

can be represented as a superposition of parabolic and elliptic distributions, which are the distributions which give the simplest explicit formulae. The method has been applied to a practical propeller case with a finite hub, and it was found that in the hub region there is slipstream expansion followed by contraction to the hub radius far downstream. The method is suitable for calculation of the slipstream effect on complex configurations by embedding it in a suitable boundary integral method, and can be extended into the compressible flow regime using compressibility corrections.

The method applied here to the actuator disk will also solve all the analogous electromagnetic problems associated with both semi-infinite and finite solenoids and radial distributions of solenoids. The magnetic fields induced by a semi-infinite solenoid distribution are exactly analogous to the slipstream solutions presented here, and the fields induced by a radial distribution of solenoids of finite length is obtained by superposition of two semi-infinite distributions of opposite sign and axial relative displacement equal to the solenoid length.

The author would like to thank Dr. Christopher Grigson and Dr. Lars Soland for reading the manuscript and for their encouragement during the course of the work. Thanks are also due to the referees for their helpful suggestions.

Appendix A. Evaluation Of Bessel-Laplace Integrals

The solution of the actuator disk problem in most cases reduces to evaluation of integrals of the form

$$I_{(\lambda, \mu, \nu)} = \int_0^{\infty} e^{-s|z|} s^{\lambda} J_{\mu}(sR_a) J_{\nu}(sr) ds. \quad (\text{A } 1)$$

where λ, μ and ν are integers. The simplest case is when $\lambda = 0$ and $\mu = \nu$. For this case, the integral reduces to the basic formula below (Watson 1944)

$$I_{(0, \nu, \nu)} = \frac{1}{\pi \sqrt{R_a r}} Q_{\nu - \frac{1}{2}}(\omega), \quad (\text{A } 2)$$

$$\text{where } \omega \equiv \frac{R_a^2 + r^2 + z^2}{2rR_a}.$$

This has the particularly simple integral representation

$$I_{(0, \nu, \nu)} = \frac{1}{\pi} \int_0^{\pi} \frac{\cos \nu t dt}{\sqrt{r^2 + R_a^2 + z^2 - 2rR_a \cos t}}. \quad (\text{A } 3)$$

Legendre functions of half-integral order such as the above can always be expressed in terms of complete elliptic integrals $\mathbf{E}(k)$ and $\mathbf{K}(k)$ where

$$k \equiv \sqrt{\frac{4rR_a}{(R_a + r)^2 + z^2}} \text{ and } \omega \equiv \frac{2 - k^2}{k^2}.$$

Abramowitz and Stegun (1972) give $Q_{-\frac{1}{2}}(\omega)$ and $Q_{\frac{1}{2}}(\omega)$ as in terms of complete elliptic integrals as:

$$Q_{-\frac{1}{2}}(\omega) = k\mathbf{K}(k) \quad (\text{A } 4)$$

and

$$Q_{\frac{1}{2}}(\omega) = \frac{(2 - k^2)}{k} \mathbf{K}(k) - \frac{2}{k} \mathbf{E}(k). \quad (\text{A } 5)$$

Elliptic integral representations for all other Legendre functions of half-integral order can be obtained from those above using the recursion relation of Hobson (1896)

$$Q_{\nu+1}(x) = \frac{(2\nu+1)}{\nu+1} x Q_{\nu}(x) - \frac{\nu}{\nu+1} Q_{\nu-1}(x). \quad (\text{A } 6)$$

This gives the recurrence below for Bessel-Laplace integrals

$$I_{(0,\nu,\nu)} = \frac{4(\nu-1)\omega}{(2\nu-1)} I_{(0,\nu-1,\nu-1)} - \frac{(2\nu-3)}{(2\nu-1)} I_{(0,\nu-2,\nu-2)}. \quad (\text{A } 7)$$

Hobson's recurrence relation is unstable for upward recursion for all $\omega > 1$, so in numerical work downward recursion using Miller's algorithm must be used (Abramowitz and Stegun 1972, Press et al 1992). The recursion sequence can be normalized using the elliptic integral formula for $I_{(0,0,0)}$.

For λ a positive integer, then $I_{(\lambda,\nu,\nu)}$ can be evaluated using the formula below given by Prudnikov et al (1992)

$$I_{(\lambda,\nu,\nu)} = \frac{(-1)^\lambda}{\pi\sqrt{R_a r}} \frac{d^\lambda}{dz^\lambda} Q_{\nu-\frac{1}{2}}(\omega). \quad (\text{A } 8)$$

This can always be expressed in terms of the Associated Legendre functions through the relation below (Gradshteyn and Ryzhik 1980)

$$Q_{\nu-\frac{1}{2}}^n(x) = (-1)^n (1-x^2)^{\frac{n}{2}} \frac{d^n}{dx^n} Q_{\nu-\frac{1}{2}}(x). \quad (\text{A } 9)$$

For the particular case of $\lambda = 1$, then this reduces to

$$I_{(1,\nu,\nu)} = -\frac{|z| k^2 (R_a r)^{-\frac{3}{2}}}{2\pi\sqrt{1-k^2}} Q_{\nu-\frac{1}{2}}^1(\omega). \quad (\text{A } 10)$$

The Associated Legendre functions above can always be reduced to Legendre functions and hence expressed as elliptic integrals through the recursion relation below (Abramowitz and Stegun 1972)

$$Q_{\nu}^{\mu+1}(x) = (x^2-1)^{-\frac{1}{2}} \{(\nu-\mu)xQ_{\nu}^{\mu}(x) - (\nu+\mu)Q_{\nu-1}^{\mu}(x)\}. \quad (\text{A } 11)$$

Hence all Bessel-Laplace integrals of the form $I_{(\lambda,\nu,\nu)}$ for λ a positive integer or zero can be expressed in terms of elliptic integrals, although the formulae may be relatively complex. Equations A 10 and A 11 give the recurrence relation:

$$I_{(1,\nu,\nu)} = \frac{(2\nu-1)|z|k^4}{8R_a r(1-k^2)} (I_{(0,\nu-1,\nu-1)} - \omega I_{(0,\nu,\nu)}). \quad (\text{A } 12)$$

For $\lambda = 0$ and $\mu \neq \nu$, formulae in terms of elliptic integrals can be derived from the formulae above using the standard recursion relations for Bessel functions if a formula can be obtained for the fundamental Bessel-Laplace integral below, which occurs naturally for the constantly loaded actuator disk

$$I_{(0,1,0)} \equiv \int_0^{\infty} e^{-s|z|} J_1(sR_a) J_0(sr) ds. \quad (\text{A } 13)$$

The formula adopted for this integral will then determine all other formulae derived from it by recursion. There are several formulae available, but the one chosen for the current study is that using Heuman's Lambda function given (with a typographical error) by

Prudnikov et al (1992). The corrected expressions are:

$$I_{(0,1,0)} = \frac{1}{R_a} \left(1 - \frac{|z|k}{2\pi\sqrt{rR_a}} \mathbf{K}(k) - \frac{\Lambda_0(\beta, k)}{2} \right) \quad [r < R_a] \quad (\text{A } 14)$$

$$I_{(0,1,0)} = \frac{1}{R_a} \left(-\frac{|z|k}{2\pi\sqrt{rR_a}} \mathbf{K}(k) + \frac{\Lambda_0(\beta, k)}{2} \right) \quad [r > R_a] \quad (\text{A } 15)$$

$$\text{where } \beta \equiv \arcsin \left(\frac{|z|}{\sqrt{(r - R_a)^2 + z^2}} \right). \quad (\text{A } 16)$$

An alternative expression for $I_{(0,1,0)}$ in terms of the Lambda function is given by Byrd and Friedman (1971). It is extremely useful to have an integral representation of $I_{(0,1,0)}$. From equation A 13 then clearly:

$$\frac{\partial I_{(0,1,0)}}{\partial r} = \pm \frac{\partial I_{(0,1,1)}}{\partial z}. \quad (\text{A } 17)$$

Substituting the integral representation for $I_{(0,1,1)}$ into equation A 17 and integrating both sides with respect to r gives finally

$$I_{(0,1,0)} = \frac{|z|}{\pi} \int_0^\pi \frac{\cos t (R_a \cos t - r) dt}{(R_a^2 \sin^2 t + z^2) \sqrt{r^2 + R_a^2 + z^2 - 2rR_a \cos t}}. \quad (\text{A } 18)$$

Exchange of r and R_a gives

$$I_{(0,0,1)} = \frac{|z|}{\pi} \int_0^\pi \frac{\cos t (r \cos t - R_a) dt}{(r^2 \sin^2 t + z^2) \sqrt{r^2 + R_a^2 + z^2 - 2rR_a \cos t}}. \quad (\text{A } 19)$$

Equation A 19 is a key element in the solution for arbitrary polynomial radial distribution of load. Applying the method of partial fractions to equation A 18, $I_{(0,1,0)}$ can be expressed in terms of Legendre complete elliptic integrals of the third kind.

The fundamental recurrence relations for Bessel functions (Watson 1944) are

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) \quad (\text{A } 20)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x). \quad (\text{A } 21)$$

These relations can be used to obtain a number of recurrence relations for Bessel-Laplace integrals. From equation A 20 it is simple to show that

$$\begin{aligned} J_{\nu+1}(sR_a)J_\nu(sr) &\equiv \frac{r}{R_a} J_\nu(sR_a)J_{\nu-1}(sr) \\ &+ \frac{sr}{2\nu} [J_{\nu+1}(sR_a)J_{\nu+1}(sr) - J_{\nu-1}(sR_a)J_{\nu-1}(sr)]. \end{aligned} \quad (\text{A } 22)$$

Substituting this into the Bessel-Laplace integral $I_{(0,\nu+1,\nu)}$ gives the recursion relation below, which is valid for $\nu \neq 0$

$$I_{(0,\nu+1,\nu)} = \frac{r}{R_a} I_{(0,\nu,\nu-1)} + \frac{r}{2\nu} [I_{(1,\nu+1,\nu+1)} - I_{(1,\nu-1,\nu-1)}]. \quad (\text{A } 23)$$

For numerical work this relation has been found to be stable for downward recurrence. Equation A 20 also yields the recurrence below which is valid for $\mu \neq 1$ and $\nu \neq 0$

$$I_{(0,\mu,\nu)} = -I_{(0,\mu-2,\nu)} + \frac{(\mu-1)r}{\nu R_a} (I_{(0,\mu-1,\nu+1)} + I_{(0,\mu-1,\nu-1)}). \quad (\text{A } 24)$$

An independent recurrence relation can be obtained using equation A 21 and integration by parts:

$$I_{(0,\mu,\nu)} = I_{(0,\mu-2,\nu)} - \frac{2|z|}{R_a} I_{(0,\mu-1,\nu)} + \frac{r}{R_a} (I_{(0,\mu-1,\nu-1)} - I_{(0,\mu-1,\nu+1)}). \quad (\text{A } 25)$$

Combining equations A 24 and A 25 give the recurrence

$$I_{(0,\mu,\nu)} = \frac{2(\nu+1)|z|}{r(\nu+1-\mu)} I_{(0,\mu,\nu+1)} - \frac{2(\nu+1)R_a}{r(\nu+1-\mu)} I_{(0,\mu-1,\nu+1)} + \frac{(\nu+1+\mu)}{(\nu+1-\mu)} I_{(0,\mu,\nu+1)}. \quad (\text{A } 26)$$

For λ a negative integer, then for all cases where the integral exists, it can be evaluated using the recurrence relation below which follows immediately from equation A 20

$$I_{(\lambda,\mu,\nu)} = \frac{R_a}{2\mu} (I_{(\lambda+1,\mu+1,\nu)} + I_{(\lambda+1,\mu-1,\nu)}). \quad (\text{A } 27)$$

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