

Yarin: Surface-tension-driven flow at low Reynolds number  
FM 9101

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$$\sigma_{r\varphi 1} = \sigma_{r\varphi 2} \quad \text{at } r = R_1; \quad (2.12c-g)$$

$$\Gamma_2: \quad v_{r2} = \frac{\partial \zeta_2^*}{\partial t}, \quad \sigma_{rr 2} = -\alpha_2, \quad \sigma_{r\varphi 2} = 0 \quad \text{at } r = R_2; \quad (2.12h-j)$$

$$q_{\alpha i} = \frac{\alpha_i}{R_i} (1 - \zeta_i) - \frac{\alpha_i}{R_i} \frac{\partial^2 \zeta_i}{\partial \varphi^2}, \quad i = 1, 2 \quad (2.12k)$$

where  $\alpha_1$  and  $\alpha_2$  are the interfacial tension at the boundary  $\Gamma_1$  and the surface tension at the boundary  $\Gamma_2$ , respectively.

We represent the perturbations of the boundaries in the form of Fourier series

$$\zeta_1 = \frac{b_{01}(t)}{2} + \sum_{n=1}^{\infty} [a_{n1}(t) \sin n\varphi + b_{n1}(t) \cos n\varphi], \quad (2.13a)$$

$$\zeta_2 = \frac{b_{02}(t)}{2} + \sum_{n=1}^{\infty} [a_{n2}(t) \sin n\varphi + b_{n2}(t) \cos n\varphi]. \quad (2.13b)$$

From (2.12a), by using (2.9a) we arrive at

$$A_n R_0^{n+1} + A_{2n} R_0^{n-1} + A_{3n} F_0^{-n-1} + A_{4n} f_1(R_0, n) = 0, \quad (2.14a)$$

$$f_1(R_0, 1) = \ln R_0, \quad f_1(R_0, n) = R_0^{-n+1}, \quad n \geq 2. \quad (2.14b,c)$$

Here and hereinafter we use the notation

$$A_{2n} = A_n C_{2n1}, \quad A_{3n} = A_n C_{3n1}, \quad A_{4n} = A_n C_{4n1}, \quad (2.15a-c)$$

$$D_{2n} = D_n C_{2n2}, \quad D_{3n} = D_n C_{3n2}, \quad D_{4n} = D_n C_{4n2}. \quad (2.15d-f)$$

From (2.12b), by using (2.9b) we arrive at

$$(n+2)A_n R_0^{n+1} + nA_{2n} R_0^{n-1} - nA_{3n} R_0^{-n-1} + A_{4n} f_2(R_0, n) = 0, \quad (2.16a)$$

$$f_2(R_0, 1) = \ln R_0 + 1, \quad f_2(R_0, n) = (-n+2)R_0^{-n+1}, \quad n \geq 2. \quad (2.16b,c)$$

In addition, here and hereinafter there are the relations obtainable from (2.14a), (2.15) and (2.16a) by replacing  $b_{n1}$ ,  $b_{n2}$ ,  $A_n$ ,  $A_{2n}$ ,  $A_{3n}$ ,  $A_{4n}$ ,  $D_n$ ,  $D_{2n}$ ,  $D_{3n}$ , and  $D_{4n}$  with  $a_{n1}$ ,  $a_{n2}$ ,  $B_n$ ,  $B_{2n}$ ,  $B_{3n}$ ,  $B_{4n}$ ,  $E_n$ ,  $E_{2n}$ ,  $E_{3n}$ , and  $E_{4n}$ , respectively; henceforth we refer to these as the complementary relations.

Using (2.9c), (2.12c,h,d) and (2.13) we obtain for  $n = 0$

$$b_{01} = b_{010}, \quad b_{02} = b_{020} \quad (2.17a,b)$$

(here and hereinafter the additional subscript 0 stands for  $t = 0$ ), whereas for  $n \geq 1$ :

$$\frac{1}{nR_1^n} \frac{db_{n1}}{dt} = A_n + A_{2n}R_1^{-2} + A_{3n}R_1^{-2n-2} + A_{4n}f_3(R_1, n), \quad (2.18a)$$

$$\frac{1}{nR_2^n} \frac{db_{n2}}{dt} = D_n + D_{2n}R_2^{-2} + D_{3n}R_2^{-2n-2} + D_{4n}f_3(R_2, n), \quad (2.18b)$$

$$\frac{1}{nR_1^n} \frac{db_{n1}}{dt} = D_n + D_{2n}R_1^{-2} + D_{3n}R_1^{-2n-2} + D_{4n}f_3(R_1, n), \quad (2.18c)$$

$$f_3(R_k, 1) = R_k^{-2} \ln R_k, \quad f_3(R_k, n) = R_k^{-2n}, \quad n \geq 2. \quad (2.18d,e)$$

and the corresponding complementary relations.

For  $n \geq 1$  condition (2.12e) yields

$$\begin{aligned} & A_n(n+2) + nA_{2n}R_1^{-2} - nA_{3n}R_1^{-2n-2} + A_{4n}f_4(R_1, n) \\ & = D_n(n+2) + nD_{2n}R_1^{-2} - nD_{3n}R_1^{-2n-2} + D_{4n}f_4(R_1, n), \end{aligned} \quad (2.19a)$$

$$f_4(R_1, 1) = R_1^{-2}(\ln R_1 + 1), \quad f_4(R_1, n) = R_1^{-2n}(-n+2), \quad n \geq 2 \quad (2.19b,c)$$

and the corresponding complementary relation.

With the aid of (2.12i,k), (2.11a) for  $\sigma_{rr2}$  and (2.13b) we find that  $K_2 = (\alpha_2/R_2)(1 - b_{020}/2)$  and for  $n \geq 1$  we have

$$\begin{aligned} & \mu_2[D_nR_2^n(-2n-4+2n^2) + D_{2n}R_2^{n-2}(2n^2-2n) \\ & + D_{3n}R_2^{-n-2}(-2n^2-2n) + D_{4n}f_5(R_2, n)] = \frac{\alpha_2}{R_2}(1-n^2)b_{n2}, \end{aligned} \quad (2.20a)$$

$$f_5(R_k, 1) = 4R_k^{-1}, \quad f_5(R_k, n) = R_k^{-n}(4-2n-2n^2), \quad n \geq 2 \quad (2.20b)$$

with the corresponding complementary relation.

Similarly, using (2.12f,k) we find that  $K_1 = (\alpha_1/R_1)(1 - b_{010}/2) + (\alpha_2/R_2)(1 - b_{020}/2)$ , and for  $n \geq 1$

$$\begin{aligned}
& \mu_1[A_n R_1^n(-2n - 4 + 2n^2) + A_{2n} R_1^{n-2}(2n^2 - 2n) \\
& + A_{3n} R_1^{-n-2}(-2n^2 - 2n) + A_{4n} f_5(R_1, n)] \\
& = \mu_2[D_n R_1^n(-2n - 4 + 2n^2) + D_{2n} R_1^{n-2}(2n^2 - 2n) \\
& + D_{3n} R_1^{-n-2}(-2n^2 - 2n) + D_{4n} f_5(R_1, n)] + \frac{\alpha_1}{R_1}(1 - n^2)b_{n1}
\end{aligned} \tag{2.21}$$

with the corresponding complementary relation.

For  $n \geq 1$  condition (2.12j) yields

$$\begin{aligned}
& D_n R_2^n(-2n^2 - 2n) + D_{2n} R_2^{n-2}(-2n^2 + 2n) + D_{3n} R_2^{-n-2}(-2n^2 - 2n) \\
& + D_{4n} R_2^{-n}(-2n^2 + 2n) = 0
\end{aligned} \tag{2.22}$$

with the corresponding complementary relation.

Condition (2.12g) yields for  $n \geq 1$

$$\begin{aligned}
& \mu_1[A_n R_1^n(-2n^2 - 2n) + A_{2n} R_1^{n-2}(-2n^2 + 2n) \\
& + A_{3n} R_1^{-n-2}(-2n^2 - 2n) + A_{4n} R_1^{-n}(-2n^2 + 2n)] \\
& = \mu_2[D_n R_1^n(-2n^2 - 2n) + D_{2n} R_1^{n-2}(-2n^2 + 2n) \\
& + D_{3n} R_1^{-n-2}(-2n^2 - 2n) + D_{4n} R_1^{-n}(-2n^2 + 2n)]
\end{aligned} \tag{2.23}$$

together with the complementary relation.

Relations (2.14a), (2.16a), (2.18a-c), (2.19a), (2.20a), (2.21) - (2.23) form a system of ten equations in ten unknowns:  $b_{n1}$ ,  $b_{n2}$ ,  $A_n$ ,  $A_{2n}$ ,  $A_{3n}$ ,  $A_{4n}$ ,  $D_n$ ,  $D_{2n}$ ,  $D_{3n}$  and  $D_{4n}$ . When  $n = 1$  this system yields

$$b_{11} = b_{110}, \quad b_{12} = b_{120} \tag{2.24a,b}$$

and when  $n \geq 2$  by successive elimination of  $A_{3n}$ ,  $A_{4n}$ , then of  $D_n$ ,  $D_{2n}$ ,  $D_{3n}$ , and  $D_{4n}$ , and finally of  $A_n$  and  $A_{2n}$ , we reduce the system to two differential equations for determining the coefficients  $b_{n1}$  and  $b_{n2}$ :

$$k_1 \frac{db_{n1}}{dt} + k_2 \frac{db_{n2}}{dt} + k_3 b_{n2} + k_4 b_{n1} = 0, \tag{2.25a}$$

$$k_5 \frac{db_{n1}}{dt} + k_6 \frac{db_{n2}}{dt} + k_7 b_{n2} + k_8 b_{n1} = 0. \tag{2.25b}$$

Here we adopt the notation given in Appendix A.

$$\bar{R}_0 = \frac{\gamma_1}{2} \left[ \frac{1 - \gamma_1^{-2}}{1 - \gamma_1^{-1} + \bar{t}/2} - (1 - \gamma_1^{-1} + \bar{t}/2) \right], \quad (3.10a)$$

$$\bar{R}_2 = \frac{1}{2} \left[ \frac{1 - \gamma_1^{-2}}{1 - \gamma_1^{-1} + \bar{t}/2} + (1 - \gamma_1^{-1} + \bar{t}/2) \right] \quad (3.10b)$$

which shows that the tube will collapse completely after a time

$$\bar{t}_* = -2(1 - \gamma_1^{-1}) + 2\sqrt{1 - \gamma_1^{-2}}. \quad (3.11)$$

In another particular case when  $\bar{\alpha}_0 = 1$ ,  $\bar{\alpha}_1 = 0$ , and  $\bar{\mu}_1$  is arbitrary, equation (3.8a) is identical with (6) of Lewis (1977) with the pressure differential equal to zero.

For all three interfaces, we consider perturbations in the form of the Fourier series (cf. (2.13)):

$$\zeta_i = \frac{b_{0i}(t)}{2} + \sum_{n=1}^{\infty} [a_{ni}(t) \sin n\varphi + b_{ni}(t) \cos n\varphi], \quad i = 0, 1, 2 \quad (3.12)$$

and use the notation (2.15).

By using (3.2), (3.3) and (3.12) we satisfy the boundary conditions (3.5) - (3.7). As a result, we obtain for  $n \geq 1$  the following set of algebraic equations:

$$A_n R_0^{n+1} + A_{2n} R_0^{n-1} + A_{3n} R_0^{-n-1} + A_{4n} \begin{cases} \ln R_0, & n = 1 \\ R_0^{-n+1}, & n \geq 2 \end{cases} = \frac{1}{n} \left( R_0 \frac{db_{n0}}{dt} + 2 \frac{dR_0}{dt} b_{n0} \right), \quad (3.13a)$$

$$\begin{aligned} \mu_1 \left[ A_n R_0^n (-2n - 4 + 2n^2) + A_{2n} R_0^{n-2} (2n^2 - 2n) + A_{3n} R_0^{-n-2} (-2n^2 - 2n) \right. \\ \left. + A_{4n} \begin{cases} 4/R_0, & n = 1 \\ R_0^{-n} (-2n + 4 - 2n^2), & n \geq 2 \end{cases} \right] \\ = -b_{n0} \left[ (1 - n^2) \frac{\alpha_0}{R_0} + \frac{4\mu_1}{R_0} \frac{dR_0}{dt} \right], \end{aligned} \quad (3.13b)$$

$$A_n R_0^n (-2n^2 - 2n) + A_{2n} R_0^{n-2} (-2n^2 + 2n) + A_{3n} R_0^{-n-2} (-2n^2 - 2n) + A_{4n} R_0^{-n} (-2n^2 + 2n) = 0, \quad (3.13c)$$

$$D_n + D_{2n} R_2^{-2} + D_{3n} R_2^{-2n-2} + D_{4n} \begin{cases} R_2^{-2} \ln R_2, & n = 1 \\ R_2^{-2n}, & n \geq 2 \end{cases} \\ = \frac{1}{n} \left( R_2^{-n} \frac{db_{n2}}{dt} + 2R_2^{-n-1} \frac{dR_2}{dt} b_{n2} \right), \quad (3.13d)$$

$$D_n + D_{2n} R_1^{-2} + D_{3n} R_1^{-2n-2} + D_{4n} \begin{cases} R_1^{-2} \ln R_1, & n = 1 \\ R_1^{-2n}, & n \geq 2 \end{cases} \\ = \frac{1}{n} \left( R_1^{-n} \frac{db_{n1}}{dt} + 2R_1^{-n-1} \frac{dR_1}{dt} b_{n1} \right), \quad (3.13e)$$

$$\mu_2 \left[ D_n R_2^n (-2n - 4 + 2n^2) + D_{2n} R_2^{n-2} (2n^2 - 2n) + D_{3n} R_2^{-n-2} (-2n^2 - 2n) \right. \\ \left. + D_{4n} \begin{cases} 4/R_2, & n = 1 \\ R_2^{-n} (-2n + 4 - 2n^2), & n \geq 2 \end{cases} \right] \\ = b_{n2} \left[ (1-n^2) \frac{\alpha_2}{R_2} - \frac{4\mu_2}{R_2} \frac{dR_2}{dt} \right], \quad (3.13f)$$

$$\mu_2 \left[ D_n R_1^n (-2n - 4 + 2n^2) + D_{2n} R_1^{n-2} (2n^2 - 2n) + D_{3n} R_1^{-n-2} (-2n^2 - 2n) \right. \\ \left. + D_{4n} \begin{cases} 4/R_1, & n = 1 \\ R_1^{-n} (-2n + 4 - 2n^2), & n \geq 2 \end{cases} \right] + b_{n1} \left[ (1-n^2) \frac{\alpha_1}{R_1} + (\mu_2 - \mu_1) \frac{4}{R_1} \frac{dR_1}{dt} \right] \\ = \mu_1 \left[ A_n R_1^n (-2n - 4 + 2n^2) + A_{2n} R_1^{n-2} (2n^2 - 2n) + A_{3n} R_1^{-n-2} (-2n^2 - 2n) \right. \\ \left. + A_{4n} \begin{cases} 4/R_1, & n = 1 \\ R_1^{-n} (-2n + 4 - 2n^2), & n \geq 2 \end{cases} \right], \quad (3.13g)$$

$$D_n R_2^n (-2n^2 - 2n) + D_{2n} R_2^{n-2} (-2n^2 + 2n) + D_{3n} R_2^{-n-2} (-2n^2 - 2n) \\ + D_{4n} R_2^{-n} (-2n^2 + 2n) = 0, \quad (3.13h)$$

$$\begin{aligned} \mu_2[D_n R_1^n(-2n - 2n^2) + D_{2n} R_1^{n-2}(-2n^2 + 2n) + D_{3n} R_1^{-n-2}(-2n^2 - 2n) \\ + D_{4n} R_1^{-n}(-2n^2 + 2n)] = \mu_1[A_n R_1^n(-2n - 2n^2) + A_{2n} R_1^{n-2}(-2n^2 + 2n) \\ + A_{3n} R_1^{-n-2}(-2n^2 - 2n) + A_{4n} R_1^{-n}(-2n^2 + 2n)], \end{aligned} \quad (3.13i)$$

$$\begin{aligned} A_n + A_{2n} R_1^{-2} + A_{3n} R_1^{-2n-2} + A_{4n} \begin{cases} R_1^{-2} \ln R_1, & n = 1 \\ R_1^{-2n}, & n \geq 2 \end{cases} \\ = \frac{1}{n} \left( R_1^{-n} \frac{db_{n1}}{dt} + 2R_1^{-n-1} \frac{dR_1}{dt} b_{n1} \right), \end{aligned} \quad (3.13j)$$

$$\begin{aligned} A_n(n+2) + nA_{2n} R_1^{-2} - nA_{3n} R_1^{-2n-2} + A_{4n} \begin{cases} R_1^{-2}(\ln R_1 + 1), & n = 1 \\ (-n+2)R_1^{-2n}, & n \geq 2 \end{cases} \\ = D_n(n+2) + nD_{2n} R_1^{-2} - nD_{3n} R_1^{-2n-2} \\ + D_{4n} \begin{cases} R_1^{-2}(\ln R_1 + 1), & n = 1 \\ (-n+2)R_1^{-2n}, & n \geq 2 \end{cases}. \end{aligned} \quad (3.13k)$$

We consider separately the equations for the  $n = 0$  mode of interface perturbations. After rearrangement, these reduce to the following differential equations:

$$\frac{db_{00}}{dt} = -\frac{2}{\bar{R}_0} \frac{d\bar{R}_0}{dt} b_{00} + \frac{\gamma_1^2}{\bar{R}_0^2} F_1, \quad (3.14a)$$

$$\frac{db_{01}}{dt} = -\frac{2}{\bar{R}_1} \frac{d\bar{R}_1}{dt} b_{01} + \frac{\gamma_*^2}{\bar{R}_1^2} F_1, \quad (3.14b)$$

$$\frac{db_{02}}{dt} = -\frac{2}{\bar{R}_2} \frac{d\bar{R}_2}{dt} b_{02} + \frac{1}{\bar{R}_2^2} F_1, \quad (3.14c)$$

$$\begin{aligned} F_1 = -\frac{1}{2} \frac{1}{\bar{\mu}_1 \gamma_*^2 \bar{R}_1^{-2} - \bar{\mu}_1 \gamma_1^2 \bar{R}_0^{-2} - \gamma_*^2 \bar{R}_1^{-2} + \bar{R}_2^{-2}} \left[ b_{00} \left( \frac{\bar{\alpha}_0 \gamma_1}{\bar{R}_0} + \frac{4\bar{\mu}_1}{\bar{R}_0} \frac{d\bar{R}_0}{dt} \right) \right. \\ \left. + b_{01} \left( \frac{\bar{\alpha}_1 \gamma_*}{\bar{R}_1} + (1 - \bar{\mu}_1) \frac{4}{\bar{R}_1} \frac{d\bar{R}_1}{dt} \right) + b_{02} \left( \frac{1}{\bar{R}_2} - \frac{4}{\bar{R}_2} \frac{d\bar{R}_2}{dt} \right) \right]. \end{aligned} \quad (3.14d)$$

For  $n = 1$ , (3.13) reduce, after some rearrangements, to the following equations