

Appendix A: The functions of F_1 and F_2

$$\begin{aligned}
 F_1 = & -2\left(\frac{m^2}{h^2} - 2m + 1\right)\frac{\partial^3\psi}{\partial r^3} + 2\kappa \sin\theta\left\{\frac{2h^2}{m^2r^2}\frac{\partial^3\psi}{\partial\theta^3} + 2\frac{\partial^3\psi}{\partial r^2\partial\theta} + \frac{2}{mr}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial\theta^2}\right. \\
 & + \frac{r}{m}\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{mr}\frac{\partial^2\psi}{\partial r\partial\theta}\frac{\partial\psi}{\partial\theta} - r^2\tau\left(\frac{\partial^2\psi}{\partial r^2}w + \frac{\partial\psi}{\partial r}\frac{\partial w}{\partial r}\right) - 2\tau\frac{\partial w}{\partial\theta} \\
 & + \frac{1}{m^3r}\left[(m^2 - 2h^2m + 3h^2)\frac{\partial^2\psi}{\partial\theta\partial r} + \frac{1}{r}(2m^2 - 2m + h^2)\left(\frac{\partial\psi}{\partial\theta}\right)^2\right] \\
 & + \left(\frac{1}{h^2} + \frac{1}{m^2} - \frac{1}{m} + \frac{h^2}{m^3}\right)\left(\frac{\partial\psi}{\partial r}\right)^2 - r\tau\left(\frac{2m}{h^2} + \frac{h^2}{m^2}\right)\frac{\partial\psi}{\partial r}w\} \\
 & + \frac{2}{r}\left(4 - \frac{2}{m} - \frac{2m}{h^2}\right)\left(\frac{h^2}{mr}\frac{\partial^3\psi}{\partial r\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2}\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial r^2}\right) \\
 & + 2(m - h^2)\left(\frac{1}{m^2r^3}\frac{\partial\psi}{\partial\theta}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\tau}{m}\frac{\partial^2\psi}{\partial r\partial\theta}w - \frac{2\tau}{h^2r}\frac{\partial\psi}{\partial\theta}w + \frac{2m\tau}{h^2}\frac{\partial w}{\partial r}\right) \\
 & - \frac{2}{r}\left(\frac{m}{h^2} + \frac{h^2}{m^2} - 2\right)\frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial r\partial\theta} - \left(\frac{4\kappa^2}{m^2} + \frac{2}{r^2}\left(\frac{6}{m} - \frac{3}{m^2} - \frac{1}{h^2} - 2\right)\right)\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\theta} \\
 & + \frac{2}{m^4r^2}\left(2\kappa \sin\theta G_1\frac{\partial\psi}{\partial\theta} - 2mG_2\frac{\partial^2\psi}{\partial\theta^2} - G_3\frac{\partial\psi}{\partial r}\right),
 \end{aligned}$$

and

$$\begin{aligned}
 F_2 = & \tau r \nabla^2\left(\frac{1}{m}\frac{\partial\psi}{\partial r}\right) + \frac{\tau}{m^2}\frac{\partial(\psi, \partial\psi/\partial r)}{\partial(r, \theta)} + \frac{2}{r}(m - h^2)\left(\frac{\partial w}{\partial r} - \frac{1}{mr}\frac{\partial\psi}{\partial\theta}w\right) \\
 & + \frac{2\tau}{m^2}\left(\left(1 - m\right)\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right) + \frac{\tau}{m^3r}\left(2(\kappa^2r^2 \sin^2\theta - m)\frac{\partial\psi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial\theta}\right) \\
 & + 2\kappa \sin\theta\left(\frac{\tau}{m^3r}\frac{\partial\psi}{\partial\theta} + \frac{\tau}{m^4}(m^2 + h^2)\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{r\tau}{2m^3}\left(\frac{\partial\psi}{\partial r}\right)^2 - \frac{h^2}{mr}\frac{\partial w}{\partial\theta} - \frac{\partial\psi}{\partial r}w\right).
 \end{aligned}$$

In the above, G_1 , G_2 , and G_3 are defined as follows:

$$G_1 = \kappa^2r^2(3h^2 - 2m^2) + m^2(2m^2 - 4m + 4) + h^2(2m - 3) - \frac{m^4}{h^2},$$

$$G_2 = \kappa^2r^2(3h^2 - 2m^2) + m^2(2m^2 - 4m + 3) + h^2(-m^2 + 4m - 3) - \frac{m^4}{h^2},$$

and

$$G_3 = \kappa^2r^2(3h^2 - m^2) + m^2(4m^2 - 5m + 4) - h^2(2m^2 - 5m + 3) - \frac{2m^5}{h^2} - \frac{m^4}{h^2}.$$

Appendix B: The solutions of w_{22} and ψ_{22}

$$\begin{aligned}
 w_{22} = & -\frac{\partial p}{4\partial s} \left(\frac{1}{4608}\right)^2 \left(\frac{1}{64}\right)^2 \left\{ \left[\frac{1}{60480} \left(-\frac{\partial p}{\partial s}\right)^6 \left(\frac{85769 r^{24}}{396396} - \frac{4204091 r^{22}}{693000} + \frac{1709429 r^{20}}{23100} \right. \right. \right. \\
 & - \frac{319489 r^{18}}{600} + \frac{8221337 r^{16}}{3150} - \frac{1629253 r^{14}}{175} + \frac{43925011 r^{12}}{1750} - \frac{21608521 r^{10}}{420} \\
 & + \frac{165815611 r^8}{2100} - \frac{1694906249 r^6}{19800} + \frac{4045025399 r^4}{69300} - \frac{361017880103 r^2}{19819800} \left. \right) \\
 & + \frac{2}{25} \left(-\frac{\partial p}{\partial s}\right)^4 \left(\frac{6628 r^{18}}{105} - \frac{1477888 r^{16}}{1323} + \frac{812603 r^{14}}{105} - 29976 r^{12} + \frac{22834454 r^{10}}{315} \right. \\
 & - \frac{11588056 r^8}{105} + \frac{31897249 r^6}{315} - \frac{2821708 r^4}{63} + \left. \frac{31030232 r^2}{6615} \right) \\
 & + 12288 \left(-\frac{\partial p}{\partial s}\right)^2 \left(\frac{4366 r^{12}}{175} + \frac{9 r^{10}}{5} - \frac{33286 r^8}{25} + 4563 r^6 - \frac{978094 r^4}{175} + \frac{11654 r^2}{5} \right) \\
 & + 18874368 (-687 r^6 + 1444 r^4 - 757 r^2) \cos 2\theta \\
 & + \frac{1}{12600} \left(-\frac{\partial p}{\partial s}\right)^6 \left(\frac{13 r^{24}}{896} - \frac{2771 r^{22}}{6160} + \frac{9379 r^{20}}{1600} - \frac{67909 r^{18}}{1512} + \frac{621169 r^{16}}{2688} - \frac{357983 r^{14}}{420} \right. \\
 & + \frac{1117313 r^{12}}{480} - \frac{834139 r^{10}}{175} + \frac{6525499 r^8}{896} - \frac{1150573 r^6}{144} + \frac{39159503 r^4}{6720} \\
 & - \left. \frac{416953 r^2}{168} + \frac{3068498717}{6652800} \right) \\
 & + \frac{2}{5} \left(-\frac{\partial p}{\partial s}\right)^4 \left(\frac{58687 r^{18}}{19845} - \frac{794 r^{16}}{15} + \frac{5392 r^{14}}{15} - \frac{63476 r^{12}}{45} + \frac{54596 r^{10}}{15} - 6526 r^8 \right. \\
 & + \left. \frac{285976 r^6}{35} - \frac{340306 r^4}{49} + \frac{344069 r^2}{105} - \frac{1460638}{2835} \right) \\
 & + 2048 \left(-\frac{\partial p}{\partial s}\right)^2 \left(\frac{554 r^{12}}{5} - \frac{11616 r^{10}}{25} - 1581 r^8 + \frac{46824 r^6}{5} - 17082 r^4 + \frac{64764 r^2}{5} - \frac{82519}{25} \right) \\
 & + 37748736 (-118 r^6 + 273 r^4 - 204 r^2 + 49) \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
 \psi_{22} = & -\frac{\partial p}{4\partial s} \left(\frac{1}{4608}\right)^2 \left(\frac{1}{64}\right)^2 \left\{ \frac{1}{35} \left(-\frac{\partial p}{\partial s}\right)^5 \left(-\frac{7649 r^{22}}{2993760} + \frac{100277 r^{20}}{1247400} - \frac{5293 r^{18}}{5600} \right. \right. \\
 & + \frac{2678 r^{16}}{405} - \frac{340799 r^{14}}{10800} + \frac{170117 r^{12}}{1575} - \frac{2918753 r^{10}}{10800} + \frac{9182261 r^8}{18900} - \frac{3669763 r^6}{6048} \\
 & + \left. \frac{1672462969 r^4}{3742200} - \frac{688195943 r^2}{4989600} \right) \\
 & + \frac{128}{5} \left(-\frac{\partial p}{\partial s}\right)^3 \left(-\frac{14144 r^{16}}{2205} + \frac{7172 r^{14}}{105} - \frac{38158 r^{12}}{175} - 47 r^{10} \right. \\
 & + \left. \frac{940018 r^8}{525} - \frac{459421 r^6}{105} + \frac{6760918 r^4}{1575} - \frac{368618 r^2}{245} \right) \\
 & + \left. \frac{196608}{5} \left(-\frac{\partial p}{\partial s}\right) (527 r^{10} - 2276 r^8 - 1183 r^6 + 7086 r^4 - 4154 r^2) \right\} \sin 2\theta.
 \end{aligned}$$