Fm 8356.

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Hydrodynamic instabilities in cylindrical thermocapillary liquid bridges

Appendix

Here we list the various abbreviations and integrals that enter the equation for the basic steady two-dimensional flow ((3.4)-(3.7)) and the linear stability problem ((4.6)-(4.16)).

The matrices $(A,B,D,E)_{klnm}$, $\hat{K}_{kl}$, and the non-linearities $C_{klijnm}\Psi_{ij}^{(o)}\Psi_{nm}^{(o)}$ and $F_{klijnm}\Psi_{ij}^{(o)}\hat{\theta}_{nm}^{(o)}$ that are required for the basic state are defined as follows. Summation is over all pairwise indices.

$$
\begin{aligned}
A_{klnm} &= \frac{1}{\Gamma}\Big\{\alpha_1(l,m)\beta_1(k,n)+2\Gamma^2\alpha_2(l,m)\big[\beta_2(k,n)+ \\
&\quad +\beta_3(k,n)-\beta_4(k,n)\big]+\Gamma^4\delta_{l,m}\Big[\beta_5(k,n)+2\beta_6(k,n)- \\
&\quad -3\beta_7(k,n)+3\beta_8(k,n)-3\frac{\pi}{2}c_k\delta_{n,k}\Big]\Big\};\ k\le N-4 \\
B_{klnm} &= \frac{Gr}{Re}\alpha_5(l,m)\beta_9(k,n);\ k\le N-4 \\
D_{klnm} &= \alpha_5(m,l)\Big\{\beta_8(k,n)+\frac{\pi}{2}c_k\delta_{k,n}\Big\};\quad k\le N-2 \\
E_{klnm} &= \frac{\Gamma}{RePr}\delta_{l,m}\Big\{\beta_{10}(k,n)+\beta_{11}(k,n)-\bar{m}^2\frac{\pi^2}{\Gamma^2}\beta_{12}(k,n)\Big\};\quad k\le N-2 \\
\hat{K}_{kl} &= -\frac{4}{\Gamma^2}\delta_{k,N}\alpha_4(l)
\end{aligned}
\qquad \text{(A.1a-e)}
$$

The non-linear terms are given by

$$
\begin{aligned}
C_{klijnm}\Psi_{ij}^{(o)}\Psi_{nm}^{(o)} &= Re\Big\{\Gamma^2\Big[\zeta_{ljn}^{(1)}\sigma_{kjn}^{(1)}+\zeta_{ljn}^{(2)}\sigma_{kjn}^{(2)}\Big]+ \\
&\quad +\zeta_{ljn}^{(3)}\sigma_{kjn}^{(3)}+\zeta_{ljn}^{(4)}\sigma_{kjn}^{(4)}\Big\}, \\
F_{klijnm}\Psi_{ij}^{(0)}\hat{\theta}_{nm}^{(o)} &= \xi_{ljn}^{(1)}\pi_{kjn}^{(1)}+\xi_{ljn}^{(2)}\pi_{kjn}^{(2)},
\end{aligned}
\qquad \text{(A.2a-b)}
$$

where

$$\zeta^{(1)}_{ljn} = \tau^{(2)}_{lmj}\psi^{(o)}_{nm}, \qquad \tau^{(1)}_{kjn} = \Gamma^{(3)}_{kin}\psi^{(o)}_{ij},$$

$$\zeta^{(2)}_{ljn} = \tau^{(2)}_{ljm}\psi^{(o)}_{nm}, \qquad \tau^{(2)}_{kjn} = -\Gamma^{(4)}_{kin}\psi^{(o)}_{ij},$$

$$\zeta^{(3)}_{ljn} = \tau^{(1)}_{ljm}\psi^{(o)}_{nm}, \qquad \tau^{(3)}_{kjn} = \Gamma^{(5)}_{kin}\psi^{(o)}_{ij},$$

$$\zeta^{(4)}_{ljn} = \tau^{(3)}_{ljm}\psi^{(o)}_{nm}, \qquad \tau^{(4)}_{kjn} = -\Gamma^{(6)}_{kin}\psi^{(o)}_{ij}, \qquad \text{(A.3a-h)}$$

and

$$\xi^{(1)}_{ljn} = \kappa^{(1)}_{ljm}\hat{\theta}^{(0)}_{nm}, \qquad \pi^{(1)}_{kjn} = -\Gamma^{(1)}_{kin}\psi^{(o)}_{ij},$$

$$\xi^{(2)}_{ljn} = \kappa^{(2)}_{ljm}\hat{\theta}^{(0)}_{nm}, \qquad \pi^{(2)}_{kjn} = \Gamma^{(2)}_{kin}\psi^{(o)}_{ij}, \qquad \text{(A.4a-d)}$$

with abbreviations

$$\left.\begin{aligned}
\Gamma^{(1)}_{kin} &= \gamma_9(k,n,i)\\
\Gamma^{(2)}_{kin} &= \gamma_9(k,i,n)+\gamma_{10}(k,i,n)
\end{aligned}\right\} \qquad k \le N-2$$

$$\left.\begin{aligned}
\Gamma^{(3)}_{kin} &= \gamma_1(k,i,n)+\gamma_6(k,i,n)+\gamma_4(k,i,n)-\\
&\quad -\gamma_7(k,i,n)+\gamma_7(k,n,i)-\gamma_8(k,i,n),\\
\Gamma^{(4)}_{kin} &= \gamma_3(k,i,n)-3\gamma_7(k,n,i)+3\gamma_8(k,i,n),\\
\Gamma^{(5)}_{kin} &= \gamma_2(k,i,n)+\gamma_5(k,i,n),\\
\Gamma^{(6)}_{kin} &= \gamma_2(k,n,i)-\gamma_5(k,i,n),
\end{aligned}\right\} \qquad k \le N-4$$

$$\Gamma^{(1)}_{kin} = \Gamma^{(2)}_{kin} = 0; \qquad k = N-1,\ N$$

$$\Gamma^{(3)}_{kin} = \Gamma^{(4)}_{kin} = \Gamma^{(5)}_{kin} = \Gamma^{(6)}_{kin} = 0; \qquad k > N-4. \qquad \text{(A.5a-h)}$$

We have used the following integrals over axial basis functions ($< > = \int_{-\frac{1}{2}}^{\frac{1}{2}} dz$ denotes the scalar product).

$$\alpha_1(l,m) = <H_l H^{IV}_m> \qquad \alpha_6(l,m) = <R_l H^{II}_m>$$

$$\alpha_2(l,m) = <H_l H^{II}_m> \qquad \alpha_7(l,m) = <R_l R^{I}_m>$$

$$\alpha_3(l,m) = <H_l R^{I}_m> \qquad \alpha_8(l,m) = <R_l H^{IV}_m>$$

$$\alpha_4(l) = <H_l> \qquad \alpha_9(l,m) = <R_l R^{III}_m>$$

$$\alpha_5(l,m) = <H_l R_m> \qquad \alpha_{10}(l,m) = <R_l H^{I}_m>$$

$$\tau_1(l,j,m) = <H_l H_j H^{III}_m> \qquad \tau_3(l,j,m) = <H_l H^{I}_j H^{II}_m>$$

$$\tau_2(l,j,m) = \langle H_l H_j^I H_m \rangle$$

$$\kappa_1(l,j,m) = \langle R_l H_j^I R_m \rangle \qquad \kappa_4(l,j,m) = \langle R_l H_j^{II} R_m \rangle$$

$$\kappa_2(l,j,m) = \langle R_l H_j R_m^I \rangle \qquad \kappa_5(l,j,m) = \langle R_l H_j^I R_m^I \rangle$$

$$\kappa_3(l,j,m) = \langle R_l H_j R_m \rangle \qquad \eta_1(l,j,m) = \langle R_l R_j R_m \rangle$$

$$\phi_1(l,j,m) = \langle R_l H_j H_m^I \rangle \qquad \phi_3(l,j,m) = \langle R_l H_j^I H_m^{II} \rangle$$

$$\phi_2(l,j,m) = \langle R_l H_j^I H_m^I \rangle \qquad \phi_4(l,j,m) = \langle R_l H_j H_m^{III} \rangle \qquad \text{(A.6a-v)}$$

The superscript denotes the axial derivatives, e.g. $R_m^I = \dfrac{dR_m}{dz}$. Integrals involving radial basis functions are

$$\beta_1(k,n) = \langle T_k \rho^4 T_n \rangle \qquad \beta_7(k,n) = \langle T_k \rho^2 T_n^{II} \rangle$$

$$\beta_2(k,n) = \langle T_k \rho^4 T_n^{II} \rangle \qquad \beta_8(k,n) = \langle T_k \rho T_n^I \rangle$$

$$\beta_3(k,n) = \langle T_k \rho^3 T_n^I \rangle \qquad \beta_9(k,n) = \langle T_k \rho^4 T_n^I \rangle$$

$$\beta_4(k,n) = \langle T_k \rho^2 T_n \rangle \qquad \beta_{10}(k,n) = \langle T_k \rho T_n^{II} \rangle$$

$$\beta_5(k,n) = \langle T_k \rho^4 T_n^{IV} \rangle \qquad \beta_{11}(k,n) = \langle T_k T_n^I \rangle$$

$$\beta_6(k,n) = \langle T_k \rho^3 T_n^{III} \rangle \qquad \beta_{12}(k,n) = \langle T_k \rho T_n \rangle \qquad \text{(A.7a-l)}$$

and

$$\gamma_1(k,i,n) = \langle \rho^4 T_k T_i^I T_n^{II} \rangle \qquad \gamma_6(k,i,n) = \langle \rho^3 T_k T_i T_n^{II} \rangle$$

$$\gamma_2(k,i,n) = \langle \rho^4 T_k T_i^I T_n \rangle \qquad \gamma_7(k,i,n) = \langle \rho^2 T_k T_i^I T_n \rangle$$

$$\gamma_3(k,i,n) = \langle \rho^4 T_k T_i T_n^{III} \rangle \qquad \gamma_8(k,i,n) = \langle \rho T_k T_i T_n \rangle$$

$$\gamma_4(k,i,n) = \langle \rho^3 T_k T_i^I T_n^I \rangle \qquad \gamma_9(k,i,n) = \langle \rho T_k T_i^I T_n \rangle$$

$$\gamma_5(k,i,n) = \langle \rho^3 T_k T_i T_n \rangle \qquad \gamma_{10}(k,i,n) = \langle T_k T_i T_n \rangle \qquad \text{(A.8a-j)}$$

Here $\rho = 1+x$, the superscript denotes radial derivatives, and $\langle\ \rangle$ is the integral with standard Chebyshev weight and normalization

$$\langle T_k T_n \rangle = \frac{\pi}{2} c_k \delta_{k,n} , \quad where \quad c_k = \begin{cases} 2, & if\ k = 0 \\ 1, & if\ k \neq 0 \end{cases} \qquad \text{(A.9)}$$

If feasible, integrals over axial functions have been calculated analytically. The remaining integrals have been computed numerically using a 3-point Gaussian integration scheme with relative accuracy of 10^{-10}. Integrals over Chebyshev functions have been calculated utilizing addition theorems and calculating derivatives in Chebyshev space.

The sub-matrices A^{ij} and B^{ij} that enter the stability equations are defined in the following, where the abbreviations $\Gamma = 2\tilde{\Gamma}$, $\rho_k = 1+x_k$ *and* $T_n(k) := T_n(x_k)$ (c.f. (4.18)) are used.

$$A^{11}(k,l,n,m) = -\alpha_7(l,m)\frac{1}{\Gamma}\left\{\rho_k^2 T_n^I(k)+\rho_k T_n(k)\right\}$$

$$A^{12}(k,l,n,m) = T_n(k)\left\{\bar{m}^2\alpha_5(m,l)-\alpha_6(l,m)\frac{\rho_k^2}{\Gamma^2}\right\}$$

$$A^{13}(k,l,n,m) = 0$$

$$A^{21}(k,l,n,m) = \alpha_7(l,m)T_n(k)$$

$$A^{22}(k,l,n,m) = -\alpha_5(m,l)\Gamma T_n^I(k)$$

$$A^{23}(k,l,n,m) = 0$$

$$A^{31}(k,l,n,m) = 0$$

$$A^{32}(k,l,n,m) = 0$$

$$A^{33}(k,l,n,m) = \delta_{l,m}T_n(k) \qquad \text{(A.10a-i)}$$

and

$$\begin{aligned} B^{11}(k,l,n,m) = {} & \Pi^{11}(k,l,n,m)+Re\psi_{ij}^{(o)}\ \{\kappa_3(l,j,m) \\ & \left[\Omega_1^{11}(k,i,n)+m^2\pi^2\Omega_2^{11}(k,i,n)\right] \\ & +\kappa_4(l,j,m)\Omega_3^{11}(k,i,n) \\ & +\kappa_5(l,j,m)\Omega_4^{11}(k,i,n)\} \end{aligned}$$

$$\begin{aligned} B^{12}(k,l,n,m) = {} & \Pi^{12}(k,l,n,m)+Re\psi_{ij}^{(o)}\ \{\phi_1(l,j,m)\Omega_1^{12}(k,i,n)+ \\ & +\ \phi_1(l,m,j)\Omega_2^{12}(k,i,n)+\phi_3(l,j,m)\Omega_3^{12}(k,i,n)+ \\ & +\phi_3(l,m,j)\Omega_4^{12}(k,i,n)+\phi_4(l,j,m)\Omega_5^{12}(k,i,n)\} \end{aligned}$$

$$B^{13}(k,l,n,m) = \bar{m}^2 \frac{Gr}{Re} \delta_{l,m} T_n(k)$$

$$B^{21}(k,l,n,m) = \Pi^{21}(k,l,n,m) + Re\hat{\psi}_{ij}^{(o)} \{\kappa_3(l,j,m) \left[\Omega_1^{21}(k,i,n) + m^2\pi^2\Omega_2^{21}(k,i,n)\right] + \kappa_4(l,j,m)\Omega_3^{21}(k,i,n) + \kappa_5(l,j,m)\Omega_4^{21}(k,i,n)\}$$

$$B^{22}(k,l,n,m) = \Pi^{22}(k,l,n,m) + Re\hat{\psi}_{ij}^{(o)} \{\phi_1(l,j,m)\Omega_1^{22}(k,i,n) + \phi_1(l,m,j)\Omega_2^{22}(k,i,n) + (\phi_3(l,m,j) + \phi_4(l,m,j))\Omega_3^{22}(k,i,n)\}$$

$$B^{23}(k,l,n,m) = -\Gamma \frac{Gr}{Re} \delta_{l,m} T_n^I(k)$$

$$B^{31}(k,l,n,m) = Re\hat{\theta}_{ij}^{(o)} \eta_1(l,j,m)\Omega_1^{31}(k,i,n)$$

$$B^{32}(k,l,n,m) = Re\{\Pi^{32}(k,l,n,m) + \kappa_2(l,m,j)\Omega_1^{32}(k,i,n)\hat{\theta}_{ij}^{(o)}\}$$

$$B^{33}(k,l,n,m) = \Pi^{33}(k,l,n,m) + Re\hat{\psi}_{ij}^{(o)} \{\kappa_1(l,j,m)\Omega_1^{33}(k,i,n) + \kappa_2(l,j,m)\Omega_2^{33}(k,i,n)\} \qquad \text{(A.11a-i)}$$

In the equations for B^{ij}, the indices j,l,m run from 1 through M and i,n run from 0 through N. The index k runs from 0 through N-2 except for B^{21}, B^{22}, and B^{23}, where it runs from 0 through N-3. The abbreviations used in the above equations for B^{ij} are

$$\Pi^{11}(k,l,n,m) = -\alpha_7(l,m)\Gamma\left[\rho_k^2 T_n^{III}(k) + 4\rho_k T_n^{II}(k) + (\bar{m}^2-1)\left(\frac{T_n(k)}{\rho_k} - T_n^I(k)\right)\right] - \frac{\alpha_9(l,m)}{\Gamma}\left[\rho_k^2 T_n^I(k) + \rho_k T_n(k)\right]$$

$$\Pi^{12}(k,l,n,m) = \bar{m}^2\alpha_5(m,l)\Gamma^2\left[T_n^{II}(k) + \frac{T_n^I(k)}{\rho_k} - \bar{m}^2\frac{T_n(k)}{\rho_k^2}\right] - \alpha_6(l,m)\left[\rho_k^2 T_n^{II}(k) + 3\rho_k T_n^I(k) - 2\bar{m}^2 T_n(k)\right] - \alpha_8(l,m)\frac{\rho_k^2 T_n(k)}{\Gamma^2}$$

$$\Pi^{21}(k,l,n,m) = \alpha_6(l,m)\Gamma\left[2\frac{T_n(k)}{\rho_k} - T_n^I(k)\right] + \alpha_5(m,l)\Gamma^3$$

$$\left[\bar{m}^2\left(\frac{T_n^I(k)}{\rho_k^2} - \frac{2T_n(k)}{\rho_k^3}\right) - T_n^{III}(k) - \frac{T_n^{II}(k)}{\rho_k} + \frac{T_n^I(k)}{\rho_k^2}\right]$$

$$\Pi^{32}(k,l,n,m) = -\alpha_5(m,l)T_n(k)$$

$$\Pi^{33}(k,l,n,m) = \frac{\Gamma^2}{Pr}\delta_{l,m}\left[T_n^{II}(k) + \frac{T_n^I(k)}{\rho_k} - \bar{m}^2\frac{T_n(k)}{\rho_k^2} - \frac{m^2\pi^2}{\Gamma^2}T_n(k)\right] \qquad \text{(A.12a-e)}$$

and

$$\Omega_1^{11}(k,i,n) = \bar{m}^2\Gamma^2\left(T_i^{II}(k) + \frac{T_i^I(k)}{\rho_k} - \frac{T_i(k)}{\rho_k^2}\right)T_n(k)$$

$$\Omega_2^{11}(k,i,n) = \left(\rho_k T_i^I(k) + T_i(k)\right)\left(\rho_k T_n^I(k) + T_n(k)\right)$$

$$\Omega_3^{11}(k,i,n) = \left(\rho_k T_n^{II}(k) + 3T_n^I(k) + \frac{T_n(k)}{\rho_k}\right)\rho_k T_i(k)$$

$$\Omega_4^{11}(k,i,n) = \rho_k\left[T_i(k)\left(\rho_k T_n^{II}(k) + 2T_n^I(k)\right) - \right.$$
$$\left. - \rho_k T_i^I(k)\left(T_n^I(k) + \frac{T_n(k)}{\rho_k}\right)\right]$$

$$\Omega_1^{12}(k,i,n) = \bar{m}^2\Gamma T_n(k)\left(T_i^I(k) + \frac{T_i(k)}{\rho_k}\right)$$

$$\Omega_2^{12}(k,i,n) = \bar{m}^2\Gamma\left[T_n(k)\left(T_i^I(k) + \frac{T_i(k)}{\rho_k}\right) - T_i(k)T_n^I(k)\right]$$

$$\Omega_3^{12}(k,i,n) = \frac{\rho_k^2}{\Gamma}\left[T_i(k)T_n^I(k) - T_i^I(k)T_n(k) + \frac{T_i(k)T_n(k)}{\rho_k}\right]$$

$$\Omega_4^{12}(k,i,n) = \frac{\rho_k}{\Gamma}T_i(k)\left[\rho_k T_n^I(k) + 2T_n(k)\right]$$

$$\Omega_5^{12}(k,i,n) = -\frac{T_n(k)}{\Gamma}\rho_k^2\left[T_i^I(k) + \frac{T_i(k)}{\rho_k}\right]$$

$$\Omega_1^{21}(k,i,n) = -\Gamma^3\left\{T_n^I(k)\left[T_i^{II}(k)+\frac{T_i^I(k)}{\rho_k}-\frac{T_i(k)}{\rho_k^2}\right]\right.$$

$$\left.+T_n(k)\left[T_i^{III}(k)+\frac{T_i^{II}(k)}{\rho_k}-2\frac{T_i^I(k)}{\rho_k^2}+2\frac{T_i(k)}{\rho_k^3}\right]\right\}$$

$$\Omega_2^{21}(k,i,n) = -\Gamma T_n(k)\left[T_i^I(k)+\frac{T_i(k)}{\rho_k}\right]$$

$$\Omega_3^{21}(k,i,n) = -\Gamma\left[T_i^I(k)T_n(k)+T_i(k)T_n^I(k)\right]$$

$$\Omega_4^{21}(k,i,n) = \Gamma T_i(k)\left[\frac{T_n(k)}{\rho_k}-T_n^I(k)\right]$$

$$\Omega_1^{22}(k,i,n) = -\Gamma^2\left\{T_n(k)\left[T_i^{II}(k)+\frac{T_i^I(k)}{\rho_k}-\frac{T_i(k)}{\rho_k^2}\right]+\right.$$

$$\left.+T_n^I(k)\left[T_i^I(k)+\frac{T_i(k)}{\rho_k}\right]\right\}$$

$$\Omega_2^{22}(k,i,n) = -\Gamma^2\left\{T_i(k)\left[\frac{T_n^I(k)}{\rho_k}-T_n^{II}(k)\right]+T_n(k)\right.$$

$$\left.\left[T_i^{II}(k)+\frac{T_i^I(k)}{\rho_k}-\frac{T_i(k)}{\rho_k^2}\right]\right\}$$

$$\Omega_3^{22}(k,i,n) = -T_i(k)T_n(k)$$

$$\Omega_1^{31}(k,i,n) = -\Gamma T_i^I(k)T_n(k)$$

$$\Omega_1^{32}(k,i,n) = T_i(k)T_n(k)$$

$$\Omega_1^{33}(k,i,n) = -\Gamma T_i(k)T_n^I(k)$$

$$\Omega_2^{33}(k,i,n) = \Gamma T_n(k)\left[T_i^I(k)+\frac{T_i(k)}{\rho_k}\right] \qquad \text{(A.13a-t)}$$