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MATERIALS OF FOOTNOTE REFERENCES

(for editor's retention only; not for publication)

Far-field conditions

$$L_1 = Q_{11}A_0^f + Q_{12}\partial_{x_1}A_0^f , \qquad L_2 = Q_{13}$$

$$L_3 = Q_{21}A_0^f + Q_{22}\partial_{x_1}A_0^f , \qquad L_4 = Q_{23}$$

where

$$Q_{11} = (\partial_{x_1}C_{0\alpha})(B_1+\alpha_0^2B_2) + C_{0\alpha}(\partial_{x_1}\alpha_0)(2\alpha_0B_2+B_3+\alpha_0^2B_4) - C_{0\alpha}\alpha_0(B_6+\alpha_0^2B_8),$$

$$Q_{12} = C_{0\alpha}(B_1+\alpha_0^2B_2), \qquad Q_{13} = -C_{0\alpha}(\partial_{x_1}\alpha_0)(B_1+\alpha_0^2B_2),$$

$$Q_{21} = (\partial_{x_1}C_{0\chi})(B_1+\chi_0^2B_2)+C_{0\chi}(\partial_{x_1}\alpha_0)(B_3+\chi_0^2B_4)+2C_{0\chi}\chi_0(\partial_{x_1}\chi_0)B_2-C_{0\chi}\chi_0(B_6+\chi_0^2B_8),$$

$$Q_{22} = C_{0\chi}(B_1+\chi_0^2B_2), \qquad Q_{23} = -C_{0\chi}(\partial_{x_1}\chi_0)(B_1+\chi_0^2B_2) .$$

$C_{0\alpha}$ and $C_{0\chi}$ are the coefficients of the exponential solution of ϕ_0 as given by (2.1.11).

$$H_1 = \beta_1L_1 + \beta_2L_2 , \qquad H_2 = \frac{1}{2}\beta_1L_2$$

$$H_3 = \beta_3L_3 + \beta_4L_4 , \qquad H_4 = \frac{1}{2}\beta_3L_4$$

where

$$\beta_1 = (2\alpha_0(\chi_0^2-\alpha_0^2))^{-1} , \qquad \beta_2 = (\chi_0^2-5\alpha_0^2)(4\alpha_0^2(\chi_0^2-\alpha_0^2)^2)^{-1}$$

$$\beta_3 = (2\chi_0(\alpha_0^2-\chi_0^2))^{-1} , \qquad \beta_4 = (\alpha_0^2-5\chi_0^2)(4\chi_0^2(\alpha_0^2-\chi_0^2)^2)^{-1}$$

$$\gamma_{\infty 1} = \begin{bmatrix}(Q_{11}\gamma_1+Q_{13}\gamma_2)exp(-\alpha_0 z) + (Q_{21}\gamma_3+Q_{23}\gamma_4)exp(-\chi_0 z)\\(Q_{11}\gamma_5+Q_{13}\gamma_6)exp(-\alpha_0 z) + (Q_{21}\gamma_7+Q_{23}\gamma_8)exp(-\chi_0 z)\end{bmatrix}$$

$$\gamma_{\infty 2} = \begin{bmatrix}Q_{12}\gamma_1exp(-\alpha_0 z) + Q_{22}\gamma_3exp(-\chi_0 z)\\Q_{12}\gamma_5exp(-\alpha_0 z) + Q_{22}\gamma_7exp(-\chi_0 z)\end{bmatrix}$$

where

$$\gamma_1 = (\chi_0-\alpha_0)\beta_1 , \qquad \gamma_2 = (\chi_0-\alpha_0)(\beta_1z+\beta_2) + \beta_1$$

$$\gamma_3 = (\alpha_0 - \chi_0)\beta_3 , \qquad \gamma_4 = (\alpha_0 - \chi_0)(\beta_3 z + \beta_4) + \beta_3$$

$$\gamma_5 = -\alpha_0 \gamma_1 , \qquad \gamma_6 = \gamma_1 - \alpha_0 \gamma_2$$

$$\gamma_7 = \chi_0 \gamma_3 , \qquad \gamma_8 = \gamma_3 + \chi_0(\gamma_4 - 2\beta_3)$$

The adjoint problem

The homogeneous boundary-value problem represented by (2.4.1) may be written in the form

$$L^f\phi = 0 \quad (z_0 < z < z_\infty) \tag{C.1}$$

$$\mathbf{U}_B\underset{\sim}{\Phi}(z_0,z_\infty) = 0 \tag{C.2}$$

where $\mathbf{U}_B$ is the 4×8 boundary condition matrix or operator of rank 4 defined by

$$\mathbf{U}_B = \begin{bmatrix} {}^2\mathbf{M}_w & \mathbf{0}_{2\times4} \\ \mathbf{0}_{2\times4} & \mathbf{M}_\infty \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\Phi}(z_0,z_\infty) = \begin{bmatrix} \underset{\sim}{\phi}(z_0) \\ \underset{\sim}{\phi}(z_\infty) \end{bmatrix} .$$

$\underset{\sim}{\Phi}(z_0,z_\infty)$ is the column vector comprising values of ϕ and its derivatives at the boundary points z_0 and z_∞, and $\mathbf{O}_{2\times4}$ is the 2×4 null matrix. The corresponding homogeneous *adjoint* problem has the form

$$L^{f+}\varphi = 0 \quad (z_0 < z < z_\infty) \tag{C.3}$$

$$\mathbf{U}^+_B\underset{\sim}{\Psi}(z_0,z_\infty) = 0 \tag{C.4}$$

where L^{f+} is the adjoint differential operator to L^f. $\mathbf{U}^+_B$ is a 4×8 matrix of rank 4 termed the *adjoint boundary-condition* matrix/operator and

$$\underset{\sim}{\Psi}(z_0,z_\infty) = \begin{bmatrix} \underset{\sim}{\varphi}(z_0) \\ \underset{\sim}{\varphi}(z_\infty) \end{bmatrix} , \qquad \underset{\sim}{\varphi} = (\varphi,\varphi',\varphi'',\varphi''')^T .$$

The adjoint boundary-condition matrix $\mathbf{U}^+_B$ may be obtained from the original boundary condition matrix $\mathbf{U}_B$ as follows. We first of all append to $\mathbf{U}_B$ a 4×8 matrix, denoted by $\mathbf{U}_{BC}$ (termed the complementary matrix), such that combined 8×8 matrix

$$\begin{pmatrix} \mathbf{U}_B \\ \mathbf{U}_{BC} \end{pmatrix} \text{ is non-singular.}$$

The requisite adjoint boundary condition matrix $\mathbf{U}^+_B$ and its complementary matrix $\mathbf{U}^+_{BC}$, both 4×8 matrices of rank 4, are then given by

$$\begin{pmatrix} U_B^+ \\ U_{BC}^+ \end{pmatrix} := \left[\begin{pmatrix} -B(z_0) & 0_{4\times4} \\ 0_{4\times4} & B(z_\infty) \end{pmatrix} \begin{pmatrix} U_B \\ U_{BC} \end{pmatrix}^{-1} \right]^{\mathcal{T}} \tag{C.5}$$

where $\mathcal{T}$ denotes the conjugate transpose of the matrix. $\mathbf{B}(z)$ is the following 4×4 non-singular matrix function of z, determined from the coefficient functions of the operator L^f:

$$\mathbf{B}(z) = \begin{bmatrix} -p'(z) & p(z) & 0 & 1 \\ -p(z) & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{C.6}$$

where $p(z) = -2\alpha_0^2 - i\alpha_0 R_\delta(U_0 - c)$, see Coddington & Livingson (1955). It can be shown for our case that U_B^+ and U_{BC}^+ have the two-point separated form

$$U_B^+ = \begin{bmatrix} 0_{2\times4} & M_\infty \\ M_w & 0_{2\times4} \end{bmatrix} \quad \text{and} \quad U_{BC}^+ = \begin{bmatrix} M_{wc} & 0_{2\times4} \\ 0_{2\times4} & M_{\infty c} \end{bmatrix}. \tag{C.7a,b}$$

The adjoint boundary condition matrices U_B^+ and U_{BC}^+ are non-unique in general. Different choices of the complementary matrix U_{BC} lead to different but equivalent adjoint boundary conditions.

The non-homogeneous case given by (2.4.2) has the form

$$L^f\phi = f(z) \qquad (z_0 < z < z_\infty) \tag{C.8a}$$

$$U_B\underset{\sim}{\Phi}(z_0, z_\infty) = \underset{\sim}{\chi} = \begin{pmatrix} {}^2\underset{\sim}{\chi}_w \\ \underset{\sim}{\chi}_\infty \end{pmatrix} \tag{C.8b}$$

where ${}^2\underset{\sim}{\chi}_w$ and $\underset{\sim}{\chi}_\infty$ are 2-vectors associated with the interface ($z = z_0$) and far field ($z = z_\infty$) boundary conditions respectively. For such a non-homogeneous boundary-value problem, it is known that a solution exists if and only if

$$\int_{z_0}^{z_\infty} f\varphi^* \, dz = \underset{\sim}{\chi} \cdot (U_{BC}^+ \underset{\sim}{\Psi}(z_0, z_\infty))$$

$$= {}^2\underset{\sim}{\chi}_w \cdot (M_{wc}^+ \varphi(z_0)) + \underset{\sim}{\chi}_\infty \cdot (M_{\infty c}^+ \varphi(z_\infty)) \tag{C.9}$$

for every solution $\varphi(z)$ of the corresponding adjoint homogeneous problem (C.3–4). The centre-dot and superscript asterisk in (C.9) denote complex inner product of vectors and complex conjugation respectively.