P. A. Davidson: Global stability of two-dimensional and axisymmetric Euler flows. FM 8926

Appendix 1: Derivation of $\delta^2 E$ for Swirling Flow.

In order to evaluate $\delta^2 E_p$ we must first find $u_p \cdot \delta^2 u_p$. From (2.4) we have,

$$\delta^2 u_p = \tfrac{1}{2}(\eta_\theta/r)\nabla(\delta^1\Gamma) - \tfrac{1}{2}(\delta^1\omega_\theta/r)\nabla\psi^* + \nabla\phi_2$$

which gives,

$$2u_p \cdot \delta^2 u_p = -(\eta_\theta/r)u_p \cdot \nabla(\eta \cdot \nabla\Gamma) - (\delta^1\omega_\theta/r)u_p \cdot \nabla\psi^* + \nabla \cdot (2\phi_2 u_p)$$

Noting that $u_p \cdot \nabla\psi^* = -\eta \cdot \nabla\psi$ we can rewrite this in the form

$$2u_p \cdot \delta^2 u_p = (\eta \cdot \nabla\Gamma)u_p \cdot \nabla(\eta_\theta/r) + (\delta^1\omega_\theta/r)\eta \cdot \nabla\psi + \nabla \cdot [(.....)u_p]$$

Now the second term on the right-hand-side may be evaluated with the aid of (8.7) to give

$$(\eta \cdot \nabla\psi)\delta^1\omega_\theta/r = (\eta \cdot \nabla\psi)\Gamma'(\psi)u \cdot \nabla(\eta_\theta/r) - (\eta \cdot \nabla\psi)\eta \cdot \nabla(\omega_\theta/r)$$

since $\omega_p = \Gamma'(\psi)u_p$. Substituting back into the expression for $u_p \cdot \delta^2 u_p$ gives,

$$2u_p \cdot \delta^2 u_p = 2(\eta \cdot \nabla\Gamma)u_p \cdot \nabla(\eta_\theta/r) - (\eta \cdot \nabla\psi)\eta \cdot \nabla(\omega_\theta/r) + \nabla \cdot [(.......)u_p]$$

from which we can evaluate $\delta^2 E_p$.

$$\delta^2 E_p = \tfrac{1}{2}\int(\delta^1 u_p)^2 dV + \int_V [(\eta \cdot \nabla\Gamma)u_p \cdot \nabla(\eta_\theta/r) - \tfrac{1}{2}(\eta \cdot \nabla\psi)\eta \cdot \nabla(\omega_\theta/r)]dV \qquad (A.1)$$

This is equation (8.11). We may derive a different form for $\delta^2 E$ by introducing the variables,

$$\varepsilon = -\eta \cdot \nabla\psi = \delta^1\Gamma/\Gamma'(\psi), \quad \phi = \delta^1\psi$$

Now consider the quantity,

$$\frac{g\varepsilon^2}{2r^2} + \frac{\varepsilon\nabla_*^2\phi}{r^2} = \tfrac{1}{2}(\eta \cdot \nabla\psi)^2(g/r^2) + (\delta^1\omega_\theta/r)(\eta \cdot \nabla\psi)$$

Using the expression

$$r^2\eta \cdot \nabla\left(\frac{\omega_\theta}{r}\right) = g\eta \cdot \nabla\psi - 2\Gamma\Gamma'(\psi)(\eta_r/r)$$

we have,

$$\frac{g\varepsilon^2}{2r^2} + \frac{\varepsilon\nabla_*^2\phi}{r^2} = (\eta \cdot \nabla\Gamma)u_p \cdot \nabla(\eta_\theta/r) - \tfrac{1}{2}(\eta \cdot \nabla\psi)\eta \cdot \nabla(\omega_\theta/r) + \tfrac{1}{2}\left(\frac{\eta_r}{r^3}\right)\eta \cdot \nabla(\Gamma^2)$$

It follows that the total change in energy is,

$$\delta^2 E_p + \delta^2 E_\theta = \tfrac{1}{2}\int_V \left[(\nabla\phi)^2 + g\varepsilon^2 + 2\varepsilon\nabla_*^2\phi\right] r^{-2} dV$$

Finally, introducing $\gamma = \varepsilon - \phi$, and noting that

$$(\varepsilon/r^2)\nabla_*^2\phi = \varepsilon\nabla\cdot\left[(\nabla\phi)/r^2\right]$$

we obtain

$$\delta^2 E = \tfrac{1}{2}\int_V \left[g\varepsilon^2 - (\nabla\varepsilon)^2 + (\nabla\gamma)^2\right] r^{-2} dV \qquad \text{(A.2)}$$

Appendix 2: First and Second Variations of the Functional $A(\Psi,\Gamma)$.

Consider the functional, A, defined by (8.18). The first variation in A is

$$\delta^1 A = \int_V \left[\frac{\nabla\psi\cdot\nabla\phi}{r^2} + \frac{\delta\Gamma}{\Gamma_0'(\psi)}\left(\frac{\Gamma_0\Gamma_0'(\psi)}{r^2} - \frac{H'(\Gamma_0)}{\psi'(\Gamma_0)} \right) - \frac{\omega_\theta}{r}\frac{\delta\Gamma}{\Gamma_0'(\psi)} + \frac{\psi\nabla_*^2\phi}{r^2} \right] dV$$

where ω_θ is the azimuthal vorticity of the Euler flow. If we now recall that ω_θ is related to H and Γ_0 through (6.5), and note that,

$$\frac{\nabla\psi\cdot\nabla\phi}{r^2} + \frac{\psi\nabla_*^2\phi}{r^2} = \nabla\cdot\left(\frac{\psi\nabla\phi}{r^2}\right)$$

it is clear that $\delta^1 A=0$. The second variation in A is,

$$\delta^2 A = \int_V \left[\frac{(\nabla\phi)^2}{2r^2} + \frac{(\delta\Gamma)^2}{2r^2} - \frac{H''(\Gamma_0)(\delta\Gamma)^2}{2} - \frac{\omega_\theta}{r}\frac{\psi''(\Gamma_0)(\delta\Gamma)^2}{2} + \frac{\nabla_*^2\phi}{r^2}\frac{\delta\Gamma}{\Gamma_0'(\psi)} \right] dV$$

This may be simplified with the aid of the expression,

$$1 - r^2H''(\Gamma_0) - \psi''(\Gamma_0)(r\omega_\theta) = g/(\Gamma_0'(\psi))^2$$

to give,

$$\delta^2 A = \tfrac{1}{2}\int_V \left[(\nabla\phi)^2 + g\varepsilon^2 + 2\varepsilon\nabla_*^2\phi \right] r^{-2} dV$$

Here ε is defined in the same way as (8.13a), $\varepsilon=\delta\Gamma/\Gamma_0'(\psi)$. Finally, we use the identity,

$$(\varepsilon/r^2)\nabla_*^2\phi = \nabla\cdot[\varepsilon(\nabla\phi)/r^2] - (\nabla\varepsilon\cdot\nabla\phi)/r^2$$

to give

$$\delta^2 A = \tfrac{1}{2}\int_V \left[g\varepsilon^2 - (\nabla\varepsilon)^2 + (\nabla\gamma)^2 \right] r^{-2} dV$$

where γ is defined as in (8.13b).

Appendix 3: Derivation of d^2E for Swirling Flows.

Let us first calculate d^2E_θ. From (2.4) we have,

$$2d^2u_\theta = -r\boldsymbol{\eta}\cdot\nabla\left(d^1u_\theta/r\right) + rd^1\boldsymbol{u}_p\cdot\nabla\left(\eta_\theta/r\right)$$

and it follows that

$$2u_\theta d^2u_\theta = -\Gamma\boldsymbol{\eta}\cdot\nabla\left(d^1u_\theta/r\right) + \Gamma d^1\boldsymbol{u}_p\cdot\nabla\left(\eta_\theta/r\right)$$

We now introduce the vector

$$A = (\eta_\theta/r)\Gamma d^1\boldsymbol{u}_p + (\eta_\theta/r)\Gamma'(\psi)\phi\boldsymbol{u}_p - u_\theta d^1u_\theta\boldsymbol{\eta}_p - (\phi/r^2)\Gamma\Gamma'(\psi)\boldsymbol{\eta}_p$$

which has divergence,

$$\nabla\cdot A = 2u_\theta d^2u_\theta + (\phi/r)^2\left(\Gamma\Gamma'(\psi)\right)' + 2\Gamma'(\psi)(\phi/r)d^1u_\theta - \left[\phi\Gamma'(\psi)/r\right]^2 - \Gamma\Gamma'(\psi)\boldsymbol{\eta}\cdot\nabla\phi/r^2$$

We also have the relationship

$$\left(d^1u_\theta\right)^2 = \left[d^1u_\theta - \Gamma'(\psi)\phi/r\right]^2 - \left[\phi\Gamma'(\psi)/r\right]^2 + 2\Gamma'(\psi)(\phi/r)d^1u_\theta$$

from which,

$$\nabla\cdot A = 2u_\theta d^2u_\theta + (\phi/r)^2\left(\Gamma\Gamma'(\psi)\right)' + \left(d^1u_\theta\right)^2 - \left[d^1u_\theta - \Gamma'(\psi)\phi/r\right]^2 - \Gamma\Gamma'(\psi)\boldsymbol{\eta}\cdot\nabla\phi/r^2$$

It follows that d^2E_θ is given by

$$d^2E_\theta = \tfrac{1}{2}\int_V\left[d^1\Gamma - \Gamma'(\psi)\phi\right]^2 r^{-2}dV - \tfrac{1}{2}\int_V\left[\phi^2\left(\Gamma\Gamma'(\psi)\right)' - \Gamma\Gamma'\boldsymbol{\eta}\cdot\nabla\phi\right]r^{-2}dV$$

In addition, from (9.1) and (9.2) we can find d^2E_p. Following the procedure in §7 we find,

$$d^2E_p = \tfrac{1}{2}\int\left[(\nabla\phi)^2 + r^2H''(\psi)\phi^2\right]r^{-2}dV - \tfrac{1}{2}\int_V\left[\Gamma\Gamma'\boldsymbol{\eta}\cdot\nabla\phi\right]r^{-2}dV$$

Combining these expressions gives the desired result.

$$d^2E = \tfrac{1}{2}\int\left[(\nabla\phi)^2 - g\phi^2\right]r^{-2}dV + \tfrac{1}{2}\int_V\left[d^1\Gamma + \boldsymbol{\eta}\cdot\nabla\Gamma\right]^2 r^{-2}dV$$