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To be held in editorial files

Supplement : Time dependent beat coefficients

To simplify their description, the integrals for the time-dependent-beat coefficients, $\alpha_1 \ldots \alpha_9$, are defined initially. The coefficients are then defined separately for the effective and the recovery strokes.

S.1 Integrals

The integrals are given in a general form and then some integrals which need to be repeatedly evaluated are given separately from the main definition of the coefficients.

S.1.1 Definition of the integrals

The total force and torque at any time are derived by integrating the equation for (a) the viscous force produced by an element of a flagellum, Equation (2.4), and (b) the cross product of the position vector of the element and the viscous force, respectively, along the lengths of both flagella. Appropriate substitutions need to be made for the position vector of the element, $\mathbf{r}$, the relative velocity of the element, $\mathbf{u}_{\mathrm{rel}}$, and the normal and tangential vectors, $\mathbf{n}$ and $\mathbf{t}$ respectively, at any time (see Equations (5.5) - (5.12)) for either of the flagella. These are described in detail in the main paper but Figure 4 are reproduced with this document to aid in the description of these vectors during the two strokes.

The integrals are evaluated in terms of the normal and tangent unit vectors, $\mathbf{n}$ and $\mathbf{t}$ and then combined in terms of the body axes, $\mathbf{p}$ and $\mathbf{q}$.

1 . The α_1 and α_4 coefficients are the coefficients of the $\mathbf{p}$ and $\mathbf{q}$ components, respectively, of the contribution of the velocity of the organism to the force balance equation. They are derived from the following integral :

$$\alpha_1 \mathrm{v_p}\,\mathbf{p} + \alpha_4 \mathrm{v_q}\,\mathbf{q} = \mu \int \mathrm{K_N}[-(\mathbf{v}.\mathbf{n})(1 - \frac{3\mathrm{a}}{4r} - \frac{\mathrm{a}^3}{4r^3}) + \frac{(\mathbf{v}.\mathbf{r})(\mathbf{r}.\mathbf{n})}{r^2}\frac{3\mathrm{a}}{4r}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{n}\,\mathrm{ds}$$
$$+\mu \int \mathrm{K_T}[-(\mathbf{v}.\mathbf{t})(1 - \frac{3\mathrm{a}}{4r} - \frac{\mathrm{a}^3}{4r^3}) + \frac{(\mathbf{v}.\mathbf{r})(\mathbf{r}.\mathbf{t})}{r^2}\frac{3\mathrm{a}}{4r}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{t}\,\mathrm{ds} \;\; - 6\pi\mu\mathrm{a}\mathbf{v}. \quad (S.1)$$

Note that the viscous force on the body has been included in these coefficients.

2. The α_5 coefficient is the coefficient of the contribution of the relative angular velocity of the body to the force balance equation. It is derived from the following integral :

$$\alpha_5(\dot\theta + \omega/2)\,\mathbf{q} = \mu \int \mathrm{K_N}[\{(\frac{\omega}{2} - \mathbf{\Omega}) \times \mathbf{r}\}.\mathbf{n}(1 - \frac{\mathrm{a}^3}{r^3})]\,\mathbf{n}\,\mathrm{ds}$$
$$+\mu \int \mathrm{K_T}[\{(\frac{\omega}{2} - \mathbf{\Omega}) \times \mathbf{r}\}.\mathbf{t}(1 - \frac{a^3}{r^3})]\,\mathbf{t}\,\mathrm{ds}\,. \quad (S.2)$$

3. The α_2 and α_6 coefficients are the coefficients fo the **p** and **q** components, respectively, of the contribution of the rate of strain to the force balance equation. They are dervied from the following integral :

$$\alpha_2 \mathrm{e}\cos(2\theta)\,\mathbf{p} + \alpha_6 \mathrm{e}\sin(2\theta)\,\mathbf{q} =$$
$$\mu \int \mathrm{K_N}[(\mathbf{E}.\mathbf{r}).\mathbf{n}\,(1 - \frac{\mathrm{a}^5}{r^5}) - \frac{5}{2}(\mathbf{r}.\mathbf{n})\frac{(\mathbf{r}.\mathbf{E}.\mathbf{r})}{r^2}\frac{\mathrm{a}^3}{r^3}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{n}\,\mathrm{ds}$$
$$+ \;\; \mu \int \mathrm{K_T}[(\mathbf{E}.\mathbf{r}).\mathbf{t}\,(1 - \frac{\mathrm{a}^5}{r^5}) - \frac{5}{2}(\mathbf{r}.\mathbf{t})\frac{(\mathbf{r}.\mathbf{E}.\mathbf{r})}{r^2}\frac{\mathrm{a}^3}{r^3}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{t}\,\mathrm{ds}\,. \quad (S.3)$$

4. The α_3 coefficient is the entire contribution to the force balance equation in the **p** direction due to the motion of the flagella relative to the body. It is derived from the following integral :

$$\alpha_3\,\mathbf{p} = \mu \int \mathrm{K_N}(\dot{\mathbf{r}}.\mathbf{n})\,\mathbf{n}\,\mathrm{ds} \;+ \mu \int \mathrm{K_T}(\dot{\mathbf{r}}.\mathbf{t})\,\mathbf{t}\,\mathrm{ds}\,. \quad (S.4)$$

5. The α_7 coefficient is the coefficient of the contribution of the velocity of the organism to the torque balance equation. It is derived from the following integral :

$$\alpha_7 \mathrm{v_q}(\mathbf{p} \times \mathbf{q}) = \mu \int \mathbf{r} \times \{\mathrm{K_N}[-(\mathbf{v}.\mathbf{n})(1 - \frac{3\mathrm{a}}{4r} - \frac{\mathrm{a}^3}{4r^3}) + \frac{(\mathbf{v}.\mathbf{r})(\mathbf{r}.\mathbf{n})}{r^2}\frac{3\mathrm{a}}{4r}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{n}$$
$$+\mathrm{K_T}[-(\mathbf{v}.\mathbf{t})(1 - \frac{3\mathrm{a}}{4r} - \frac{\mathrm{a}^3}{4r^3}) + \frac{(\mathbf{v}.\mathbf{r})(\mathbf{r}.\mathbf{t})}{r^2}\frac{3\mathrm{a}}{4r}(1 - \frac{\mathrm{a}^2}{r^2})]\,\mathbf{t}\}\,\mathrm{ds}\,. \quad (S.5)$$

6. . The α_8 coefficient is the coefficient of the contribution of the relative angular velocity of the body to the torque balance equation. It is derived from the following integral :

$$\alpha_8(\dot{\theta}+\omega/2)(\mathbf{p}\times\mathbf{q}) = \mu\int \mathbf{r}\times\{\mathrm{K_N}[\{(\frac{\omega}{2}-\boldsymbol{\Omega})\times\mathbf{r}\}.\mathbf{n}\,(1-\frac{\mathrm{a}^3}{r^3})]\,\mathbf{n}$$
$$+\mathrm{K_T}[\{(\frac{\omega}{2}-\boldsymbol{\Omega})\times\mathbf{r}\}.\mathbf{t}\,(1-\frac{\mathrm{a}^3}{r^3})]\,\mathbf{t}\,\}\,\mathrm{ds}\;-8\pi\mu\mathrm{a}^3(\boldsymbol{\Omega}-\frac{\omega}{2}). \tag{S.6}$$

Note that this includes the viscous torque acting upon the body.

7. The α_9 coefficient is the coefficient of the contribution of the rate of strain to the torque balance equation. It is dervied from the following integral :

$$\alpha_9\mathrm{e}\sin(2\theta)(\mathbf{p}\times\mathbf{q}) = \mu\int \mathbf{r}\times\{\mathrm{K_N}[(\mathbf{E}.\mathbf{r}).\mathbf{n}\,(1-\frac{\mathrm{a}^5}{r^5})-\frac{5}{2}(\mathbf{r}.\mathbf{n})\,\frac{(\mathbf{r}.\mathbf{E}.\mathbf{r})}{r^2}\frac{\mathrm{a}^3}{r^3}(1-\frac{\mathrm{a}^2}{r^2})]\,\mathbf{n}$$
$$+\mathrm{K_T}[(\mathbf{E}.\mathbf{r}).\mathbf{t}\,(1-\frac{\mathrm{a}^5}{r^5})-\frac{5}{2}(\mathbf{r}.\mathbf{t})\,\frac{(\mathbf{r}.\mathbf{E}.\mathbf{r})}{r^2}\frac{\mathrm{a}^3}{r^3}(1-\frac{\mathrm{a}^2}{r^2})]\,\mathbf{t}\,\}\,\mathrm{ds}\,. \tag{S.7}$$

When these have been evaluated, they become Equations (5.16) - (5.18) and describe the motion of the organism fully.

In the full definition of the coefficients below, some core integrals appear repeatedly. They are listed here; $i = 1$ refers to the effective stroke and $i = 2,3$ to the recovery stroke. Full explanation of the subscripts and definition of the limits of integration and the quantity A_i are specified in S.1.2 below.

$$\begin{aligned} I_{1,i} &= \int \frac{1}{r}\,\mathrm{ds} \\ &= \int \frac{1}{(r^2-A_i^2)^{\frac{1}{2}}}\,\mathrm{dr} \end{aligned} \tag{S.8}$$

$$\begin{aligned} I_{2,i} &= \int \frac{1}{r^3}\,\mathrm{ds} \\ &= \int \frac{1}{r^2(r^2-A_i^2)^{\frac{1}{2}}}\,\mathrm{dr} \end{aligned} \tag{S.9}$$

$$\begin{aligned} I_{3,i} &= \int \frac{1}{r^5}\,\mathrm{ds} \\ &= \int \frac{1}{r^4(r^2-A_i^2)^{\frac{1}{2}}}\,\mathrm{dr} \end{aligned} \tag{S.10}$$

$$
\begin{aligned}
I_{4,i} &= \int \frac{1}{r^7}\,\mathrm{ds} \\
&= \int \frac{1}{r^6(r^2 - A_i^2)^{\frac{1}{2}}}\,\mathrm{dr} \qquad \text{(S.11)}
\end{aligned}
$$

$$
\begin{aligned}
J_{1,i} &= \int \frac{1}{r^2} - \frac{1}{r^4}\,dr \qquad \text{(S.12)} \\
J_{2,i} &= \int \frac{1}{r^2}\,dr \qquad \text{(S.13)} \\
J_{3,i} &= \int \frac{1}{r^4}\,dr \qquad \text{(S.14)} \\
J_{4,i} &= \int \frac{1}{r^4} - \frac{1}{r^6}\,dr \qquad \text{(S.15)} \\
J_{5,i} &= \int 1 + \frac{1}{3r^2}\,dr \qquad \text{(S.16)}
\end{aligned}
$$

S.1.2 Setup

The index i here refers to the different sections of the idealised model of the flagellum. The integrals must be evaluated at their lower and upper limits with the appropriate value of the variable A_i.

The transformation from s to r means that, during the recovery stroke, sometimes dr can initially be negative and then change to positive. This occurs on all integrals during the recovery stroke when the position $r = A_i$ lies within the range of integration. In this case, the integral, $f(r)$ say, must be divided into 2 parts; integrated negatively up to $r = A_i$ and then positively from there to the end of the range of integration. Since the integrand evaluated at A_i is always zero, the result is the same as before but with the integrand evaluated at the two limits and then *added* together.

For the effective stroke, $i = 1$ and the variables are defined by

1. At the lower limit $r = \mathrm{a}$
2. At the upper limit $r = \{\mathrm{a}^2 \sin^2(\chi) + [\mathrm{a}\cos(\chi) + l]^2\}^{\frac{1}{2}}$
3. The value $A_1 = \mathrm{a}\sin(\chi)$.

For the section of the recovery stroke parallel to the body axis, $i = 2$ and the variables are defined by

1. At the lower limit $r = \mathrm{a}$
2. At the upper limit $r = \mathrm{a} + \mathrm{wt}$
3. The value $A_2 = 0$,

where w is the velocity at which the bending point moves along the flagellum and t is the time since the start of the recovery stroke. For the part of the recovery stroke at an angle χ to the body axis, [See Figure 4(b)], $i = 3$ and the variables are defined by

1. The lower limit $r = \{(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) + \mathrm{wt}\cos^2(\chi))^{\frac{1}{2}}\}$
2. The upper limit $= r = \{(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) + (\mathrm{wt}\cos(\chi) + (l - \mathrm{wt}))^2)^{\frac{1}{2}}\}$
3. The value $A_3 = (\mathrm{a} + \mathrm{wt})\sin(\chi)$

The core integrals are then given by :

$$I_{1,i} = \begin{cases} \ln\left|\frac{r+(r^2-A_i^2)^{\frac{1}{2}}}{A_i}\right| & \text{For } A_i \neq 0 \\ \ln r & \text{For } A_i = 0 \end{cases} \tag{S.17}$$

$$I_{2,i} = \begin{cases} \frac{1}{A_i^2}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r}] & \text{For } A_i \neq 0 \\ -\frac{1}{2r^2} & \text{For } A_i = 0 \end{cases} \tag{S.18}$$

$$I_{3,i} = \begin{cases} \frac{1}{A_i^4}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r} - \frac{1}{3}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r}]^3] & \text{For } A_i \neq 0 \\ -\frac{1}{4r^4} & \text{For } A_i = 0 \end{cases} \tag{S.19}$$

$$I_{4,i} = \begin{cases} \frac{1}{A_i^6}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r} - \frac{2}{3}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r}]^3 + \frac{1}{5}[\frac{(r^2-A_i^2)^{\frac{1}{2}}}{r}]^5] & \text{For } A_i \neq 0 \\ -\frac{1}{6r^6} & \text{For } A_i = 0 \end{cases} \tag{S.20}$$

$$J_{1,i} = -\frac{1}{r} + \frac{1}{3r^3} \tag{S.21}$$

$$J_{2,i} = -\frac{1}{r} \tag{S.22}$$

$$J_{3,i} = -\frac{1}{3r^3} \tag{S.23}$$

$$J_{4,i} = -\frac{1}{3r^3} + \frac{1}{5r^5} \tag{S.24}$$

$$J_{5,i} = r - \frac{1}{3r} \tag{S.25}$$

S.2 Coefficients

The time dependent beat coefficients are defined separately for the effective and the recovery strokes.

S.2.1 Effective stroke

$$\begin{aligned}
\alpha_1 = \; & \mu[-2l + \frac{3}{2}(I_{1,1} + \frac{1}{3}I_{2,1})](\mathrm{K_N}\sin^2(\chi) + \mathrm{K_T}\cos^2(\chi)) \\
& + \frac{3\mu}{2}(\mathrm{K_N}\sin(\chi)\{\sin^3(\chi)(I_{2,1} - I_{3,1}) \\
& + \cos(\chi)\sin(\chi)J_{1,1}\} \\
& + \mathrm{K_T}\cos(\chi)\{\sin^2(\chi)J_{1,1} \\
& + \cos(\chi)\}[(I_{1,1} - I_{2,1}) - \sin^2(\chi)(I_{2,1} - I_{3,1})]) - 6\pi\mu\mathrm{a}
\end{aligned} \tag{S.26}$$

$$\begin{aligned}
\alpha_2 = \; & 2\mu[\mathrm{K_N}\sin(\chi)\{\cos(2\chi)\sin(\chi)(l - I_{3,1}) \\
& + \sin(2\chi)(\cos(\chi)l + l^2/2 - J_{3,1}\} \\
& + \mathrm{K_T}\cos(\chi)\{+\sin(2\chi)\sin(\chi)(l - I_{3,1}) \\
& - \cos(2\chi)(\cos(\chi)l + l^2/2 - J_{3,1}\}] \\
& -5\mu[\mathrm{K_N}\sin(\chi)\{\cos(2\chi)\sin^3(\chi)(I_{3,1} - I_{4,1}) \\
& + 2\sin(2\chi)\sin^2(\chi)J_{4,1} \\
& - \cos(2\chi)\sin(\chi)[(I_{2,1} - I_{3,1}) - \sin^2(\chi)(I_{3,1} - I_{4,1})]\} \\
& + \mathrm{K_T}\cos(\chi)\{\cos(2\chi)\sin^2(\chi)J_{4,1}
\end{aligned}$$

$$+2\sin(2\chi)\sin(\chi)[(I_{2,1} - I_{3,1}) - \sin^2(\chi)(I_{3,1} - I_{4,1})]$$

$$- \cos(2\chi)(J_{1,1} - \sin^2(\chi)J_{4,1})\}] \tag{S.27}$$

$$\alpha_3 = -\mathrm{K_N}\mu \sin(\chi) l^2 \dot{\chi} \tag{S.28}$$

$$\begin{aligned} \alpha_4 = {} & \mu[-2l + \frac{3}{2}(I_{1,1} + \frac{1}{3}I_{2,1})](\mathrm{K_N}\cos^2(\chi) + \mathrm{K_T}\sin^2(\chi)) \\ & + \frac{3\mu}{2}(\mathrm{K_N}\cos(\chi)\{\cos(\chi)\sin^2(\chi)(I_{2,1} - I_{3,1}) \\ & - \sin^2(\chi)J_{1,1}\} \\ & + \mathrm{K_T}\sin(\chi)\{-\sin(\chi)\cos(\chi)J_{1,1} \\ & + \sin(\chi)[(I_{1,1} - I_{2,1}) - \sin^2(\chi)(I_{2,1} - I_{3,1})]\}) - 6\pi\mu \mathrm{a} \end{aligned} \tag{S.29}$$

$$\begin{aligned} \alpha_5 = {} & -2\mu[\mathrm{K_N}\cos(\chi)(\cos(\chi)l + l^2/2 - J_{2,1}) \\ & + \mathrm{K_T}\sin^2(\chi)(l - I_{2,1})] \end{aligned} \tag{S.30}$$

$$\begin{aligned} \alpha_6 = {} & 2\mu[\mathrm{K_N}\cos(\chi)\{-\sin(2\chi)\sin(\chi)(l - I_{3,1}) \\ & + \cos(2\chi)(\cos(\chi)l + l^2/2 - J_{3,1}\} \\ & + \mathrm{K_T}\sin(\chi)\{-\cos(2\chi)\sin(\chi)(l - I_{3,1}) \\ & - \sin(2\chi)(\cos(\chi)l + l^2/2 - J_{3,1}\}] \\ & -5\mu[\mathrm{K_N}\cos(\chi)\{-\sin(2\chi)\sin^3(\chi)(I_{3,1} - I_{4,1}) \\ & + 2\cos(2\chi)\sin^2(\chi)J_{4,1} \\ & + \sin(2\chi)\sin(\chi)[(I_{2,1} - I_{3,1}) - \sin^2(\chi)(I_{3,1} - I_{4,1})]\} \\ & + \mathrm{K_T}\sin(\chi)\{\sin(2\chi)\sin^2(\chi)J_{4,1} \\ & -2\cos(2\chi)\sin(\chi)[(I_{2,1} - I_{3,1}) - \sin^2(\chi)(I_{3,1} - I_{4,1})] \end{aligned}$$

$$- \sin(2\chi)(J_{1,1} - \sin^2(\chi)J_{4,1})\}] \tag{S.31}$$

$$\begin{aligned}
\alpha_7 &= -2\mu[\mathrm{K_T}\sin^2(\chi)l + \mathrm{K_N}\cos(\chi)(\cos(\chi)l + l^2/2)] \\
&\quad + \frac{3\mu}{2}[\mathrm{K_T}\sin^2(\chi)(I_{1,1} + I_{2,1}/3) + \mathrm{K_N}\cos(\chi)J_{5,1} \\
&\quad + (\mathrm{K_N} - \mathrm{K_T})(\cos(\chi)\sin^2(\chi)J_{1,1} - \sin^2(\chi) \\
&\quad [(I_{1,1} - I_{2,1}) - \sin^2(\chi)(I_{2,1} - I_{3,1})]\,)]
\end{aligned} \tag{S.32}$$

$$\begin{aligned}
\alpha_8 &= -2\mu[\mathrm{K_T}\sin^2(\chi)(l - I_{2,1}) \\
&\quad + \mathrm{K_N}\{\cos^2(\chi)l + \cos(\chi)l^2 + l^3/3 - (I_{1,1} - \sin^2(\chi)I_{2,1})\}] - 8\pi\mu \mathrm{a}^3
\end{aligned} \tag{S.33}$$

$$\begin{aligned}
\alpha_9 &= 2\mu\{\mathrm{K_T}[-\cos(2\chi)\sin^2(\chi)(l - I_{3,1}) \\
&\quad - \sin(2\chi)\sin(\chi)(\cos(\chi)l + l^2/2 - J_{3,1})] \\
&\quad - \mathrm{K_N}[\sin(2\chi)\sin(\chi)(\cos(\chi)l + l^2/2 - J_{3,1}) \\
&\quad - \cos(2\chi)(\cos^2(\chi)l + \cos(\chi)l^2 + l^3/3 - (I_{2,1} - \sin^2(\chi)I_{3,1}))]\} \\
&\quad - 5\mu(\mathrm{K_N} - \mathrm{K_T})\{-\sin^3(\chi)\sin(2\chi)J_{4,1} \\
&\quad + 2\sin^2(\chi)\cos(2\chi)[(I_{2,1} - I_{3,1}) - \sin^2(\chi)(I_{3,1} - I_{4,1})] \\
&\quad + \sin(2\chi)\sin(\chi)[J_{1,1} - \sin^2(\chi)J_{4,1}]\}
\end{aligned} \tag{S.34}$$

S.2.2 Recovery Stroke

$$\begin{aligned}
\alpha_1 &= \mu\mathrm{K_T}[-2\mathrm{wt} + (3I_{1,2} - I_{2,2})] \\
&\quad + \mu[-2(l - \mathrm{wt}) + \frac{3}{2}(I_{1,3} + \frac{1}{3}I_{2,3})](\mathrm{K_N}\sin^2(\chi) + \mathrm{K_T}\cos^2(\chi)) \\
&\quad + \frac{3\mu}{2}(\mathrm{K_N}\sin(\chi)\{(\mathrm{a} + \mathrm{wt})^2\sin^3(\chi)(I_{2,3} - I_{3,3}) \\
&\quad + (\mathrm{a} + \mathrm{wt})\sin(\chi)\cos(\chi)J_{1,3}\}
\end{aligned}$$

$$+\mathrm{K_T}\cos(\chi)\{(\mathrm{a}+\mathrm{wt})\sin^2(\chi)J_{1,3}$$

$$+\cos(\chi)[(I_{1,3}-I_{2,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{2,3}-I_{3,3})]\})-6\pi\mu\mathrm{a} \tag{S.35}$$

$$\begin{aligned}
\alpha_2 \;=\;& -2\mu\mathrm{K_T}(\mathrm{wt}+\mathrm{wt}^2/2-J_{3,2})+5\mu\mathrm{K_T}J_{1,2}\\
&+2\mu[\mathrm{K_N}\sin(\chi)\{(\mathrm{a}+\mathrm{wt})\sin(\chi)\cos(2\chi)((l-\mathrm{wt})-I_{3,3})\\
&+\sin(2\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2-J_{3,3}\}\\
&+\mathrm{K_T}\cos(\chi)\{(\mathrm{a}+\mathrm{wt})\sin(\chi)\sin(2\chi)((l-\mathrm{wt})-I_{3,3})\\
&-\cos(2\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2-J_{3,3}\}]\\
&-5\mu[\mathrm{K_N}\sin(\chi)\{\cos(2\chi)(\mathrm{a}+\mathrm{wt})^3\sin^3(\chi)(I_{3,3}-I_{4,3})\\
&+2\sin(2\chi)(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)J_{4,3}\\
&+\cos(2\chi)(\mathrm{a}+\mathrm{wt})\sin(\chi)[(I_{2,3}-I_{3,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{3,3}-I_{4,3})]\}\\
&+\mathrm{K_T}\cos(\chi)\{-\cos(2\chi)\sin^2(\chi)J_{4,3}\\
&+2\sin(2\chi)(\mathrm{a}+\mathrm{wt})\sin(\chi)[(I_{2,3}-I_{3,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{3,3}-I_{4,3})]\\
&+\cos(2\chi)(J_{1,3}-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)J_{4,3})\}]
\end{aligned} \tag{S.36}$$

$$\begin{aligned}
\alpha_3 \;=\;& 2\mu[\mathrm{K_N}\sin(\chi)(\mathrm{w}(l-\mathrm{wt})\sin(\chi)-\dot{\chi}(l-\mathrm{wt})^2/2)\\
&+\mathrm{K_T}\cos(\chi)\mathrm{w}(l-\mathrm{wt})(\cos(\chi)-1)]
\end{aligned} \tag{S.37}$$

$$\begin{aligned}
\alpha_4 \;=\;& \mu\mathrm{K_N}[-2\mathrm{wt}+\frac{3}{2}(I_{1,2}+\frac{1}{3}I_{2,2})]\\
&+\mu[-2(l-\mathrm{wt})+\frac{3}{2}(I_{1,3}+\frac{1}{3}I_{2,3})]\\
&(\mathrm{K_N}\cos^2(\chi)+\mathrm{K_T}\sin^2(\chi))\\
&+\frac{3\mu}{2}(\mathrm{K_N}\cos(\chi)\{(a+\mathrm{wt})^2\sin^2(\chi)\cos(\chi)(I_{2,3}-I_{3,3})\\
&-(\mathrm{a}+\mathrm{wt})\sin^2(\chi)J_{1,3}\}
\end{aligned}$$

$$+\mathrm{K_T}\sin(\chi)\{-(\mathrm{a}+\mathrm{wt})\sin(\chi)\cos(\chi)J_{1,3}$$

$$+\sin(\chi)[(I_{1,3}-I_{2,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{2,3}-I_{3,3})]\})-6\pi\mu\mathrm{a} \qquad \text{(S.38)}$$

$$\alpha_5 = 2\mu[\mathrm{K_N}(\mathrm{wt}+\mathrm{wt}^2/2-J_{2,2})]$$

$$+2\mu[\mathrm{K_N}\cos(\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2-J_{2,3})$$

$$+\mathrm{K_T}(\mathrm{a}+\mathrm{wt})\sin^2(\chi)((l-\mathrm{wt})-I_{2,3})] \qquad \text{(S.39)}$$

$$\alpha_6 = 2\mu[-\mathrm{K_N}(\mathrm{wt}+\mathrm{wt}^2/2-J_{3,2}\}$$

$$+2\mu[\mathrm{K_N}\cos(\chi)\{-(\mathrm{a}+\mathrm{wt})\sin(\chi)\sin(2\chi)((l-\mathrm{wt})-I_{3,3})$$

$$+\cos(2\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2-J_{3,3}\}$$

$$+\mathrm{K_T}\sin(\chi)\{-(\mathrm{a}+\mathrm{wt})\sin(\chi)\cos(2\chi)((l-\mathrm{wt})-I_{3,3})$$

$$-\sin(2\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2-J_{3,3}\}]$$

$$-5\mu[\mathrm{K_N}\cos(\chi)\{-\sin(2\chi)(\mathrm{a}+\mathrm{wt})^3\sin^3(\chi)(I_{3,3}-I_{4,3})$$

$$+2\cos(2\chi)(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)J_{4,3}$$

$$+\sin(2\chi)(\mathrm{a}+\mathrm{wt})\sin(\chi)[(I_{2,3}-I_{3,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{3,3}-I_{4,3})]\}$$

$$+\mathrm{K_T}\sin(\chi)\{\sin(2\chi)(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)J_{4,3}$$

$$-2\cos(2\chi)(\mathrm{a}+\mathrm{wt})\sin(\chi)[(I_{2,3}-I_{3,3})-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)(I_{3,3}-I_{4,3})]$$

$$-\sin(2\chi)(J_{1,3}-(\mathrm{a}+\mathrm{wt})^2\sin^2(\chi)J_{4,3})\}] \qquad \text{(S.40)}$$

$$\alpha_7 = -2\mu[\mathrm{K_N}(\mathrm{wt}+\mathrm{wt}^2/2)]+\frac{3}{2}\mathrm{K_N}J_{5,2}$$

$$-2\mu[\mathrm{K_T}(\mathrm{a}+\mathrm{wt})\sin^2(\chi)(l-\mathrm{wt})$$

$$+\mathrm{K_N}\cos(\chi)(\mathrm{wt}\cos(\chi)(l-\mathrm{wt})+(l-\mathrm{wt})^2/2)]$$

$$+\frac{3\mu}{2}[\mathrm{K_T}(\mathrm{a}+\mathrm{wt})\sin^2(\chi)(I_{1,3}+I_{2,3}/3)+\mathrm{K_N}\cos(\chi)J_{5,3}$$

$$+(\mathrm{K_N} - \mathrm{K_T})(\cos(\chi)(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) J_{1,3}$$

$$-(\mathrm{a} + \mathrm{wt}) \sin^2(\chi)[(I_{1,3} - I_{2,3}) - (\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi)(I_{2,3} - I_{3,3})])] \tag{S.41}$$

$$\begin{aligned}
\alpha_8 \quad = \quad & -2\mu[\mathrm{K_N}\{\mathrm{wt} + \mathrm{wt}^2 + \mathrm{wt}^3/3 - I_{1,2}\}] \\
& -2\mu[\mathrm{K_T}(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi)((l - \mathrm{wt}) - I_{2.3}) \\
& +\mathrm{K_N}\{\mathrm{wt}^2 \cos^2(\chi)(l - \mathrm{wt}) + \mathrm{wt}\cos(\chi)(l - \mathrm{wt})^2 + (l - \mathrm{wt})^3/3 \\
& -(I_{1,3} - (\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) I_{2,3})\}] - 8\pi\mu\mathrm{a}^3
\end{aligned} \tag{S.42}$$

$$\begin{aligned}
\alpha_9 \quad = \quad & +2\mu\{\mathrm{K_N}[\mathrm{wt} + \mathrm{wt}^2 + \mathrm{wt}^3/3 - I_{2,2}]\} \\
& +2\mu\{\mathrm{K_T}[-\cos(2\chi)(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi)((l - \mathrm{wt}) - I_{3,3}) \\
& - \sin(2\chi)(\mathrm{a} + \mathrm{wt}) \sin(\chi)(\mathrm{wt}\cos(\chi)(l - \mathrm{wt}) + (l - \mathrm{wt})^2/2 - J_{3,3})] \\
& -\mathrm{K_N}[\sin(2\chi)(\mathrm{a} + \mathrm{wt}) \sin(\chi)(\mathrm{wt}\cos(\chi)(l - \mathrm{wt}) + (l - \mathrm{wt})^2/2 - J_{3,3}) \\
& - \cos(2\chi)(\mathrm{wt}^2 \cos^2(\chi)(l - \mathrm{wt}) + \mathrm{wt}\cos(\chi)(l - \mathrm{wt})^2 + (l - \mathrm{wt})^3/3 \\
& -(I_{2,3} - (\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) I_{3,3}))]\} \\
& -5\mu(\mathrm{K_T} - \mathrm{K_N})\{(\mathrm{a} + \mathrm{wt})^3 \sin^3(\chi)\sin(2\chi) J_{4,3} \\
& -2(\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi)\cos(2\chi)[(I_{2,3} - I_{3,3}) - (\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi)(I_{3,3} - I_{4,3})] \\
& - \sin(2\chi)(\mathrm{a} + \mathrm{wt}) \sin(\chi)[J_{1,3} - (\mathrm{a} + \mathrm{wt})^2 \sin^2(\chi) J_{4,3}]\}
\end{aligned} \tag{S.43}$$