m} - (3G_\alpha^{(1,0,0)} + H_\alpha^{(1,0,0)})Y_{n,m} \end{pmatrix}, \tag{A.14}$$

$$\sum_{\{n,m\}}^{(K)} \equiv \begin{pmatrix} \text{sum with respect to} \quad n_1, \ldots, n_K, m_1, \ldots, m_K \\ \text{satisfying} \\ n_1 + n_2 + \ldots + n_K = n \ , \ m_1 + m_2 + \ldots + m_K = m \end{pmatrix},$$

and $\{\boldsymbol{A}\}^{(K)}$ indicates the terms of degree K in $\boldsymbol{A}_{p,q}$. If we define

$$\sum_{i=K}{}' \equiv \sum_{j=0}^{[K/2]} \sum_{k=0}^{K-2j} \sum_{r=0}^{j} \sum_{s=0}^{j-r} \ , \tag{A.15}$$

$$C^{(K)} \equiv C(K,j) \begin{pmatrix} K-2j \\ k \end{pmatrix} \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} j-r \\ s \end{pmatrix}, \tag{A.16}$$

$$G_\alpha^{(K)} \equiv G_\alpha^{(K,j,k)}, \ G_1^{(K)} \equiv G_1^{(K,j,k)}, \ H_{\{\alpha,\beta\}}^{(K)} \equiv H_{\{\alpha,\beta\}}^{(K,j,k)}, \ H_1^{(K)} \equiv H_1^{(K,j,k)}, \tag{A.17}$$

and

$$\begin{aligned} U^K &\equiv U^{(K,k,r,s)} \\ &= X_{n_1,m_1} \cdots X_{n_\eta,m_\eta} \cdot Y_{n_{\eta+1},m_{\eta+1}} \cdots Y_{n_\mu,m_\mu} \\ &\quad \cdot Z_{n_{\mu+1},m_{\mu+1}} \cdots Z_{n_K,m_K}, \end{aligned} \tag{A.18}$$

where $\eta = k + 2s$ and $\mu = K - 2r$, then $\{\boldsymbol{A}\}^{(K)}$ for $K \geq 2$ in (A.13) can be expressed as

$$\begin{aligned} \{\boldsymbol{A}\}_x^{(K)} = &- \sum_{i=K-1}{}' C^{(K-1)} G_1^{(K)} U^{K-1} Z - \sum_{i=K-1}{}' C^{(K-1)} H_\beta^{(K)} U^{K-1} Z^\dagger \\ &+ \sum_{i=K-2}{}' C^{(K-2)} H_1^{(K)} U^{K-2} Y Z^\dagger - \sum_{i=K-2}{}' C^{(K-2)} H_1^{(K)} U^{K-2} Z Y^\dagger, \end{aligned}$$

(A.19a)

$$\{\boldsymbol{A}\}_y^{(K)} = \sum_{i=K-1}' C^{(K-1)} H_\alpha^{(K)} U^{K-1} Z^\dagger + \sum_{i=K-2}' C^{(K-2)} H_1^{(K)} U^{K-2} Z X^\dagger - \sum_{i=K-2}' C^{(K-2)} H_1^{(K)} U^{K-2} X Z^\dagger, \quad \text{(A.19b)}$$

$$\{\boldsymbol{A}\}_z^{(K)} = -\sum_{i=K}' C^{(K)} G_\alpha^{(K)} U^K + \sum_{i=K-1}' C^{(K-1)} G_1^{(K)} U^{K-1} X - \sum_{i=K-1}' C^{(K-1)} H_\beta^{(K)} U^{K-1} Z^\dagger + \sum_{i=K-1}' C^{(K-1)} H_1^{(K)} U^{K-1} X^\dagger + \sum_{i=K-2}' C^{(K-2)} H_1^{(K)} U^{K-2} X Y^\dagger - \sum_{i=K-2}' C^{(K-2)} H_1^{(K)} U^{K-2} Y X^\dagger,$$

(A.19c)

where we have omitted the subscript of X, Y or Z and attached dagger to the term which comes from s_{2x}, s_{2y} or s_{2z}. Thus we have obtained the expression of infinite system (2.21) except for evaluation of the integration coefficients (A.11a)-(A.11d).

Appendix B *Evaluation of the integration coefficients (A.11a)-(A.11d)*

Here we evaluate the integrals (A.11a)-(A.11d) which appear in Appendix A. Let us begin with $G_\alpha^{(i,j,k)}$. From (A.11a), we have

$$G_\alpha^{(i,j,k)} = -\frac{1}{4\pi}\text{p.v.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\alpha^{k+1}\beta^{i-2j-k}}{(\alpha^2+\beta^2)^{i-j+\frac{3}{2}}} I^{(i)}(\alpha,\beta)\mathrm{d}\alpha\mathrm{d}\beta,$$

By putting $\alpha = R\cos\theta$ and $\beta = R\sin\theta$, we have

$$G_\alpha^{(i,j,k)} = -\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}R\frac{(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}}{R^{i+1}} g^{(i)}(R,\theta), \quad \text{(B.1)}$$

where

$$\text{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}R \equiv \lim_{\epsilon\to 0}\int_0^{2\pi}\mathrm{d}\theta\int_\epsilon^{\infty}\mathrm{d}R,$$

$$\begin{aligned} g^{(i)}(R,\theta) &\equiv I^{(i)}(R\cos\theta, R\sin\theta) \\ &= \{1-\exp\mathrm{i}R(n_1\cos\theta+\delta m_1\sin\theta)\}\cdots \\ &\quad\cdot\{1-\exp\mathrm{i}R(n_i\cos\theta+\delta m_i\sin\theta)\}. \end{aligned} \quad \text{(B.2)}$$

Since $j \le [i/2]$ and $k \le i-2j$, (B.1) for $i=0$ becomes

$$G_\alpha^{(0,0,0)} = -\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}R\frac{\cos\theta}{R} = 0.$$

For $i \geq 1$, integration of (B.1) by part with respect to R gives

$$G_\alpha^{(i,j,k)} = \frac{1}{4\pi i}\lim_{\rho\to 0}\int_0^{2\pi} \mathrm{d}\theta(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}\left[\frac{g^{(i)}(R,\theta)}{R^i}\right]_\rho^\infty$$
$$-\frac{1}{4\pi i}\mathrm{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \mathrm{d}R\frac{(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}}{R^i}\frac{\partial g^{(i)}(R,\theta)}{\partial R}. \qquad \text{(B.3)}$$

When ρ is small, $g^{(i)}(\rho,\theta)/\rho^i$ can be expanded as

$$\frac{g^{(i)}(\rho,\theta)}{\rho^i} = (-\mathrm{i})^i(n_1\cos\theta+\delta m_1\sin\theta)\cdots(n_i\cos\theta+\delta m_i\sin\theta)+O(\rho).$$

Expanding the first term of the right hand side of this expression gives terms that are proportional to $\cos^\mu\theta\sin^{i-\mu}\theta$ ($\mu = 0,1,\cdots,i$). As a result, we have the following integrals from (B.3);

$$\int_0^{2\pi}(\cos\theta)^{k+\mu+1}(\sin\theta)^{2i-2j-(k+\mu)}\mathrm{d}\theta, \quad (\mu=0,1,\cdots,i),$$

each of which is zero because $\int_0^{2\pi}\cos^N\theta\sin^M\theta\mathrm{d}\theta$ is not zero only if both $N(\geq 0)$ and $M(\geq 0)$ are even. Because of the above fact and $g^{(i)}(R,\theta)/R^i \to 0$ as $R\to\infty$, the first term of the right hand side of (B.3) is equal to zero. Repeating the partial integrations i times over, we have

$$G_\alpha^{(i,j,k)} = -\frac{1}{4\pi i!}\mathrm{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \mathrm{d}R(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}\frac{1}{R}\left(\frac{\partial}{\partial R}\right)^i g^{(i)}(R,\theta). \qquad \text{(B.4)}$$

When we substitute (B.2) into (B.4), we have integrals in the following form:

$$I = -\frac{1}{4\pi i!}\mathrm{p.v.}\int_0^{2\pi}\mathrm{d}\theta\int_0^\infty \mathrm{d}R(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}$$
$$\times\frac{1}{R}\left(\frac{\partial}{\partial R}\right)^i\exp\{\mathrm{i}R(n\cos\theta+\delta m\sin\theta)\}.$$

Since

$$\exp\{\mathrm{i}R(n\cos\theta+\delta m\sin\theta)\} = \exp\{\mathrm{i}Ra\sin(\theta+\Pi)\}$$
$$= \sum_{\mu=-\infty}^{\infty} J_\mu(Ra)\exp\{\mathrm{i}\mu(\theta+\Pi)\},$$

where

$$a \equiv (n^2+(\delta m)^2)^{\frac{1}{2}},$$

Π is defined by

$$\exp\{\mathrm{i}\Pi\} = \delta m/a + \mathrm{i}n/a,$$

and J_μ is the Bessel function of the first kind, I is expressed as

$$I = -\frac{1}{4\pi i!}\mathrm{p.v.}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}R(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k} \times \sum_{\mu=-\infty}^{\infty} \exp\{\mathrm{i}\mu(\theta+\Pi)\}\frac{1}{R}\left(\frac{\mathrm{d}}{\mathrm{d}R}\right)^i J_\mu(Ra).$$

Using $\mathrm{d}J_\mu(R)/\mathrm{d}R = \{J_{\mu-1}(R) - J_{\mu+1}(R)\}/2$ repeatedly, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}R}\right)^i J_\mu(Ra) = \left(\frac{a}{2}\right)^i \sum_{l=0}^{i}(-1)^l \begin{pmatrix} i \\ l \end{pmatrix} J_{\mu-i+2l}(Ra).$$

Then we have

$$I = -\frac{1}{4\pi i!}\left(\frac{a}{2}\right)^i \mathrm{p.v.}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}z(\cos\theta)^{k+1}(\sin\theta)^{i-2j-k} \times \sum_{\mu=-\infty}^{\infty} \exp\{\mathrm{i}\mu(\theta+\Pi)\} \sum_{l=0}^{i}(-1)^l \begin{pmatrix} i \\ l \end{pmatrix} \frac{J_{\mu-i+2l}(z)}{z},$$

where we have put $z = Ra$. When $\mu = i - 2l$, the integrand diverges at $z = 0$. In order to evaluate the principal value of the integral, we need to estimate the following integral with respect to θ:

$$c_\alpha(\mu) \equiv \int_0^{2\pi} (\cos\theta)^{k+1}(\sin\theta)^{i-2j-k}\mathrm{e}^{\mathrm{i}\mu\theta}\mathrm{d}\theta. \tag{B.5}$$

It can be readily seen that if $i-2j+1+\mu\{=(k+1)+(i-2j-k)+\mu\}$ is an odd number or if $|\mu| > i-2j+1$, $c_\alpha(\mu)$ is equal to zero. Since $c_\alpha(i-2l) = 0$, the principal value is equal to zero when $\mu = i - 2l$. Using $c_\alpha(\mu)$ defined above, we have

$$I = -\frac{1}{4\pi i!}\left(\frac{a}{2}\right)^i \sum_{\mu=-(i-2j+1)}^{(i-2j+1)} c_\alpha(\mu)\exp\{\mathrm{i}\mu\Pi\} \sum_{l=0}^{i}(-1)^l \begin{pmatrix} i \\ l \end{pmatrix} \int_0^{\infty} \mathrm{d}z \frac{J_{\mu-i+2l}(z)}{z}.$$

Here $c_\alpha(\mu)$ is non-zero only if μ is every other integer from $-(i-2j+1)$ to $i-2j+1$. Noting that for such values of μ, $(\mu - i + 2l)$ is an odd number, we have

$$\int_0^\infty \mathrm{d}z \frac{J_{\mu-i+2l}(z)}{z} = \frac{1}{2(\mu-1+2j)} \int_0^\infty \{J_{\mu-i+2l-1}(z) + J_{\mu-i+2l+1}(z)\}\mathrm{d}z$$
$$= \frac{1}{2(\mu-1+2j)} \int_0^\infty \{J_{|\mu-i+2l-1|}(z) + J_{|\mu-i+2l+1|}(z)\}\mathrm{d}z$$
$$= \frac{1}{\mu-1+2j},$$

where we have used the following relations among the Bessel functions

$$\frac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \quad J_{-\nu}(z) = (-1)^\nu J_\nu(z),$$

and

$$\int_0^\infty J_\nu(z)\mathrm{d}z = 1 \quad \text{for} \qquad \nu > -1.$$

Consequently, we have

$$I = -\frac{1}{4\pi i!}\left(\frac{a}{2}\right)^i \sum_{\mu=-(i-2j+1)}^{(i-2j+1)} c_\alpha(\mu)\exp\{\mathrm{i}\mu\Pi\} \sum_{l=0}^{i}(-1)^l \begin{pmatrix} i \\ l \end{pmatrix} \frac{1}{\mu-1+2j}. \tag{B.6}$$

Let $\boldsymbol{q}$ be an i-dimensional vector whose component q_η for $(\eta = 1, \cdots, i)$ is 0 or 1, and $\|\boldsymbol{q}\|$ denote $\sum_{\eta=1}^{i} q_\eta$. Then $g^{(i)}(R,\theta)$ defined by (B.2) can be expressed as

$$g^{(i)}(R,\theta) = \sum_{\nu=0}^{i}(-1)^\nu \sum_{\|\boldsymbol{q}\|=\nu} \exp\{\mathrm{i}R[\boldsymbol{q}\cdot\boldsymbol{n}\cos\theta + \delta\boldsymbol{q}\cdot\boldsymbol{m}\sin\theta]\}, \tag{B.7}$$

where

$$\boldsymbol{n} \equiv (n_1, n_2, \cdots, n_i), \quad \boldsymbol{m} \equiv (m_1, m_2, \cdots, m_i),$$

and the dot between the vectors implies the inner product. By substituting this expression into (B.4), we obtain an expression for $G_\alpha^{(i,j,k)}$:

$$G_\alpha^{(i,j,k)} = -\frac{1}{4\pi i!}\sum_{\nu=0}^{i}(-1)^\nu \sum_{\|\boldsymbol{q}\|=\nu}\left(\frac{a_q}{2}\right)^i$$
$$\sum_{\mu=-(i-2j+1)}^{i-2j+1} c_\alpha(\mu)\mathrm{e}^{\mathrm{i}\mu\Pi_q} \sum_{l=0}^{i}(-1)^l \begin{pmatrix} i \\ l \end{pmatrix} \frac{1}{\mu-i+2l}. \tag{B.8}$$

where

$$a_q \equiv [(\boldsymbol{q}\cdot\boldsymbol{n})^2 + \delta^2(\boldsymbol{q}\cdot\boldsymbol{m})^2]^{\frac{1}{2}},$$

and Π_q is defined by

$$e^{i\Pi_q} \equiv \frac{\delta \boldsymbol{q} \cdot \boldsymbol{m}}{a_q} + i\frac{\boldsymbol{q} \cdot \boldsymbol{n}}{a_q}.$$

In the same way we can obtain expressions for the other integrals. Substituting $\alpha = R\cos\theta$ and $\beta = R\sin\theta$ into (A.11b)-(A.11d) yield

$$G_1^{(i+1,j,k)} = -\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi} d\theta \int_0^{\infty} dR \frac{(\cos\theta)^k (\sin\theta)^{i-2j-k}}{R^{i+2}} g^{(i+1)}(R,\theta), \tag{B.9}$$

$$\begin{aligned} H_\alpha^{(i+1,j,k)} = &-\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi} d\theta \int_0^{\infty} dR \frac{(\cos\theta)^{k+1} (\sin\theta)^{i-2j-k}}{R^{i+1}} \\ &\times i\delta m_{i+1} e^{i(n_{i+1}\alpha + \delta m_{i+1}\beta)} g^{(i)}(R,\theta), \end{aligned} \tag{B.10}$$

$$\begin{aligned} H_\beta^{(i+1,j,k)} = &-\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi} d\theta \int_0^{\infty} dR \frac{(\cos\theta)^{k} (\sin\theta)^{i-2j-k+1}}{R^{i+1}} \\ &\times i\delta m_{i+1} e^{i(n_{i+1}\alpha + \delta m_{i+1}\beta)} g^{(i)}(R,\theta), \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} H_1^{(i+2,j,k)} = &-\frac{1}{4\pi}\text{p.v.}\int_0^{2\pi} d\theta \int_0^{\infty} dR \frac{(\cos\theta)^{k} (\sin\theta)^{i-2j-k}}{R^{i+2}} \\ &\times i\delta m_{i+2} e^{i(n_{i+2}\alpha + \delta m_{i+2}\beta)} g^{(i+1)}(R,\theta). \end{aligned} \tag{B.12}$$

The right hand side of (B.10) is different from that of (B.1) only by the factor

$$(i\delta m_{i+1}) \exp\{i(n_{i+1}\alpha + \delta m_{i+1}\beta)\}.$$

Hence $H_\alpha^{(i+1,j,k)}$ is given by the right hand side of (B.4) with $g^{(i)}(R,\theta)$ replaced by

$$(i\delta m_{i+1}) \sum_{\nu=0}^{i} (-1)^\nu \sum_{\|\boldsymbol{q}\|=\nu} \exp\left\{iR[(\boldsymbol{q}\cdot\boldsymbol{n} + n_{i+1})\cos\theta + \delta(\boldsymbol{q}\cdot\boldsymbol{m} + m_{i+1})\sin\theta]\right\},$$

(cf. (B.7)), and we have the following expression:

$$\begin{aligned} H_\alpha^{(i,j,k)} = &-\frac{i m_{i+1}\delta}{4\pi i!} \sum_{\nu=0}^{i} (-1)^\nu \sum_{|\boldsymbol{q}|=\nu} \left(\frac{a'_q}{2}\right)^i \\ &\sum_{\mu=-(i-2j+1)}^{i-2j+1} c_\alpha(\mu) e^{i\mu\Pi'_q} \sum_{l=0}^{i} (-1)^l {}_iC_l \frac{1}{\mu - i + 2l}, \end{aligned} \tag{B.13}$$

where

$$a'_q \equiv [(\boldsymbol{q}\cdot\boldsymbol{n}+n_{i+1})^2+\delta^2(\boldsymbol{q}\cdot\boldsymbol{m}+m_{i+1})^2]^{\frac{1}{2}}$$

and Π'_q is defined by

$$\mathrm{e}^{\mathrm{i}\Pi'_q}=\frac{\delta(\boldsymbol{q}\cdot\boldsymbol{m}+m_{i+1})}{a'_q}+\mathrm{i}\frac{\boldsymbol{q}\cdot\boldsymbol{n}+n_{i+1}}{a'_q}.$$

The expression for $H_\beta^{(i,j,k)}$ is the same as (B.13) except the coefficient $c_\alpha(\mu)$. If we define

$$c_\beta(\mu)\equiv\int_0^{2\pi}(\cos\theta)^k(\sin\theta)^{i-2j-k+1}\mathrm{e}^{\mathrm{i}\mu\theta}\mathrm{d}\theta,$$

and replace $c_\alpha(\mu)$ in (B.13) by $c_\beta(\mu)$, then we obtain the expression for $H_\beta^{(i,j,k)}$.

In order to obtain the expression for $G_1^{(i+1,j,k)}$, we define

$$c_1(\mu)\equiv\int_0^{2\pi}(\cos\theta)^k(\sin\theta)^{i-2j-k}\mathrm{e}^{\mathrm{i}\mu\theta}\mathrm{d}\theta,$$

and express $g^{(i+1)}(R,\theta)$ as

$$g^{(i+1)}(R,\theta)=\sum_{\nu=0}^{i+1}(-1)^\nu\sum_{||\boldsymbol{q}||=\nu}\exp\left\{\mathrm{i}R[\boldsymbol{q}\cdot\boldsymbol{n}\cos\theta+\delta\boldsymbol{q}\cdot\boldsymbol{m}\sin\theta]\right\},$$

where the definition of the vectors $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{q}$ is the same as in (B.7) but the dimension of each vector is $i+1$. Then the expression for $G_1^{(i+1,j,k)}$ is obtained just by replacing $c_\alpha(\mu)$ and i in (B.8) by $c_1(\mu)$ and $i+1$, respectively. That is

$$\begin{aligned}G_1^{(i+1,j,k)}=&-\frac{1}{4\pi(i+1)!}\sum_{\nu=0}^{i+1}(-1)^\nu\sum_{||\boldsymbol{q}||=\nu}\left(\frac{a_q}{2}\right)^{i+1}\\&\sum_{\mu=-(i-2j)}^{i-2j}c_1(\mu)\mathrm{e}^{\mathrm{i}\mu\Pi_q}\sum_{l=0}^{i+1}(-1)^l{}_{i+1}C_l\frac{1}{\mu-(i+1)+2l},\end{aligned}\qquad\text{(B.14)}$$

where the definition of a_q and Π_q is the same as in (B.8) except the dimensions of $\boldsymbol{n}$ etc. In (B.14) we have reduced the upper and lower limit of sum with respect to μ because $c_1(\mu)$ is non-zero only if μ is every other integer from $-(i-2j)$ to $i-2j$.

The expression of $H_1^{(i+2,j,k)}$ is obtained by modifying (B.14). Regarding $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{q}$ as $(i+1)$-dimensional vectors, we have

$$\begin{aligned}H_1^{(i+2,j,k)}=&-\frac{\mathrm{i}m_{i+2}\delta}{4\pi(i+1)!}\sum_{\nu=0}^{i+1}(-1)^\nu\sum_{||\boldsymbol{q}||=\nu}\left(\frac{a''_q}{2}\right)^{i+1}\\&\sum_{\mu=-(i-2j)}^{i-2j}c_1(\mu)\mathrm{e}^{\mathrm{i}\mu\Pi''_q}\sum_{l=0}^{i+1}(-1)^l{}_{i+1}C_l\frac{1}{\mu-(i+1)+2l},\end{aligned}\qquad\text{(B.15)}$$

where

$$a''_q \equiv [(\boldsymbol{q}\cdot\boldsymbol{n}+n_{i+2})^2+\delta^2(\boldsymbol{q}\cdot\boldsymbol{m}+m_{i+2})^2]^{\frac{1}{2}},$$

and Π''_q is defined by

$$\mathrm{e}^{\mathrm{i}\Pi''_q} = \frac{\delta(\boldsymbol{q}\cdot\boldsymbol{m}+m_{i+2})}{a''_q} + \mathrm{i}\frac{\boldsymbol{q}\cdot\boldsymbol{n}+n_{i+2}}{a''_q}.$$

Here we have completed the evaluation of the integrals (A.11a)-(A.11d).

Using the expressions obtained so far, we have

$$\begin{aligned} G_1^{(1,0,0)} &= -\frac{1}{2}(n^2+\delta^2m^2)^{\frac{1}{2}}, \\ G_\alpha^{(1,0,1)} &= -\frac{1}{6}(2n^2+\delta^2m^2)(n^2+\delta^2m^2)^{-\frac{1}{2}}, \\ G_\alpha^{(1,0,0)} &= -\frac{1}{6}n(\delta m)(n^2+\delta^2m^2)^{-\frac{1}{2}}, \\ H_\alpha^{(1,0,0)} &= \frac{1}{2}n(\delta m)(n^2+\delta^2m^2)^{-\frac{1}{2}}, \end{aligned}$$

and

$$H_\beta^{(1,0,0)} = \frac{1}{2}\delta^2m^2(n^2+\delta^2m^2)^{-\frac{1}{2}}.$$

Substituting these into (A.14), we have

$$\boldsymbol{L}\boldsymbol{A}_{n,m} = \frac{1}{2}\begin{pmatrix} 0 & 0 & n^2(n^2+\delta^2m^2)^{-\frac{1}{2}} \\ 0 & 0 & \delta nm(n^2+\delta^2m^2)^{-\frac{1}{2}} \\ (n^2+\delta^2m^2)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}\begin{pmatrix} X_{n,m} \\ Y_{n,m} \\ Z_{n,m} \end{pmatrix}. \tag{B.16}$$

Similarly we can construct the infinite system (2.21).

Note: By using the algebraic manipulation language (Mathematica), it has been checked that the equations for $\boldsymbol{A}_{0,m}$ $(m=0,1,\cdots,10)$ on the basis of the above analysis reduce to the equations for $A[m]^{(0)}$ $(m=0,1,\cdots,10)$ in §5 at the leading order.

Appendix C *Construction of the truncated equation in §3*

The truncated equation numerically solved in §3 can, in principle, be obtained by taking account of

$$\boldsymbol{A}_{n,m} = O(\epsilon^{|n|+|m|}), \tag{C.1}$$

and discarding terms of order less than ϵ^M in (2.21). In practice, a certain systematic treatment is required for doing it by using a algebraic manipulation language. Here we give an outline of the construction of the truncated equations in §3.

Estimation (C.1) implies that terms $\{\boldsymbol{A}\}^K$ summed up in (2.24), i.e.

$$\sum_{\{n,m\}}^{(K)} \equiv \left(\begin{array}{l} \text{sum with respect to} \quad n_1, \ldots, n_K, m_1, \ldots, m_K \\ \text{satisfying} \\ n_1 + n_2 + \ldots + n_K = n \ , \ m_1 + m_2 + \ldots + m_K = m \end{array} \right),$$

are $O(\epsilon^L)$ where $L \equiv |n_1| + \ldots + |n_K| + |m_1| + \ldots + |m_K|$. Noting $\min(L) = |n| + |m|$, we divide (2.24) into the following sums

$$\sum_{\{n,m\}}^{(K)} = \sum_{\{n,m\}}^{(K,0)} + \sum_{\{n,m\}}^{(K,2)} + \ldots + \sum_{\{n,m\}}^{(K,2J)} + \ldots, \tag{C.2}$$

where J is a non-negative integer and

$$\sum_{\{n,m\}}^{(K,2J)} \equiv \left(\begin{array}{l} \text{sum with respect to} \quad n_1, \ldots, n_K, m_1, \ldots, m_K \\ \text{satisfying} \\ n_1 + n_2 + \ldots + n_K = n \ , \ m_1 + m_2 + \ldots + m_K = m, \\ |n_1| + \ldots + |n_K| + |m_1| + \ldots + |m_K| = |n| + |m| + 2J, \end{array} \right) \tag{C.3}$$

which collects all the terms of order $O(\epsilon^{|n|+|m|+2J})$ in $\{\boldsymbol{A}\}^K$. Here the values of J and K are so chosen that they satisfy $1 \leq |n| + |m| + 2J \leq M$ and $1 \leq K \leq |n| + |m| + 2J$.

In the following, we construct the terms of the order $O(\epsilon^{|n|+|m|+2J})$ in $\{\boldsymbol{A}\}^{(K)}$ for fixed values of J and K. First, we generate a list of all the combinations of (n_ν, m_ν)'s that satisfy the condition in (C.3) as

$$\{\{(p_1, q_1), (p_2, q_2), \ldots, (p_K, q_K)\}, \{(p_1', q_1'), (p_2', q_2'), \ldots, (p_K', q_K')\}, \ldots\}. \tag{C.4}$$

The number of combinations in the list depends on (n, m), K and J. For example, for $(n, m) = (1, 2), K = 2$ and $J = 1$, we have a list composed of five combinations:

$$\{\{(-1,0),(2,2)\},\{(-1,1),(2,1)\},\{(-1,2),(2,0)\},$$
$$\{(0,-1),(1,3)\},\{(0,3),(1,-1)\}\}.$$

Next, we construct the right hand side of (A.19a)-(A.19c) for each combination in (C.4). In order to illustrate the construction for a given combination, say, $\{ (p_1,q_1), (p_2,q_2), \ldots, (p_K,q_K) \}$, let us consider typical two types in the right hand side of (A.19a)-(A.19c):

$$\sum_{K-1}{}' C^{(K-1)} G_1^K U^{K-1} X, \tag{C.5}$$

$$\sum_{K-2}{}' C^{(K-2)} H_1^K U^{K-2} X Y^{\dagger}. \tag{C.6}$$

For (C.5), we can evaluate $C^{(K-1)}$ as a function of K, j, k, l, r and s, and G_1^K as a function of K, j, k and $\{(p_1,q_1),(p_2,q_2),\ldots,(p_K,q_K)\}$ (see (A.16), (A.3), (A.17) and (B.9)). For the term $U^{K-1}X$, we rearrange X, Y and Z as

$$X_{n_1,m_1}\cdots X_{n_\eta,m_\eta}\cdot Y_{n_{\eta+1},m_{\eta+1}}\cdots Y_{n_\mu,m_\mu}$$
$$\cdot Z_{n_{\mu+1},m_{\mu+1}}\cdots Z_{n_K,m_K}, \tag{C.7}$$

where $\eta = k+2s+1$ and $\mu = K-1-2r$, and generate all possible permutations of the elements of $\{(p_1,q_1),(p_2,q_2),\ldots,(p_K,q_K)\}$. For example, for a combination $\{(1,0),(1,0),(1,1)\}$, we generate $\{(1,0),(1,0),(1,1)\}$, $\{(1,0),(1,1),(1,0)\}$ and $\{(1,1),(1,0),(1,0)\}$. Then we replace $\{(n_1,m_1),(n_2,m_2),\ldots,(n_K,m_K)\}$ in (C.7) by the permutations in its order. By repeating the above process for all sets of (j,k,l,r,s), we complete the construction of (C.5) for the combination $\{(p_1,q_1),(p_2,q_2),\ldots,(p_K,q_K)\}$.

For (C.6), the value of $C^{(K-2)}$ can be obtained from (A.16) and (A.3). The main difference from (C.5) arise from the term attached the dagger. The term $U^{K-2}XY^{\dagger}$ in (C.6) is represented as

$$X_{n_1,m_1}\cdots X_{n_\eta,m_\eta}\cdot Y_{n_{\eta+1},m_{\eta+1}}\cdots Y_{n_\mu,m_\mu}$$
$$\cdot Z_{n_{\mu+1},m_{\mu+1}}\cdots Z_{n_{K-1},m_{K-1}} Y_{n_K,m_K}, \tag{C.8}$$

where $\eta = k+2s+1$ and $\mu = K-2r-1$. As can be seen from (A.17) and (B.12), the integrals $H^{(K)}_{\{1,\alpha,\beta\}}$ depend on not only (K, j, k) and $A \equiv \{(p_1, q_1), (p_2, q_2), \ldots, (p_K, q_K)\}$ but also the pair substituted into (n_K, m_K), in contrast to the integrals $G^{(K)}_{1,\alpha}$. For the combination A, we therefore consider a set $B \equiv \{(\bar{p}_1, \bar{q}_1), (\bar{p}_2, \bar{q}_2), \ldots, (\bar{p}_\kappa, \bar{q}_\kappa)\}$ which satisfies the following conditions: (1) $(\bar{p}_\mu, \bar{q}_\mu) \neq (\bar{p}_\nu, \bar{q}_\nu)$ when $\mu \neq \nu$ and (2) if $(p_\mu, q_\mu) \in A$ then $(p_\mu, q_\mu) \in B$. We call B the union of the elements of A. For example, the union of the elements of $\{(1,0), (1,0), (1,1)\}$ is $\{(1,0), (1,1)\}$. For each element of B, we substitute it into (n_K, m_K) and evaluate $H^{(K)}_1$. At the same time, we exclude the element from A and generate all possible permutations of the remainder of A. By replacing $\{(n_1, m_1), (n_2, m_2), \ldots, (n_{K-1}, m_{K-1})\}$ in (C.8) by the permutations in its order, we complete the construction of (C.6) for the combination A.

The other terms on the right hand side of (A.19a)-(A.19c) are treated similarly. Thus we complete the summation of the terms of order $O(\epsilon^{|n|+|m|+2J})$ and of degree K in $\boldsymbol{A}_{p,q}$ on the right hand side of the equation for $\boldsymbol{A}_{n,m}$.