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THEORETICAL ANALYSIS OF INERTIALLY IRROTATIONAL AND SOLENOIDAL FLOW IN TWO-DIMENSIONAL RADIAL-FLOW PUMP AND TURBINE IMPELLERS WITH EQUIANGULAR BLADES

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APPENDIX

Asymptotic solutions for logarithmically bladed pump impellers

DETAILED VERSION OF SECTION 6

6. Asymptotic solutions for logarithmically bladed pump impellers

In this section pump impellers will be considered only, because solutions in closed–form have not been obtained yet for turbine impellers that are fitted with curved blades. This is no major imperfection since most radial turbine impellers found in practice have straight radial blades – discussed intensively in the previous section – or are closely represented by these blades, in particular, at the entrance section. In radial turbomachines blade curvature is often applied to pumps and fans. Solutions in closed–form will be presented for the fluid velocity tangential at the blades of pump impellers fitted with logarithmical spiral blades having a low inlet–to–outlet radius ratio, that is, the case in which the approximation $\mu = (r_1/r_2)^n \longrightarrow 0$ is justified.

6.1. *Solutions from method of conformal mapping*

With $\delta \sim \pi - 2\beta$, and using equation (3.15) we obtain from equations (4.7), (4.13), and (4.17)

$$v^{\Omega}_{t\zeta}(\theta) \sim \frac{M}{2\pi n} \int_{-\pi-2\beta}^{\pi-2\beta} \left[cos(\beta+\tfrac{1}{2}\lambda)\right]^{-1+4\cos^2(\beta)/n} e^{\lambda \sin(2\beta)/n}\, sin(\tfrac{1}{2}\lambda) cotan(\tfrac{1}{2}\theta-\tfrac{1}{2}\lambda)\, d\lambda \tag{6.1}$$

$$v^{Q}_{t\zeta}(\theta) \sim -\frac{Q}{2\pi n}\, tan(\beta+\tfrac{1}{2}\theta) \tag{6.2}$$

$$v^{\Gamma}_{t\zeta}(\theta) \sim \frac{\Gamma_1}{2\pi n} \tag{6.3}$$

as $\mu \longrightarrow 0$, where

$$M = \Omega r_2^2 \left[cos(\beta)\right]^{1-4\cos^2(\beta)/n} \tag{6.4}$$

Equations (6.2) and (6.3) are both simple expressions and need no further explanation. Equation (6.1) will be evaluated somewhat further.

Substituting the trigonometric identity

$$sin(\tfrac{1}{2}\lambda) cotan(\tfrac{1}{2}\theta-\tfrac{1}{2}\lambda) = \frac{2 sin(\tfrac{1}{2}\theta)\, sin(\tfrac{1}{2}\theta+\tfrac{1}{2}\lambda)}{cos(\lambda)-cos(\theta)} - cos(\tfrac{1}{2}\lambda) \tag{6.5}$$

equation (6.1) becomes

$$v^{\Omega}_{t\zeta}(\theta) \sim \frac{M}{2\pi n}\Big[2sin(\tfrac{1}{2}\theta)J(\theta) - J_0\Big] \tag{6.6}$$

as $\mu \longrightarrow 0$, in which

$$J(\theta) = \int_{-\pi-2\beta}^{\pi-2\beta} \Big[cos(\beta+\tfrac{1}{2}\lambda)\Big]^{-1+4\cos^2(\beta)/n} e^{\lambda\sin(2\beta)/n} \frac{sin(\tfrac{1}{2}\theta+\tfrac{1}{2}\lambda)}{cos(\lambda)-cos(\theta)} d\lambda \tag{6.7}$$

$$J_0 = \int_{-\pi-2\beta}^{\pi-2\beta} \Big[cos(\beta+\tfrac{1}{2}\lambda)\Big]^{-1+4\cos^2(\beta)/n} e^{\lambda\sin(2\beta)/n} cos(\tfrac{1}{2}\lambda)\, d\lambda \tag{6.8}$$

Next using the transformations $y = \lambda + 2\beta$ and $\alpha = \beta + \frac{1}{2}\lambda$, the integrals (6.7) and (6.8) become respectively

$$J(\theta) = e^{-2\beta\sin(2\beta)/n} \int_{-\pi}^{\pi} \Big[cos(\tfrac{1}{2}y)\Big]^{-1+4\cos^2(\beta)/n} e^{y\sin(2\beta)/n} \frac{sin(\tfrac{1}{2}\theta+\tfrac{1}{2}y+\beta)}{cos(y)-cos(\theta+2\beta)} dy \tag{6.9}$$

$$J_0 = 2e^{-2\beta\sin(2\beta)/n} \int_{-\pi/2}^{\pi/2} \Big[cos(\alpha)\Big]^{-1+4\cos^2(\beta)/n} e^{2\alpha\sin(2\beta)/n} cos(\alpha-\beta)\, d\alpha \tag{6.10}$$

Since equation (6.10) is rather simple to evaluate we will treat this particular integral first; after that equation (6.9) will be dealt with.

Substituting the trigonometrical identity $cos(\alpha-\beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$ in equation (6.10) and then integrating by parts we obtain

$$J_0 = \frac{2e^{-2\beta\sin(2\beta)/n}}{cos(\beta)} \int_{-\pi/2}^{\pi/2} \Big[cos(\alpha)\Big]^{4\cos^2(\beta)/n} e^{2\alpha\sin(2\beta)/n}\, d\alpha \tag{6.11}$$

This equation can next be stated equivalently by the beta function. The equivalent reads (see also Gradshteyn & Ryzhik 1980 p.476)

$$J_o = \frac{2\pi e^{-2\beta \sin(2\beta)/n}}{cos(\beta)\left\{2^{4\cos^2(\beta)/n}\left[1+\frac{4\cos^2(\beta)}{n}\right] B(\chi,\bar{\chi})\right\}} \tag{6.12}$$

in which $B(-)$ is beta function (see for instance Abramowitz & Stegun 1972), and where

$$\chi = 1 + \frac{2cos^2(\beta)}{n} + i\,\frac{sin(2\beta)}{n} \tag{6.13}$$

Note that χ is a complex number; the beta function $B(\chi,\bar{\chi})$, however, is strictly real valued.

To evaluate integral $J(\theta)$ we rewrite equation (6.9) as

$$J(\theta) = e^{-2\beta\sin(2\beta)/n}\left[sin(\beta+\tfrac{1}{2}\theta)J_1(\theta) + cos(\beta+\tfrac{1}{2}\theta)J_2(\theta)\right] \tag{6.14}$$

where

$$J_1(\theta) = \fint_{-\pi}^{\pi} \left[cos(\tfrac{1}{2}y)\right]^{4\cos^2(\beta)/n} e^{y\sin(2\beta)/n}\,\frac{dy}{cos(y)-cos(\theta+2\beta)} \tag{6.15}$$

$$J_2(\theta) = \fint_{-\pi}^{\pi} \left[cos(\tfrac{1}{2}y)\right]^{-1+4\cos^2(\beta)/n} e^{y\sin(2\beta)/n}\,\frac{sin(\tfrac{1}{2}y)}{cos(y)-cos(\theta+2\beta)}\,dy \tag{6.16}$$

Integrating equation (6.16) by parts we may write the integral $J_2(\theta)$ alternatively as

$$J_2(\theta) = tan(\beta)J_1(\theta) + \frac{n}{2cos^2(\beta)}\,J_2^*(\theta) \tag{6.17}$$

in which

$$J_2^*(\theta) = \fint_{-\pi}^{\pi} \left[cos(\tfrac{1}{2}y)\right]^{4\cos^2(\beta)/n} e^{y\sin(2\beta)/n}\,\frac{sin(y)}{\left\{cos(y)-cos(\theta+2\beta)\right\}^2}\,dy \tag{6.18}$$

Then expanding the leading part of the integrals (6.15) and (6.18) in a Fourier series, i.e.

$$\left[cos(\tfrac{1}{2}y)\right]^{4\cos^2(\beta)/n} e^{y\sin(2\beta)/n} = \tfrac{1}{2}B_0 + \sum_{k=1}^{\infty} B_k cos(ky) + \sum_{k=1}^{\infty} C_k sin(ky) \tag{6.19}$$

where

$$B_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[cos(\tfrac{1}{2}\tau)\right]^{4\cos^2(\beta)/n} e^{\tau\sin(2\beta)/n} cos(k\tau)\, d\tau \tag{6.20}$$

$$C_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[cos(\tfrac{1}{2}\tau)\right]^{4\cos^2(\beta)/n} e^{\tau\sin(2\beta)/n} sin(k\tau)\, d\tau \tag{6.21}$$

and using equation (5.16) we first of all obtain

$$J_1(\theta) = \frac{2\pi}{sin(\theta+2\beta)} \sum_{k=1}^{\infty} B_k sin(k\theta+2k\beta) \tag{6.22}$$

Next, to evaluate the integral $J_2(\theta)$, or actually $J_2^*(\theta)$, we consider the principal value integral

$$Л(\vartheta) = \fint_{-\pi}^{\pi} \frac{sin(ky)\, sin(y)}{\left\{cos(y) - cos(\vartheta)\right\}^2}\, dy \tag{6.23}$$

Integrating by parts and using equation (5.16) we obtain

$$Л(\vartheta) = -\frac{2k\pi}{sin(\vartheta)}\; sin(k\vartheta) \tag{6.24}$$

Then substituting Fourier series (6.19) in equation (6.18) we get, using equation (6.24)

$$J_2^*(\theta) = -\frac{2\pi}{sin(\theta+2\beta)} \sum_{k=1}^{\infty} kC_k sin(k\theta+2k\beta) \tag{6.25}$$

so that equation (6.17) becomes, substituting equations (6.22) and (6.25)

$$J_2(\theta) = \frac{2\pi}{cos(\beta) sin(\theta+2\beta)} \left\{ sin(\beta) \sum_{k=1}^{\infty} B_k sin(k\theta+2k\beta) - \frac{n}{2cos(\beta)} \sum_{k=1}^{\infty} kC_k sin(k\theta+2k\beta) \right\} \tag{6.26}$$

Finally, substituting equations (6.22) and (6.26) in equation (6.14) we obtain

$$J(\theta)=\frac{\pi e^{-2\beta\sin(2\beta)/n}}{cos(\beta)sin(\beta+\frac{1}{2}\theta)}\times$$

$$\left\{\frac{sin(2\beta+\frac{1}{2}\theta)}{cos(\beta+\frac{1}{2}\theta)}\sum_{k=1}^{\infty}B_k sin(k\theta+2k\beta)-\frac{n}{2cos(\beta)}\sum_{k=1}^{\infty}kC_k sin(k\theta+2k\beta)\right\} \qquad (6.27)$$

The Fourier coefficients B_k and C_k in this equation, as given by equations (6.20) and (6.21), can be worked out somewhat further, simplifying their computation considerably. Combining the Fourier coefficients to the complex number B_k+iC_k yields

$$B_k+iC_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\left[cos(\tfrac{1}{2}\tau)\right]^{4\cos^2(\beta)/n}e^{\{\sin(2\beta)/n+ik\}\tau}d\tau \qquad (6.28)$$

This integral can be expressed alternatively by the beta function, which gives (see also Gradshteyn & Ryzhik 1980 p.476)

$$B_k+iC_k=\frac{2^{1-4\cos^2(\beta)/n}}{\left[1+\frac{4\cos^2(\beta)}{n}\right]B(\chi_1^{(k)},\chi_2^{(k)})} \qquad (6.29)$$

where

$$\chi_1^{(k)}=1+k+\frac{2cos^2(\beta)}{n}-i\frac{sin(2\beta)}{n} \qquad (6.30)$$

$$\chi_2^{(k)}=1-k+\frac{2cos^2(\beta)}{n}+i\frac{sin(2\beta)}{n} \qquad (6.31)$$

Hence, the Fourier coefficients B_k and C_k are conveniently given as the real and the imaginary part of the right-hand side of equation (6.29), being readily computable.

Finally, substituting equations (6.4), (6.12) and (6.27) in equation (6.6) we obtain

$$v^{\Omega}_{t\zeta}(\theta) \sim \frac{\Omega r_2^2}{n}\left[cos(\beta)\right]^{-4\cos^2(\beta)/n} e^{-2\beta\sin(2\beta)/n} \times$$

$$\left[\frac{sin(\frac{1}{2}\theta)}{sin(\beta+\frac{1}{2}\theta)}\left\{\frac{sin(2\beta+\frac{1}{2}\theta)}{cos(\beta+\frac{1}{2}\theta)}\sum_{k=1}^{\infty}B_k sin(k\theta+2k\beta)-\frac{n}{2cos(\beta)}\sum_{k=1}^{\infty}kC_k sin(k\theta+2k\beta)\right\}\right.$$

$$\left.-\frac{1}{2^{4\cos^2(\beta)/n}\left[1+\frac{4\cos^2(\beta)}{n}\right]B(\chi,\bar{\chi})}\right] \tag{6.32}$$

as $\mu \longrightarrow 0$, with χ according to equation (6.13) and the Fourier coefficients as given by equation (6.29).

Equation (6.32) gives the solution in closed-form of the fluid velocity tangential at the unit circle in the ζ-plane, due to the rotation of the impeller, for logarithmically bladed pump impellers as $\mu \longrightarrow 0$. Summing the individual contributions (6.2), (6.3), and (6.32) the overall velocity along the unit circle in the ζ-plane is obtained. Then by transformation (4.8) we obtain the fluid velocity in the physical plane; i.e. for logarithmic spiral blades

$$\frac{v_{tz}}{v_{t\zeta}} \sim \frac{n cos(\beta+\frac{1}{2}\theta)}{R(\theta)r_2}\frac{1}{|sin(\frac{1}{2}\theta)|} \tag{6.33}$$

as $\mu \longrightarrow 0$, where $R(\theta)$ is given by equation (3.15)

Next, substituting equations (6.2), (6.3), and (6.32) in equation (4.22) we obtain that the blade circulation of logarithmically bladed pump impellers is given by

$$n\Gamma_{bp} \sim \sigma_{p\Omega}\, 2\pi\Omega r_2^2 + Q tan(\beta) - \Gamma_1 \tag{6.34}$$

as $\mu \longrightarrow 0$, where the slip factor reads

$$\sigma_{p\Omega} \sim \frac{e^{-2\beta\sin(2\beta)/n}}{\left[1+\frac{4\cos^2(\beta)}{n}\right]\left[2cos(\beta)\right]^{4\cos^2(\beta)/n}B(\chi,\bar{\chi})} \tag{6.35}$$

as $\mu \longrightarrow 0$; this slip factor will be further discussed in section 7.

Summing equations (4.20), (6.2), (6.3), and (6.32), and using equation (6.34), we obtain the overall fluid velocity tangential at the unit circle in the ζ-plane

$$v_{t\zeta}(\theta) \sim -\frac{Q sin(\frac{1}{2}\theta)}{2\pi n cos(\beta) cos(\beta+\frac{1}{2}\theta)} + \frac{\Omega r_2^2}{n}\left[cos(\beta)\right]^{-4\cos^2(\beta)/n} e^{-2\beta\sin(2\beta)/n} \times$$

$$\frac{sin(\frac{1}{2}\theta)}{sin(\beta+\frac{1}{2}\theta)}\left\{\frac{sin(2\beta+\frac{1}{2}\theta)}{cos(\beta+\frac{1}{2}\theta)}\sum_{k=1}^{\infty} B_k sin(k\theta+2k\beta) - \frac{n}{2cos(\beta)}\sum_{k=1}^{\infty} kC_k sin(k\theta+2k\beta)\right\} \tag{6.36}$$

as $\mu \longrightarrow 0$. Note that in this equation the contribution of the vortex has vanished completely. This, however, is not surprising since we are dealing with pump impellers fitted with blades rising from the center.

Employing transformation (6.33), recalling that $r = r(\theta) = r_2 R(\theta)$, and using the auxiliary relations $v^+_{tz} = -v_s$ and $v^-_{tz} = +v_s$, equation (6.36) becomes

$$v_{sp}(\theta) \sim \frac{Q}{2\pi r cos(\beta)} - \frac{\Omega r_2^2}{r}\left[cos(\beta)\right]^{-4\cos^2(\beta)/n} e^{-2\beta\sin(2\beta)/n} cotan(\beta+\tfrac{1}{2}\theta) \times$$

$$\left\{\frac{sin(2\beta+\frac{1}{2}\theta)}{cos(\beta+\frac{1}{2}\theta)}\sum_{k=1}^{\infty} B_k sin(k\theta+2k\beta) - \frac{n}{2cos(\beta)}\sum_{k=1}^{\infty} kC_k sin(k\theta+2k\beta)\right\} \tag{6.37}$$

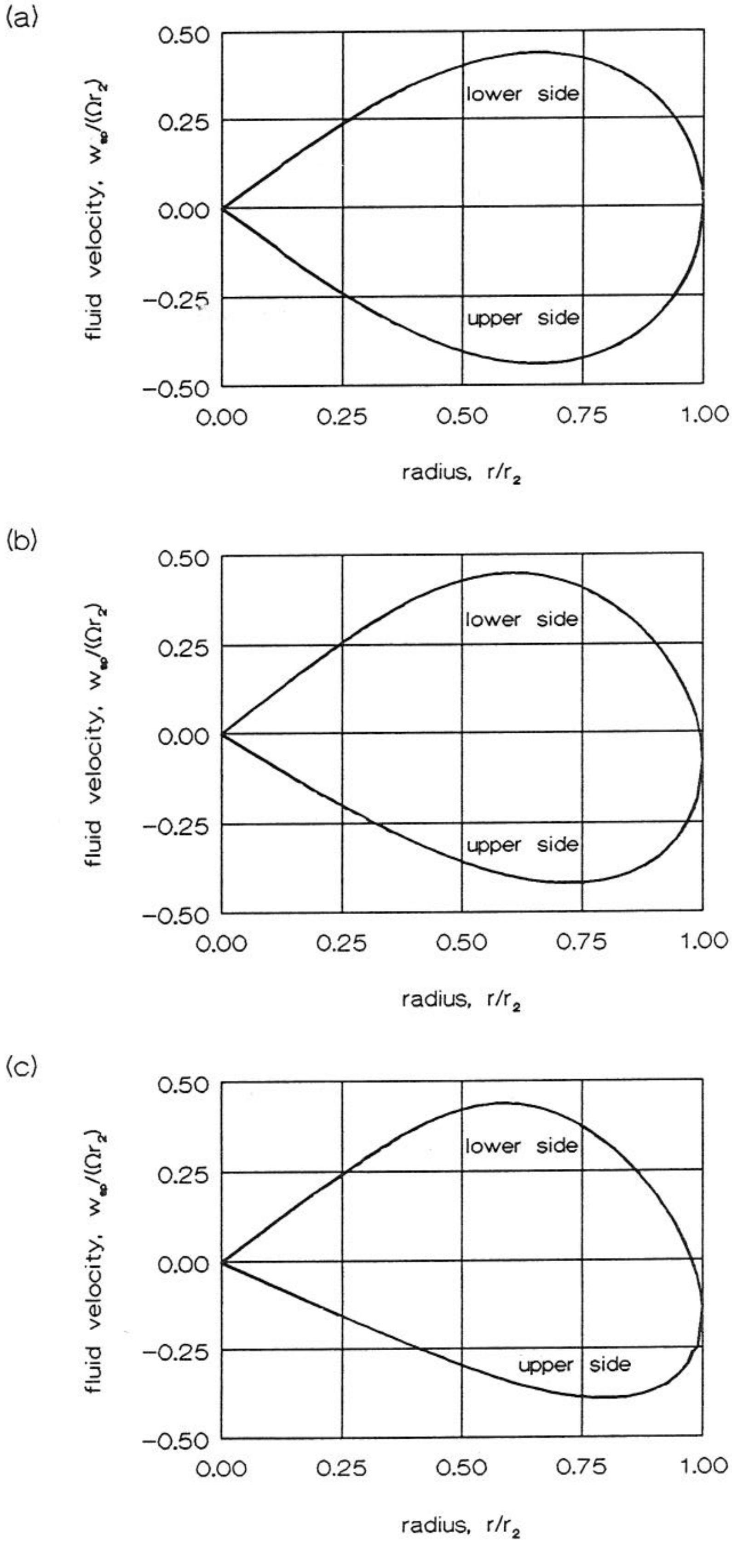
as $\mu \longrightarrow 0$, where v_{sp} is the *absolute* fluid velocity tangential at the blades of *pump* impellers, directed outwardly. The corresponding *relative* fluid velocity w_{sp} follows readily from the transformation $w_{sp} = v_{sp} - \Omega r sin(\beta)$. Hence we obtain, employing a dimensionless notation,

$$R(\theta)\frac{w_{sp}}{\Omega r_2} \sim \frac{\Phi}{cos(\beta)} - \{R(\theta)\}^2 sin(\beta) - \left[cos(\beta)\right]^{-4\cos^2(\beta)/n} e^{-2\beta\sin(2\beta)/n} cotan(\beta+\tfrac{1}{2}\theta) \times$$

$$\left\{\frac{sin(2\beta+\frac{1}{2}\theta)}{cos(\beta+\frac{1}{2}\theta)}\sum_{k=1}^{\infty} B_k sin(k\theta+2k\beta) - \frac{n}{2cos(\beta)}\sum_{k=1}^{\infty} kC_k sin(k\theta+2k\beta)\right\} \tag{6.38}$$

as $\mu \longrightarrow 0$, in which Φ is flow coefficient and $R(\theta)$ is dimensionless radius, as introduced before; note that equation (6.38) readily reduces to equation (5.31) in case of straight radial blades (i.e. $\beta = 0$).

Based on equation (6.38) we have plotted in figure 8 the relative fluid velocity (w_{sp}) due to rotation, that is, at zero throughput (i.e. $\Phi = 0$), for several 8–bladed pump impellers. The graphs of this figure are appropriate for both backwardly (i.e. $\Omega < 0$) and forwardly (i.e. $\Omega > 0$) curved blades. The figure clearly shows that the displacement flow velocity diminishes as the blade angle increases. This, and other features, will be discussed further in section 7.



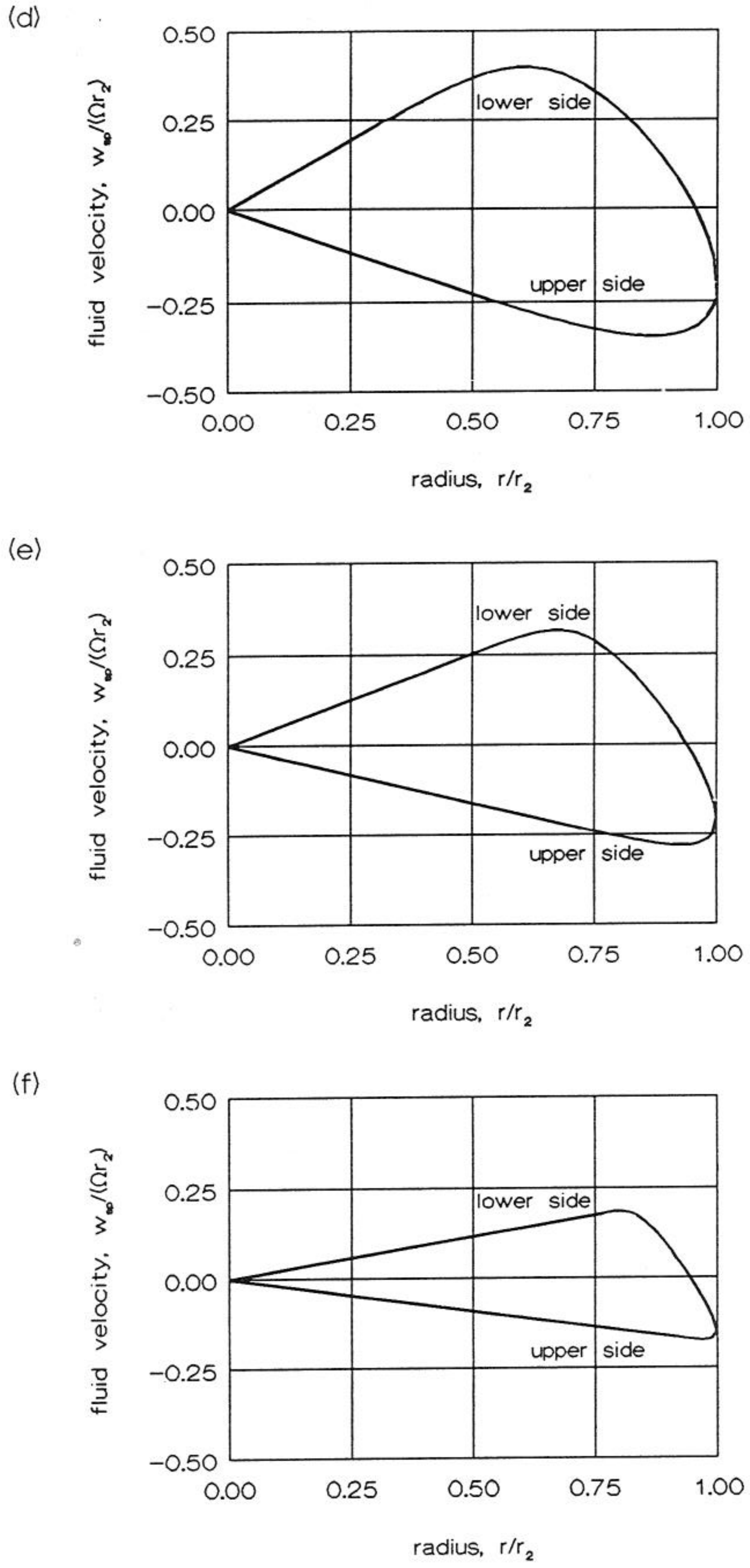


Figure 8. Displacement flow velocity along logarithmically curved blades of 8–bladed impellers as $\mu \rightarrow 0$.
(a) $\beta=0$; (b) $\beta=15^\circ$; (c) $\beta=30^\circ$; (d) $\beta=45^\circ$; (e) $\beta=60^\circ$; (f) $\beta=75^\circ$.

6.2. *Solutions from asymptotic expansion of Poisson equation*

In addition to the previous section, we shall discuss here the asymptotic behaviour of the solution for the relative fluid velocity directly, that is, by asymptotic expansion of the Poisson equation. In particular the behaviour in a region located remotely, i.e. $(1+\frac{2\pi\cos(\beta)}{n})r_1 \ll r \ll (1-\frac{2\pi\cos(\beta)}{n})r_2$, between the blade tips of logarithmically bladed impellers will be examined. To that end we will consider the Laplacian of the stream function for the relative flow. Unlike the absolute flow, this flow is stationary, and, hence, it is easier to describe than the absolute flow, which is periodical.

Denoting the relative stream function by κ, it is found that (see for instance Kucharski 1918 p. 73 or Vavra 1960 p. 226)

$$\nabla^2\kappa = 2\Omega \tag{6.39}$$

Next, we introduce the transformation

$$\xi = cos(\beta)ln\left[\frac{r}{r_0}\right] + \phi sin(\beta) \tag{6.40}$$

$$\eta = -sin(\beta)ln\left[\frac{r}{r_0}\right] + \phi cos(\beta) \tag{6.41}$$

in which r_0 is arbitrary scale factor. By this transformation, logarithmical spiral blades are conveniently described by $\eta = \text{constant}$, following readily from equations (2.2) and (6.41). In particular, we obtain from equations (6.40) and (6.41) after substituting equation (2.2) that the j^{th}-blade is described in the (ξ,η)-plane by

$$\xi^{j}(r) = \frac{1}{cos(\beta)}\, ln\left[\frac{r}{r_0}\right] + D_\xi(j) \tag{6.42}$$

and

$$\eta^{j} = D_\eta(j) \tag{6.43}$$

in which $D_\xi(j)$ and $D_\eta(j)$ are constant for a particular blade, viz.

$$D_\xi(j) = \left\{\phi_{01} + 2\pi\frac{j-1}{n} + tan(\beta)ln\left[\frac{r_0}{r_2}\right]\right\} sin(\beta) \tag{6.44}$$

$$D_\eta(j)=\left\{\phi_{01}+2\pi\frac{j-1}{n}+tan(\beta)ln\left[\frac{r_0}{r_2}\right]\right\}cos(\beta) \tag{6.45}$$

With transformation (6.40) – (6.41), equation (6.39) becomes

$$\frac{\partial^2\kappa}{\partial\xi^2}+\frac{\partial^2\kappa}{\partial\eta^2}=2\Omega r_0^2e^{2\xi\cos(\beta)-2\eta\sin(\beta)} \tag{6.46}$$

Then employing scale coordinates $(\xi_*,\eta_*)\in[0,1]$, e.g. defined by

$$\xi_*=\frac{\xi-\xi_1}{\xi_2-\xi_1} \tag{6.47}$$

$$\eta_*=\frac{\eta-\eta^{\mathrm{j}}}{\eta^{\mathrm{j}+1}-\eta^{\mathrm{j}}} \tag{6.48}$$

where $\xi_1=\xi^{\mathrm{j}}(r_1)$ and $\xi_2=\xi^{\mathrm{j}}(r_2)$, yielding

$$\xi=\xi_1+\frac{\xi_*}{cos(\beta)}ln\left[\frac{r_2}{r_1}\right] \tag{6.49}$$

$$\eta=\eta^{\mathrm{j}}+\frac{2\pi cos(\beta)}{n}\eta_* \tag{6.50}$$

which readily results, using equations (6.42), (6.43), (6.44), and (6.45), we may write equation (6.46) alternatively in a dimensionless form as

$$\left[\frac{n}{2\pi cos(\beta)}\right]^2\left\{\left[\frac{2\pi cos^2(\beta)}{nln(r_2/r_1)}\right]^2\frac{\partial^2\kappa_*}{\partial\xi_*^2}+\frac{\partial^2\kappa_*}{\partial\eta_*^2}\right\}=2\Omega_*\frac{e^{2\xi_1\cos(\beta)+2\xi_*\ln(r_2/r_1)}}{e^{2\eta^{\mathrm{j}}\sin(\beta)+2\pi\eta_*\sin(2\beta)}} \tag{6.51}$$

where we have put additionally $\kappa=\kappa_*\Omega_0r_0^2$ and $\Omega=\Omega_*\Omega_0$.

Finally, incorporating

$$\left[\frac{2\pi cos^2(\beta)}{nln(r_2/r_1)}\right]^2\sim 0 \tag{6.52}$$

as $\mu\longrightarrow 0$, we get from equation (6.51), restoring physical dimensions,

$$\frac{\partial^2 \kappa}{\partial \eta^2} \sim 2\Omega r_0^2 e^{2\xi\cos(\beta)-2\eta\sin(\beta)} \tag{6.53}$$

as $\mu \longrightarrow 0$.

The solution we obtain reads

$$\kappa(\xi,\eta) \sim \kappa^{j} + \frac{Q}{2\pi}\left[\frac{\eta}{cos(\beta)} - \phi_0^{j}\right] + \frac{\Omega r_2^2 e^{2\xi\cos(\beta)}}{2sin^2(\beta)}\left[e^{-2\eta\sin(\beta)} - e^{-\phi_0^{j}\sin(2\beta)}\right] +$$

$$\frac{n\Omega r_2^2 e^{2\xi\cos(\beta)}}{4\pi sin^2(\beta)}\left[\frac{\eta}{cos(\beta)} - \phi_0^{j}\right]\left[e^{-\phi_0^{j}\sin(2\beta)} - e^{-\phi_0^{j+1}\sin(2\beta)}\right] \tag{6.54}$$

as $\mu \longrightarrow 0$, in which we have used the boundary conditions $\kappa(\xi,\eta^{j}) = \kappa^{j}$ and $\kappa(\xi,\eta^{j+1}) = \kappa^{j+1}$, with $\kappa^{j+1} - \kappa^{j} = Q/n$, and where we have put $r_0 = r_2$ for convenience, so that $D_\xi(j) = \phi_0^{j} sin(\beta)$ and $D_\eta(j) = \phi_0^{j} cos(\beta)$.

Equation (6.54) gives the asymptotic behaviour of the stream function for the relative flow between consecutive blades, viz. the blades j and $j+1$, of logarithmically bladed impellers, in a region at a distance from the blade tips, which is (much) larger than the distance between consecutive blades; in fact $(1+\frac{2\pi\cos(\beta)}{n})r_1 \ll r \ll (1-\frac{2\pi\cos(\beta)}{n})r_2$.

Next, the tangential and normal fluid velocities w_t and w_n respectively, defined by

$$w_t = w_r cos(\beta) + w_\phi sin(\beta) \tag{6.55}$$

$$w_n = -w_r sin(\beta) + w_\phi cos(\beta) \tag{6.56}$$

where $w_r = \frac{1}{r}\frac{\partial \kappa}{\partial \phi}$ and $w_\phi = -\frac{\partial \kappa}{\partial r}$, may be obtained from

$$w_t = -\frac{1}{r}\frac{\partial \kappa}{\partial \eta} \tag{6.57}$$

$$w_n = --\frac{1}{r}\frac{\partial \kappa}{\partial \xi} \tag{6.58}$$

which follows directly from transformation (6.40) – (6.41).

For convenience we will give the velocity distribution along an imaginary, logarithmical spiral lying between consecutive impeller blades, say j and $j+1$, being curved exactly like the impeller blades. Form equations (6.42), (6.43), (6.44), and (6.45) we find that a point on such a spiral may be characterized by, again putting $r_0 = r_2$, so that $D_\xi(j) = \phi_0^j sin(\beta)$ and $D_\eta(j) = \phi_0^j cos(\beta)$,

$$\xi^\alpha = \frac{1}{cos(\beta)} ln\left[\frac{r}{r_2}\right] + \phi_0^\alpha sin(\beta) \tag{6.59}$$

and

$$\eta^\alpha = \phi_0^\alpha cos(\beta) \tag{6.60}$$

where $j \leq \alpha \leq j+1$.

The fluid velocities then become

$$w_t(r,\lambda) \sim \frac{\Omega r}{sin(\beta)}\left[\frac{sinh(t)}{t} e^{\lambda \sin(2\beta) - t} - 1\right] + \frac{Q}{2\pi r cos(\beta)} \tag{6.61}$$

$$w_n(r,\lambda) \sim -\frac{\Omega r cos(\beta)}{sin^2(\beta)}\left\{1 - e^{\lambda \sin(2\beta)} + \frac{n\lambda}{\pi} sinh(t)\, e^{\lambda \sin(2\beta) - t}\right\} \tag{6.62}$$

as $\mu \longrightarrow 0$, in which $\lambda = \phi_0^\alpha - \phi_0^j$ and $t = \pi sin(2\beta)/n$, and where we have used $\phi_0^{j+1} - \phi_0^j = 2\pi/n$.

Finally, we derive from equations (6.61) and (6.62) the fluid velocities along the blades. Taking successively $\alpha = j$ (i.e. $\lambda = 0$) and $\alpha = j+1$ (i.e. $\lambda = 2\pi/n$) we find for the upper blade surfaces, employing a dimensionless notation,

$$\frac{w_t^+}{\Omega r_2} \sim \frac{\Phi}{R cos(\beta)} + \frac{R}{sin(\beta)}\left[\frac{sinh(t)}{t} e^{-t} - 1\right] \tag{6.63}$$

$$\frac{w_n^+}{\Omega r_2} \sim 0 \tag{6.64}$$

and for the lower blade surfaces

$$\frac{w_t^-}{\Omega r_2} \sim \frac{\Phi}{R cos(\beta)} + \frac{R}{sin(\beta)}\left[\frac{sinh(t)}{t} e^{t} - 1\right] \tag{6.65}$$

$$\frac{w_n^-}{\Omega r_2} \sim 0 \tag{6.66}$$

as $\mu \longrightarrow 0$, where $R = r/r_2$.

Note, in particular, that for $r \ll r_2$ (i.e. $R \ll 1$) solutions (6.63) and (6.65) perfectly agree with solution (6.38), which was illustrated in figure 8. Furthermore, it is seen that the normal velocities (w_n) comply properly with the boundary conditions.

From equations (6.63) and (6.65) we further obtain, restoring physical dimensions

$$w_t^+ \sim \frac{Q}{2\pi r \cos(\beta)} - \frac{2\pi \Omega r \cos(\beta)}{n} \tag{6.67}$$

$$w_t^- \sim \frac{Q}{2\pi r \cos(\beta)} + \frac{2\pi \Omega r \cos(\beta)}{n} \tag{6.68}$$

as $n \longrightarrow \infty$ (i.e. $t \longrightarrow 0$).

Equations (6.67) and (6.68) clearly indicate that a negative velocity contribution is to be expected along the (pressure side of the) blades due to the revolution of the impeller. This is commonly interpreted as being the result of a relative eddy located between consecutive blades, which, basically, originates from the irrotationality of the absolute flow. Consequently, the relative flow possesses a constant vorticity equal to -2Ω (i.e. $\nabla \times w = -2\Omega$).