FM 8016.

Appendix to

The Concentration Distribution Near a Continuous Point Source in Steady Homogeneous Shear

by
James D. Bowen and Keith D. Stolzenbach

July 17, 1991

submitted for publication in the *Journal of Fluid Mechanics*

Appendix. The velocity gradient tensor in the equilibrium orientation

In this appendix we give the form of the velocity gradient tensor when the coordinate system is defined by the equilibrium orientation of an instantaneous pulse. This specification allows us to calculate how the extension rates along the equilibrium orientation vary with the intensity of a general rotation vector, without requiring that we know the orientation of the principal axes of strain relative to the equilibrium orientation. This is consistent with the emphasis placed on the shape rather than the absolute orientation of concentration distributions in shearing flow. To find the form of the velocity gradient tensor we first use the definition of an equilibrium orientation to give a condition on the evolution of the second moment tensor of an instantaneous pulse, **I**. Using the governing equations for the deformation of an instantaneous pulse (Eq. 2.3) together with an assumed form of the velocity gradient tensor **G** we then prove that the condition on **I** is satisfied.

An equilibrium orientation is the set of three asymptotic, orthogonal, and real eigenvectors for the second moment tensor **I** of an instantaneous pulse, eigenvectors that will exist for weakly rotating flows (criteria given by Eq. 5.6). This definition requires that I defined relative to the equilibrium orientation have the following evolution in time

$$\frac{I_{pq}}{I_{pp}-I_{qq}} \to 0 \quad \text{as } t \to \infty \tag{1}$$

This condition comes from requiring that the angle between the eigenvectors of **I** and the corresponding position vectors of the equilibrium orientation approach zero as time increases.

Since the trace of the velocity gradient tensor is zero, the criteria for an equilibrium orientation need only be specified for two distinct forms of G. As in the irrotational case we require that $G_{11} > G_{22} > G_{33}$. Concentration distributions in flows with $G_{11} > G_{22} > 0$

are disks, distributions in flows with $G_{22} = 0$ are diffusive tubes, and distributions in flows with $0 > G_{22} > G_{33}$ are tubes. Generalization of the irrotational case is based on the kinematic analysis giving the deformation of the pulse as dependent on the extension rates along the axes of the pulse, which are equal to G_{ii} if the velocity gradient tensor is defined in the principal axes of the pulse. If the velocity gradient tensor is defined relative to the equilibrium orientation which is the asymptotic orientation of the pulse, then G_{ii} gives the asymptotic extension rates of the instantaneous pulse. Seen from the equilibrium orientation, the asymptotic behavior of a pulse in weakly rotating flows is analogous to the behavior of a pulse in irrotational shear seen from the principal axes of strain. The forms of the velocity gradient tensors in the equilibrium orientation for disks, diffusive tubes, and tubes are given below.

Disk

To determine the velocity gradient tensor in the equilibrium orientation for a disk, we assume that the pulse is sufficiently distorted by the rotational shear so that $I_{11}>I_{22}>I_{33}$. The condition given by Eq. B 1 requires that the tensor **I** relative to the equilibrium orientation have the following asymptotic form.

$$\begin{bmatrix} I_{11} & \ll I_{11} & \ll I_{11} \\ \ll I_{11} & I_{22} & \ll I_{22} \\ \ll I_{11} & \ll I_{22} & I_{33} \end{bmatrix} \qquad (2)$$

We will show that this condition will be met if the velocity gradient tensor in the equilibrium orientation is upper triangular, as

$$\begin{bmatrix} E_1 & -\omega_3 & \omega_2 \\ 0 & E_2 & -\omega_1 \\ 0 & 0 & E_3 \end{bmatrix} \qquad (3)$$

where E_1, E_2 and E_3 are the extension rates along the equilibrium orientation axes. Note that since the velocity gradient tensor is upper triangular, E_1, E_2 and E_3 are the eigenvalues of the velocity gradient tensor **G**

With the assumed upper triangular form of the velocity gradient tensor **G** we can then solve solve the ordinary differential equation for the moment tensor **I** which is given as

$$\frac{dI_{pq}}{dt} - G_{pj}I_{qj} - G_{qj}I_{pj} = 2\kappa\delta_{pq} \tag{4}$$

Integrating the differential equations for the diagonal elements of **I** gives

$$I_{ii} = \delta_{ij}\,\frac{\gamma}{E_i}\,[\exp(2E_j\ t) - 1] \tag{5}$$

The differential equations for the off-diagonal elements are

$$\begin{aligned}
&\frac{dI_{23}}{dt} - (E_2+E_3)I_{23} + \omega_1 I_{33} = 0\\
&\frac{dI_{13}}{dt} - (E_1+E_3)I_{13} + \omega_3 I_{23} - \omega_2 I_{33} = 0\\
&\frac{dI_{12}}{dt} - (E_1+E_2)I_{12} + \omega_3 I_{22} - \omega_2 I_{23} + \omega_1 I_{13} = 0
\end{aligned} \tag{6}$$

where the symmetry of the tensor gives only three independent equations. Substituting the solutions for the diagonal elements gives solutions for the off-diagonal elements that are sums of exponentials with leading terms of

$$\begin{aligned}
&I_{23} \sim A\ \exp(E_2+E_3)t\\
&I_{13} \sim B\ \exp(E_1+E_3)t\\
&I_{12} \sim C\ \exp(E_1+E_2)t
\end{aligned} \tag{7}$$

where the constants A,B, and C are constants of order $[\kappa/(G_{ij}G_{ij})^{\frac{1}{2}}]$. Examination of Eq.

B 7 and B 5, with the specification that $E_1 > E_2 > E_3$ and $E_2 > 0$ shows that the elements of **I** satisfy the condition given by Eq. B 2 in the limit of large times. An orthogonal coordinate system with an upper triangular velocity gradient tensor **G** gives the asymptotic orientation of an instantaneous pulse in a weakly rotating shear flow.

Diffusive Tube

When $G_{22}=0$ the diagonal element of the second moment tensor I_{22} will not increase exponentially as was found for the case when $G_{22}>0$. Nonetheless the condition for an equilibrium orientation will be satisfied if the velocity gradient tensor **G** is upper triangular as

$$\begin{bmatrix} E_1 & -\omega & \omega_2 \\ 0 & 0 & -\omega_1 \\ 0 & 0 & E_3 \end{bmatrix} \tag{8}$$

where $E_1 = -E_3$ from continuity of an incompressible fluid.

With the above form of the velocity gradient tensor the differential equation for I_{pq} can be integrated. The diagonal elements I_{11} and I_{33} are unchanged from the disk (Eq. B 5). Likewise, the off-diagonal elements I_{12}, I_{23}, and I_{23} evolve in time as in the previous case (Eq. B 7) with the specification $E_2=0$. The asymptotic evolution of the I_{22} can be found using the asymptotic value for I_{23} ($I_{23} \rightarrow \Omega_1\kappa/E_3^2$ as $t \rightarrow \infty$) which gives

$$I_{22} \propto 2\kappa t\, [1 + (\Omega_1/-E_3)^2] \tag{9}$$

Examination of Eqs. B 6, B 7, and B 9 with the specification that $E_1>0$ and $E_3< 0$ shows that the elements of the second moment tensor **I** meet the criteria for an equilibrium orientation in the limit of large times. Diffusive tube concentration distributions will deform asymptotically according to the upper triangular velocity gradient tensor given by Eq. B 8.

Tube

If the diagonal elements of the velocity gradient tensor G_{22} and G_{33} are negative, then the moments I_{22} and I_{33} will asymptotically approach a constant value. The condition for an equilibrium orientation (Eq. B 1) therefore requires that I have the following asymptotic form

$$\begin{bmatrix} I_{11} & \ll I_{11} & \ll I_{11} \\ \ll I_{11} & I_{22} & 0 \\ \ll I_{11} & 0 & I_{33} \end{bmatrix} \qquad (10)$$

As in the analysis for the disk we solve the ordinary differential equation for I using a proposed form of the velocity gradient tensor, which can be given as

$$\begin{bmatrix} E_1 & -\omega_3 & \omega_2 \\ 0 & E_2 & \frac{-\omega_1}{E_2+E_3} \\ 0 & \frac{\omega_1}{E_2+E_3} & E_3 \end{bmatrix} \qquad (11)$$

Comparison of the velocity gradient tensors (Eqs. B 11 and B 4) shows that the elements of the tensor I will evolve in time as in the previous case, except for I_{23}, which evolves according to the differential equation

$$\frac{dI_{23}}{dt} - (E_2+E_3)I_{23} + \frac{\omega_1}{E_2+E_3}(E_3I_{33} - E_2I_{22}) = 0 \qquad (12)$$

Substituting the asymptotic values for I_{33} and I_{22} in B 12 with the specification that $E_2 <$

$E_3 < 0$ gives

$$I_{23} \sim \exp(E_2+E_3)t \to 0 \quad \text{as} \quad t\to\infty \tag{13}$$

The elements of the tensor **I** have the form required by Eq. B 10. Velocity gradient tensors of the form given by Eq. B 11 give instantaneous pulses with an asymptotic orientation aligned with the coordinate axes. The asymptotic extension rate of these pulses will be given by the diagonal components of the velocity gradient tensor as defined by Eq. B 11.