Appendices to accompany the paper "On the stability of vertical double-diffusive interfaces. Part 2: Two parallel interfaces" by I. A. Eltayeb and D. E. Loper which is to appear in the *Journal of Fluid Mechanics*.

Appendix A: Solution details for the thin plume

This appendix contains an outline of the calculations necessary to find the functions $\Omega_1^\beta(m, n; x_0, \sigma)$ and $\Omega_2^\beta(m, n; x_0, \sigma)$ for the thin plume, i.e. for $x_0 << 1$, using (4.3) - (4.6).

A.1 The zeroth-order problem

The governing equations are (3.11) and (3.13) – (3.15) with terms involving either R or $\overline{\Omega}$ eliminated. These equations are to be solved subject to the decay conditions (3.17*a*) and continuity conditions (4.1) or (4.2). This problem is virtually identical to the zeroth-order problem solved in Appendix B of Part 1; the solutions for the varicose and sinuous modes may be expressed as:

$$\left\{u_0^\beta, v_0^\beta, w_0^\beta, p_0^\beta, T_0^\beta\right\} = \sum_{j=1}^{3} \left\{-n\lambda_j, -mn, \mu_j^3, \mu_j, \mu_j^2\right\} 2 A_j \Lambda_j^\beta \exp\left(-\lambda_j x^*\right), \quad \text{(A 1)}$$

where A_j, μ_j, and λ_j are given by (2.19) – (2.21),

$$\Lambda_j^{v} = 1, \qquad \Lambda_j^{s} = \lambda_j x_0. \quad \text{(A 2)}$$

and

$$x^* = x - x_0. \quad \text{(A 3)}$$

It follows from (4.10) that all of the zeroth-order variables are real.

Now we may use (4.3) and (4.5) to find Ω_1 for the two modes. The results are presented in eqs. (4.7) – (4.9).

A.2 Formulation of first-order problem

At this order, it is sufficient to consider the solution of the set of four variables v_1, w_1, p_1, T_1, since expressions (4.4) and (4.6) for Ω_2^β do not contain u_1. The governing equations for both varicose and sinuous modes are, from (3.12) – (3.15):

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)p_1^\beta - T_1^\beta = F_\mathrm{p}^\beta, \quad \text{(A 4)}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)w_1^\beta + T_1^\beta + n^2 p_1^\beta = F_\mathrm{w}^\beta, \tag{A 5}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)T_1^\beta - w_1^\beta = F_\mathrm{T}^\beta, \tag{A 6}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)v_1^\beta = -mnp_1^\beta + \left(\Omega_1^\beta + \mathrm{i}n\overline{w}\right)v_0^\beta, \tag{A 7}$$

where

$$F_\mathrm{p}^\beta = -2\mathrm{i}u_0^\beta\left(\mathrm{d}\overline{w}/\mathrm{d}x\right), \tag{A 8}$$

$$F_\mathrm{w}^\beta = \left[\Omega_1^\beta + \mathrm{i}n\overline{w}\right]w_0^\beta + \mathrm{i}u_0^\beta\left(\mathrm{d}\overline{w}/\mathrm{d}x\right), \tag{A 9}$$

and

$$F_\mathrm{T}^\beta = \sigma\left[\Omega_1^\beta + \mathrm{i}n\overline{w}\right]T_0^\beta + \mathrm{i}\sigma u_0^\beta\left(\mathrm{d}\overline{T}/\mathrm{d}x\right), \tag{A 10}$$

for $\beta = \mathrm{v, s}$. Equations (A 4) – (A 7) are to be solved subject to the decay conditions (3.17*a*) and continuity conditions (4.1) and (4.2).

Since the basic-state and zeroth-order variables are real while Ω_1 is imaginary, the forcing terms F_y^β are imaginary. The auxiliary conditions (3.17*a*), (4.1) and (4.2) are homogeneous at this order. It follows that the first-order variables are imaginary and Ω_2, as given by (4.4) or (4.6), is real. Thus stability should be determined at this order.

With $x_0 << 1$, expressions (3.6) for $\overline{w}$ and $\overline{T}$, correct to dominant order in x_0, are

$$\begin{aligned} \overline{w} &= x_0 \mathrm{Im}\left[k\exp\left(-kx^*\right)\right], \\ \overline{T} &= -x_0 \mathrm{Im}\left[\mathrm{i}k\exp\left(-kx^*\right)\right]. \end{aligned} \tag{A 11}$$

Using (A 1), (A 11) and (4.7), the forcing functions defined by (A 8) – (A 10) may be expressed as

$$F_\mathrm{y}^\beta = -2\mathrm{i}nx_0\left\{\widetilde{\Omega}_1^\beta \sum_{j=1}^{3} A_j \Lambda_j^\beta \Gamma_\mathrm{y} \exp\left(-\lambda_j x^*\right) - \mathrm{Im}\left[k\sum_{j=1}^{3} A_j \Lambda_j^\beta \Phi_\mathrm{y}\exp\left(-\zeta_j x^*\right)\right]\right\}, \tag{A 12}$$

where the coefficients Λ_j^β are given by (A 2),

$$\Gamma_\mathrm{w} = \mu_j^3, \qquad \Gamma_\mathrm{p} = 0, \qquad \Gamma_\mathrm{T} = \sigma\mu_j^2, \tag{A 13}$$

$$\Phi_\mathrm{w} = \mu_j^3 + k\lambda_j, \qquad \Phi_\mathrm{p} = -2k\lambda_j, \qquad \Phi_\mathrm{T} = \sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right), \tag{A 14}$$

and

$$\zeta_j = \lambda_j + k. \tag{A 15}$$

The set of functions w_1, p_1, T_1 is uncoupled from v_1, so that they may be found first without consideration of the latter variable. This is done in §§ A.3 - A.5. Once this is accomplished, the solution for v_1 is found in § A.6. Finally an expression for Ω_2^{β} is obtained in § A.7.

A.3 Homogeneous solution for w_1, p_1 and T_1

The solutions of (A 4) – (A 6) satisfying the decay condition (3.17*a*) may found using the procedure of Appendix B of Part 1, and may be expressed as the sum of a homogeneous (subscript h) and a particular (subscript p) part:

$$y_1^{\beta} = y_{\rm h}^{\beta} + y_{\rm p}^{\beta}, \qquad \text{(A 16)}$$

for $y = w, p, T$ and $\beta = $ v, s. It is easily verified by substitution into the homogeneous version of (A 4) – (A 6) that

$$\left\{ w_{\rm h}^{\beta}, p_{\rm h}^{\beta}, T_{\rm h}^{\beta} \right\} = 2inx_0 \sum_{j=1}^{3} \left\{ \mu_j^3, \mu_j, \mu_j^2 \right\} \frac{\Lambda_j^{\beta}}{\lambda_j} B_j^{\beta} \exp(-\lambda_j x^*). \qquad \text{(A 17)}$$

The coefficients B_j^{β}, which are equivalent to coefficients B_j introduced by (1.B 11), are to be determined by conditions (4.1) or (4.2) once the particular solution has been found.

A.4 Particular solution for w_1, p_1 and T_1

The particular solution of (A 4) – (A 6) may be expressed as

$$\begin{aligned} y_{\rm p}^{\beta} &= 2inx_0 \tilde{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j \Lambda_j^{\beta} \left[y_{{\rm a}j} x^* + \tilde{y}_{{\rm a}j} \right] \exp(-\lambda_j x^*) \\ &\quad + 2inx_0 \,\mathrm{Im}\left[k \sum_{j=1}^{3} A_j \Lambda_j^{\beta} y_{{\rm b}j} \exp(-\zeta_j x^*) \right]. \end{aligned} \qquad \text{(A 18)}$$

for $y = w, p, T$, where $\tilde{\Omega}_1^{\beta}$, Λ_j^{β}, A_j are given by (4.8) or (4.9), (A 2) and (2.19), respectively. The coefficients $y_{{\rm a}j}$, $\tilde{y}_{{\rm a}j}$, and $y_{{\rm b}j}$ are to be found. The form of (A 18) has been chosen so that these unknown coefficients are the same for both varicose and sinuous modes; consequently these variables lack the superscript β. The term in (A 18) containing the variable $\tilde{y}_{{\rm a}j}$ is functionally identical to the homogeneous solution. Thus one variable ($y = w, p, T$) may be set to zero arbitrarily. We choose

$$\widetilde{p}_{aj} = 0 . \tag{A 19}$$

Substitution of (A 18) into (A 4) – (A 6) and making use of (A 12) – (A 14) yields the following algebraic equations:

$$\mu_j p_{aj} - T_{aj} = \mu_j w_{aj} + T_{aj} + n^2 p_{aj} = \mu_j T_{aj} - w_{aj} = 0, \tag{A 20}$$

$$\left.\begin{aligned} 2\lambda_j p_{aj} + \widetilde{T}_{aj} &= 0, \\ 2\lambda_j w_{aj} - \mu_j \widetilde{w}_{aj} - \widetilde{T}_{aj} &= \mu_j^3, \\ 2\lambda_j T_{aj} - \mu_j \widetilde{T}_{aj} + \widetilde{w}_{aj} &= \sigma \mu_j^2, \end{aligned}\right\} \tag{A 21}$$

$$\left.\begin{aligned} \gamma_j p_{bj} - T_{bj} &= -2k\lambda_j, \\ \gamma_j w_{bj} + n^2 p_{bj} + T_{bj} &= \left(\mu_j^3 + k\lambda_j\right), \\ -w_{bj} \qquad + \gamma_j T_{bj} &= \sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right), \end{aligned}\right\} \tag{A 22}$$

where

$$\gamma_j = \zeta_j^2 - a^2 = \left(\lambda_j + k\right)^2 - a^2. \tag{A 23}$$

Note that (A 20) is identical to the first of (1.B 6) and (A 22) is identical to (1.B 18).

The solutions of (A 20) and (A 21) are

$$\left.\begin{aligned} &\{w_{aj}, p_{aj}, T_{aj}\} = -(1+\sigma)A_j\left\{\mu_j^4, \mu_j^2, \mu_j^3\right\}, \\ &\widetilde{w}_{aj} = \frac{\mu_j^2\left(n^2\sigma - 2n^2 - 2\mu_j\right)}{3n^2 + 2\mu_j}, \quad \widetilde{T}_{aj} = \frac{(1+\sigma)\mu_j^4}{3n^2 + 2\mu_j}, \end{aligned}\right\} \tag{A 24}$$

while the solution of (A 22) is

$$\left.\begin{aligned} w_{bj} &= \frac{1}{D_j}\left[\qquad n^2\gamma_j 2k\lambda_j + \gamma_j^2\left(\mu_j^3 + k\lambda_j\right) - \left(\gamma_j + n^2\right)\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right], \\ p_{bj} &= \frac{1}{D_j}\left[-\left(1 + \gamma_j^2\right)2k\lambda_j + \quad \left(\mu_j^3 + k\lambda_j\right) + \qquad \gamma_j\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right], \\ T_{bjj} &= \frac{1}{D_j}\left[\qquad n^2 2k\lambda_j + \gamma_j\left(\mu_j^3 + k\lambda_j\right) + \qquad \gamma_j^2\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right], \end{aligned}\right\} \tag{A 25}$$

where $$D_j \;=\; \gamma_j^3 + \gamma_j + n^2. \qquad \text{(A 26)}$$

If we label the roots μ_j of (2.21) such that μ_1 is real while μ_2 and μ_3 are complex conjugates, it follows from (2.19) and (2.20) that the solutions given in (A 24) have the same complex behavior: y_{a1} and $\tilde{y}_{a1}$ are real while the pairs y_{a2}, y_{a3} and $\tilde{y}_{a2}$, $\tilde{y}_{a3}$ are complex conjugates. The complex character of the coefficients given by (A 25) is not simple.

A.5 Satisfaction of boundary conditions for w_1, p_1 and T_1

At order R and to dominant order in x_0, conditions (4.1) on the varicose mode are

At $x = x_0$: $$\mathrm{d}w_1^{\mathrm{v}}/\mathrm{d}x \;=\; \mathrm{d}p_1^{\mathrm{v}}/\mathrm{d}x \;=\; \mathrm{d}T_1^{\mathrm{v}}/\mathrm{d}x \;=\; 0. \qquad \text{(A 27)}$$

Using (A 2*a*) and (A 16) – (A 18), conditions (A 27) are satisfied provided the coefficients B_j^{v} satisfy

$$\sum_{j=1}^{3} \left\{\mu_j^3,\, \mu_j,\, \mu_j^2\right\} B_j^{\mathrm{v}} \;=\; \left\{\hat{w}^{\mathrm{v}},\, \hat{p}^{\mathrm{v}},\, \hat{T}^{\mathrm{v}}\right\}, \qquad \text{(A 28)}$$

where

$$\hat{y}^{\mathrm{v}} \;= \tilde{\Omega}_1^{\mathrm{v}} \sum_{j=1}^{3} A_j\left(y_{aj} - \lambda_j \tilde{y}_{aj}\right) - \mathrm{Im}\left[k \sum_{j=1}^{3} \zeta_j A_j y_{bj}\right], \qquad \text{(A 29)}$$

for $y = w, p, T$. Note that $\hat{y}^{\mathrm{v}}$ is real. The solution of (A 28) is

$$B_j^{\mathrm{v}} \;=\; \frac{\left[-\mu_j \hat{w}^{\mathrm{v}} + n^2 \hat{p}^{\mathrm{v}} - \mu_j^2 \hat{T}^{\mathrm{v}}\right]}{\mu_j\left(3n^2 + 2\mu_j\right)}. \qquad \text{(A 30)}$$

This may be verified with the aid of (1.B 29). Compare (A 30) with (1.B 30).

At order R and to dominant order in x_0, conditions (4.2) on the sinuous mode are

At $x = x_0$: $$w_1^{\mathrm{s}} \;=\; p_1^{\mathrm{s}} \;=\; T_1^{\mathrm{s}} \;=\; 0. \qquad \text{(A 31)}$$

Using (A 2*b*) and (A 16) – (A 18), conditions (A 31) are satisfied provided the coefficients B_j^{s} satisfy

$$\sum_{j=1}^{3} \left\{\mu_j^3,\, \mu_j,\, \mu_j^2\right\} B_j^{\mathrm{s}} \;=\; \left\{\hat{w}^{\mathrm{s}},\, \hat{p}^{\mathrm{s}},\, \hat{T}^{\mathrm{s}}\right\}, \qquad \text{(A 32)}$$

where

$$\hat{y}^{\mathrm{s}} \;= -\,\Omega_1^{\mathrm{s}} \sum_{j=1}^{3} \lambda_j A_j \tilde{y}_{aj} - \mathrm{Im}\left[k \sum_{j=1}^{3} \lambda_j A_j y_{bj}\right] \qquad \text{(A 33)}$$

for $y = w, p, T$. Note that $\hat{y}^s$ is real. The solution of (A 32) is

$$B_j^s = \frac{\left[-\mu_j \hat{w}^s + n^2 \hat{p}^s - \mu_j^2 \hat{T}^s\right]}{\mu_j\left(3n^2 + 2\mu_j\right)} . \tag{A 34}$$

This may be verified with the aid of (1.B 29). Again, compare this with (1.B 30).

A.6 The first-order problem - solution for v_1

We now wish to solve (A 7) for v_1^β, subject to conditions (3.17*a*), (4.1*d*) and (4.2*d*). Using (4.7), (A 1), (A 11*a*) and (A 16) - (A 19), this equation may be expressed as

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - a^2\right) v_{\mathrm{p}}^\beta = {} & 2\mathrm{i}mn^2 x_0 \sum_{j=1}^{3}\left[\tilde{\Omega}_1^\beta A_j\left(1 - p_{\mathrm{a}j}x^*\right) - \frac{\mu_j}{\lambda_j} B_j^\beta\right] \Lambda_j^\beta \exp\left(-\lambda_j x^*\right) \\ & - 2\mathrm{i}mn^2 x_0 \,\mathrm{Im}\left[k \sum_{j=1}^{3} A_j \Lambda_j^\beta \left(p_{\mathrm{b}j} + 1\right) \exp\left(-\zeta_j x^*\right)\right].\end{aligned} \tag{A 35}$$

The solution of this equation which decays as $x \to \infty$ is

$$\begin{aligned} v_1^\beta = {} & \mathrm{i}aB_4^\beta \exp(-ax^*) - 2\mathrm{i}mn^2 x_0 \sum_{j=1}^{3} \frac{\Lambda_j^\beta}{\lambda_j} B_j^\beta \exp\left(-\lambda_j x^*\right) \\ & + 2\mathrm{i}nx_0 \left\{\tilde{\Omega}_1^\beta \sum_{j=1}^{3} A_j \Lambda_j^\beta \left(v_{\mathrm{a}j} x^* + \tilde{v}_{\mathrm{a}j}\right) \exp\left(-\lambda_j x^*\right) + \mathrm{Im}\left[k \sum_{j=1}^{3} A_j \Lambda_j^\beta v_{\mathrm{b}j} \exp\left(-\zeta_j x^*\right)\right]\right\} \end{aligned} \tag{A 36}$$

where

$$v_{\mathrm{a}j} = -mn\frac{p_{\mathrm{a}j}}{\mu_j}, \quad \tilde{v}_{\mathrm{a}j} = \frac{mn}{\mu_j}\left[1 - \frac{2\lambda_j}{\mu_j} p_{\mathrm{a}j}\right], \quad v_{\mathrm{b}j} = -mn\frac{\left(p_{\mathrm{b}j} + 1\right)}{\zeta_j^2 - a^2} . \tag{A 37}$$

The jump condition on the varicose mode (4.1*d*) is satisfied provided

$$B_4^{\mathrm{v}} = \frac{2x_0}{a^2}\left[n\hat{v}^{\mathrm{v}} - m\left(\hat{w}^{\mathrm{v}} + \hat{p}^{\mathrm{v}}\right)\right] \tag{A 38}$$

where $\hat{v}$ is given by (A 29). This may be verified with the aid of (1.B 29).

The jump condition on the sinuous mode (4.2*d*) is satisfied provided

$$B_4^{\mathrm{s}} = \frac{2x_0^2}{a}\left[n\hat{v}^{\mathrm{s}} - m\left(\hat{w}^{\mathrm{s}} + \hat{p}^{\mathrm{s}}\right)\right]. \tag{A 39}$$

A.7 Expressions for growth rate

Now we may use (A 16) - (A 18) and (A 36) to write (4.4) as

$$\Omega_2^{\mathrm{v}} = x_0^2\,\widetilde{\Omega}_2^{\mathrm{v}} \tag{A 40}$$

where

$$\widetilde{\Omega}_2^{\mathrm{v}} = \frac{2m^2}{a}\hat{J} - 2n^2\sum_{j=1}^{3}\lambda_j B_j^{\mathrm{v}}$$

$$+ 2n\widetilde{\Omega}_1^{\mathrm{v}}\sum_{j=1}^{3} A_j(m\widetilde{v}_{\mathrm{a}j} + n\widetilde{w}_{\mathrm{a}j}) + 2n\,\mathrm{Im}\left[k\sum_{j=1}^{3} A_j(mv_{\mathrm{b}j} + nw_{\mathrm{b}j})\right]. \tag{A 41}$$

The quantity J is defined by

$$J = \frac{n}{m}v - (w + p). \tag{A 42}$$

We have also used the fact that

$$\mu_j^3 - m^2 = -\lambda_j^2. \tag{A 43}$$

Combining (A 30) and (A 43) we may write

$$\sum_{j=1}^{3}\lambda_j B_j^{\mathrm{v}} = -\hat{s}^{\mathrm{v}}$$

where s is defined by

$$s = M_1 w - n^2 M_0 p + M_2 T. \tag{A 44}$$

and M_{k} is given by (1.4.32).

Using (A 29) with $y = J$ we may write

$$\widetilde{\Omega}_2^{\mathrm{v}} = \widetilde{\Omega}_1^{\mathrm{v}}\sum_{j=1}^{3} A_j Q_\Omega^{\mathrm{v}} + \mathrm{Im}\left[k\sum_{j=1}^{3} A_j Q_{\mathrm{I}}^{\mathrm{v}}\right], \tag{A 45}$$

where

$$Q_\Omega^{\mathrm{v}} = 2n(m\widetilde{v}_{\mathrm{a}j} + n\widetilde{w}_{\mathrm{a}j}) + 2\frac{m^2}{a}\left(J_{\mathrm{a}j} - \lambda_j\widetilde{J}_{\mathrm{a}j}\right) + 2n^2\left(s_{\mathrm{a}j} - \lambda_j\widetilde{s}_{\mathrm{a}j}\right), \tag{A 46}$$

and

$$Q_{\mathrm{I}}^{\mathrm{v}} = 2n(mv_{\mathrm{b}j} + nw_{\mathrm{b}j}) - 2n^2\zeta_j s_{\mathrm{b}j} - 2\frac{m}{a}\zeta_j J_{\mathrm{b}j}. \tag{A 47}$$

The following equations, obtained from combinations of (2.21) and (A 20) - (A 22), are of use in evaluating the coefficients J_{aj}, $\widetilde{J}_{aj}$ and J_{bj}.

$$w_{aj} - \mu_j^2 p_{aj} = 0, \tag{A 48}$$

$$2\lambda_j(w_{aj} + p_{aj}) - \mu_j \widetilde{w}_{aj} = \mu_j^3, \tag{A 49}$$

$$\gamma_j(w_{bj} + p_{bj}) + n^2 p_{bj} = \mu_j^3 - k\lambda_j. \tag{A 50}$$

Using these and the definition (A 42), we obtain

$$J_{aj} = 0, \qquad \widetilde{J}_{aj} = -1, \qquad J_{bj} = 1 - \frac{k}{\gamma_j}\zeta_j. \tag{A 51}$$

Noting that

$$\sum_{j=1}^{3} \lambda_j A_j = 0, \tag{A 52}$$

we see that the terms involving J in (A 46) do not contribute to $\widetilde{\Omega}_2^{\mathrm{v}}$ and may be ignored; now

$$Q_\Omega^{\mathrm{v}} = 2n(m\widetilde{v}_{aj} + n\widetilde{w}_{aj}) + 2n^2\left(s_{aj} - \lambda_j \widetilde{s}_{aj}\right), \tag{A 53}$$

and

$$Q_{\mathrm{I}}^{\mathrm{v}} = 2n(mv_{bj} + nw_{bj}) - 2n^2 \zeta_j s_{bj} - 2\frac{m^2}{a}\zeta_j\left[1 - \frac{k}{\gamma_j}\zeta_j\right]. \tag{A 54}$$

Further using (A 37), (A 43), (A 49) and (A 52) we have that

$$Q_\Omega^{\mathrm{v}} = 2n^2\left(\frac{\lambda_j^2}{\mu_j} - \frac{2a^2\lambda_j}{\mu_j^2}p_{aj} + s_{aj} - \lambda_j\widetilde{s}_{aj}\right), \tag{A 55}$$

and

$$Q_{\mathrm{I}}^{\mathrm{v}} = -2n^2\zeta_j\left(\frac{\lambda_j + \zeta_j p_{bj}}{\gamma_j} + s_{bj}\right) + \frac{2akm^2}{\gamma_j}. \tag{A 56}$$

Note that the first term on the right hand side of (A 55), when summed will yield a term proportional to M_2.

One of the two desired relations $\Omega_2^{\beta}(m, n; x_0, \sigma)$ is given by (A 40) combined with (A 45), (A 55) and (A 56). Note that it is linear in σ and second order in x_0.

Consider now the sinuous mode. We may write (4.6) as

$$\Omega_2^{\mathrm{s}} = -\frac{\mathrm{i}n}{a^2}\left[\frac{\mathrm{d}G_1^{\mathrm{s}}}{\mathrm{d}x}\right]_{x=x_0} - \frac{\mathrm{i}}{a^2}\left(\Omega_1^{\mathrm{s}} + \frac{\mathrm{i}nx_0}{\sqrt{2}}\right)u_0^{\mathrm{s}}(x_0). \qquad \text{(A 57)}$$

where G is defined by

$$G = \frac{m}{n}v + w + p. \qquad \text{(A 58)}$$

We have from (A 1), (A 2*b*) and (1.4.32) that

$$u_0^{\mathrm{s}}(x_0) = -2nx_0\sum_{j=1}^{3}\lambda_j^2 A_j = -nx_0M_3. \qquad \text{(A 59)}$$

Also (4.7) and (4.9) may be combined to yield

$$\Omega_1^{\mathrm{s}} + \frac{\mathrm{i}nx_0}{\sqrt{2}} = -\mathrm{i}nx_0M_3. \qquad \text{(A 60)}$$

Combining (A 16) - (A 18) and (A 36) we obtain

$$G_1^{\mathrm{s}} = -2\mathrm{i}x_0^2\left[m^2\hat{J}^{\mathrm{s}}\exp(-ax^*) + a^2n^2\sum_{j=1}^{3}B_j^{\mathrm{s}}\exp(-\lambda_jx^*)\right]$$
$$+ 2\mathrm{i}n^2x_0^2\tilde{\Omega}_1^{\mathrm{s}}\sum_{j=1}^{3}\lambda_jA_j\left(G_{\mathrm{a}j}x^* + \tilde{G}_{\mathrm{a}j}\right)\exp(-\lambda_jx^*) + 2\mathrm{i}n^2x_0^2\,\mathrm{Im}\left[k\sum_{j=1}^{3}\lambda_jA_jG_{\mathrm{b}j}\exp(-\zeta_jx^*)\right]. \qquad \text{(A 61)}$$

Defining

$$\Omega_2^{\mathrm{s}} = x_0^2\tilde{\Omega}_2^{\mathrm{s}}, \qquad \text{(A 62)}$$

we may use (A 57), and (A 59) - (A 61) to write

$$\tilde{\Omega}_2^{\mathrm{s}} = \frac{n^2}{a^2}M_3^2 - \frac{2m^2}{a}\hat{J}^{\mathrm{s}} + 2n^2\sum_{j=1}^{3}B_j^{\mathrm{s}}\lambda_j$$
$$+ \frac{2n^2}{a^2}\tilde{\Omega}_1^{\mathrm{s}}\sum_{j=1}^{3}\lambda_jA_j\left(G_{\mathrm{a}j} - \lambda_j\tilde{G}_{\mathrm{a}j}\right) - \frac{2n^2}{a^2}\mathrm{Im}\left[k\sum_{j=1}^{3}\lambda_jA_jG_{\mathrm{b}j}\zeta_j\right]. \qquad \text{(A 63)}$$

Combining (A 34) and (A 44) we have that

$$\sum_{j=1}^{3}\lambda_jB_j^{\mathrm{s}} = -\hat{s}^{\mathrm{s}}.$$

Using this and (A 33) we may write

$$\tilde{\Omega}_2^{s} = \frac{n^2}{a^2} M_3^2 + \tilde{\Omega}_1^{s} \sum_{j=1}^{3} \lambda_j A_j Q_{\Omega}^{s} + \mathrm{Im}\left[k \sum_{j=1}^{3} \lambda_j A_j Q_{\mathrm{I}}^{s} \right] \quad \text{(A 64)}$$

where

$$Q_{\Omega}^{s} = \frac{2n^2}{a^2}\left(G_{aj} - \lambda_j \tilde{G}_{aj}\right) + \frac{2m^2}{a} \tilde{J}_{aj} + 2n^2 \tilde{s}_{aj} \quad \text{(A 65)}$$

and

$$Q_{\mathrm{I}}^{s} = -\frac{2n^2}{a^2} \zeta_j G_{bj} + \frac{2m^2}{a} J_{bj} + 2n^2 s_{bj}. \quad \text{(A 66)}$$

As in the varicose case, the term involving $\tilde{J}_{aj}$ does not contribute to $\widetilde{\Omega_2}^{m}$ and may be ignored. Using (A 37) and (A 48) - (A 50) we have that

$$\left.\begin{aligned} G_{aj} &= -\frac{a^2}{\mu_j} p_{aj}, \qquad \tilde{G}_{aj} = \frac{\left(\mu_j + a^2\right)}{\mu_j} - \frac{2a^2 \lambda_j}{\mu_j^2} p_{aj}, \\ G_{bj} &= -\frac{\left(\lambda_j \zeta_j + a^2 p_{bj}\right)}{\gamma_j}. \end{aligned}\right\} \quad \text{(A 67)}$$

With this result and (A 52) we may rewrite (A 65) and (A 66) as

$$Q_{\Omega}^{s} = 2n^2 \left[\frac{\left(\mu_j + 2a^2\right)}{\mu_j^2} p_{aj} - \frac{\lambda_j \left(\mu_j + a^2\right)}{a^2 \mu_j} + \tilde{s}_{aj} \right] \quad \text{(A 68)}$$

and

$$Q_{\mathrm{I}}^{s} = 2n^2 \left(\frac{\lambda_j}{a^2} + \frac{\lambda_j}{\gamma_j} + \frac{\zeta_j}{\gamma_j} p_{bj} + s_{bj} \right) - \frac{2km^2}{a \gamma_j} \zeta_j. \quad \text{(A 69)}$$

(he second of the relations for $\Omega_2^{\beta}(m, n; x_0, \sigma)$ is given by (A 62) combined with (A 64), (A 68) and (A 69). It too is linear in σ and second order in x_0.

Appendix B: Solution details for the plume of arbitrary width

In this appendix we shall outline the solution of (3.11) – (3.15) subject to conditions (3.17), expanded in powers of R as in (3.19). The analysis parallels that of Appendix A except that here we must solve the equations in two regions, $|x| < x_0$ and $x_0 < |x|$, rather than one.

B.1 Zeroth-order problem

At this order, the governing equations are (3.11) – (3.15) with terms involving R or $\overline{\Omega}$ neglected. The dominant-order varicose ($\beta = \mathrm{v}$) and sinuous ($\beta = \mathrm{s}$) modes satisfying conditions (3.17) may be expressed as follows.

For $0 < x < x_0$:

$$\left\{v_0^\beta,\, w_0^\beta,\, p_0^\beta,\, T_0^\beta\right\} = \sum_{j=1}^{3} \left\{-mn,\, \mu_j^3,\, \mu_j,\, \mu_j^2\right\} 2A_j \Lambda_j^\beta\, Q_j^\beta(x), \tag{B 1}$$

$$u_0^\beta = 2n \sum_{j=1}^{3} A_j \Lambda_j^\beta \lambda_j R_j^\beta(x), \tag{B 2}$$

for $x_0 < x$:

$$\left\{v_0^\beta,\, w_0^\beta,\, p_0^\beta,\, T_0^\beta\right\} = \sum_{j=1}^{3} \left\{-mn,\, \mu_j^3,\, \mu_j,\, \mu_j^2\right\} 2A_j \Lambda_j^\beta \exp\left(-\lambda_j x^*\right), \tag{B 3}$$

$$u_0^\beta = -2n \sum_{j=1}^{3} A_j \Lambda_j^\beta \lambda_j \exp\left(-\lambda_j x^*\right), \tag{B 4}$$

where A_j, λ_j, μ_j, and $x*$ are defined by (2.19), (2.20), (2.21) and (A 3) respectively,

$$\Lambda_j^{\mathrm{v}} = \frac{1}{2} + \frac{1}{2}\exp\left(-2\lambda_j x_0\right), \qquad \Lambda_j^{\mathrm{s}} = \frac{1}{2} - \frac{1}{2}\exp\left(-2\lambda_j x_0\right), \tag{B 5}$$

$$Q_j^{\mathrm{v}}(x) = \frac{\cosh\left(\lambda_j x\right)}{\cosh\left(\lambda_j x_0\right)}, \qquad Q_j^{\mathrm{s}}(x) = \frac{\sinh\left(\lambda_j x\right)}{\sinh\left(\lambda_j x_0\right)}, \tag{B 6}$$

and

$$R_j^{\mathrm{v}}(\mathrm{x}) = \frac{\sinh\left(\lambda_j x\right)}{\cosh\left(\lambda_j x_0\right)}, \qquad R_j^{\mathrm{s}}(x) = \frac{\cosh\left(\lambda_j x\right)}{\sinh\left(\lambda_j x_0\right)}. \tag{B 7}$$

The coefficients defined by (B 5) reduce to those defined by (A 2) as $x_0 \to 0$, while in the limit $x_0 \to \infty$*, $\Lambda_j^\beta \to 1/2$ and solutions (B 3) and (B 4) tend toward those of the single plane interface; see (1.4.18). Continuity of u_0^β at $x = x_0$ is verified by noting that $\sum_{j=1}^{3} \lambda_j A_j = 0$ and $\Lambda_j^\beta R_j^\beta(x_0) = 1 - \Lambda_j^\beta$. Note that these solutions have the same complex

* More precisely we need $(m^2+n^6)x_0^2 >> 1$ to approach the indicated limit.

character as those for the single plume; u_0, v_0, w_0, p_0, and T_0 are real for both varicose and sinuous modes.

Ω_1 may be determined from (3.21), with $\overline{w}$ given by (3.5). Noting that $\sum_{j=1}^{3} \lambda_j A_j = 0$ we obtain expressions (5.1) - (5.3) for Ω_1^{v} and Ω_1^{s}.

B.2 First-order problem - formulation

The equations of the first-order problem are

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)p_1^\beta - T_1^\beta = F_\mathrm{p}^\beta, \tag{B 8}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)w_1^\beta + T_1^\beta + n^2 p_1^\beta = F_\mathrm{w}^\beta, \tag{B 9}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)T_1^\beta - w_1^\beta = F_\mathrm{T}^\beta, \tag{B 10}$$

$$\left(\mathrm{d}^2/\mathrm{d}x^2 - a^2\right)v_1^\beta = -mnp_1^\beta + \left(\Omega_1^\beta + \mathrm{i}n\overline{w}\right)v_0^\beta, \tag{B 11}$$

$$\mathrm{d}u_1^\beta/\mathrm{d}x = -mv_1^\beta - nw_1^\beta. \tag{B 12}$$

with the forcing functions given by (A 8) – (A 10) and $\overline{w}$ and $\overline{T}$ by (3.5) and (3.6). These equations are to be solved, for both the varicose (β = v) and sinuous (β = s) modes, separately in $0 < x < x_0$ and $x_0 < x$, subject to conditions (3.17*a*), (3.17*b*), (3.17*f*) or (3.17*g*)and the homogeneous versions of (3.17*c*) and (3.17*d*):

$$\begin{aligned} \left(\mathrm{d}w_1^\beta/\mathrm{d}x\right)_{x=x_0-} &= \left(\mathrm{d}w_1^\beta/\mathrm{d}x\right)_{x=x_0+}, \\ \left(\mathrm{d}p_1^\beta/\mathrm{d}x\right)_{x=x_0-} &= \left(\mathrm{d}p_1^\beta/\mathrm{d}x\right)_{x=x_0+}. \end{aligned} \tag{B 13}$$

Since $\overline{w}$, $\overline{T}$, u_0^β and T_0^β are real and Ω_1^β is imaginary, the forcing terms are imaginary. It follows from this, and the fact that the boundary conditions are homogeneous to this order, that the first-order variables are imaginary.

Using (3.5), (3.6) and (B 1) – (B 4), the forcing functions (A 8) – (A 10) may be expressed as follows:

For $0 < x < x_0$:

$$F_{\mathrm{y}}^{\beta} = -2in\hat{\Omega}_1^{\beta}\sum_{j=1}^{3} A_j\Lambda_j^{\beta}\Gamma_{\mathrm{y}}Q_j^{\beta}(x) - in\,\mathrm{Im}\left\{\sum_{j=1}^{3} A_j\left[Z_j^{\beta}\Phi_{\mathrm{y}}Q_+^{\beta}(x) + \left(\Lambda_j^{\beta} - K\right)\Psi_{\mathrm{y}}Q_-^{\beta}(x)\right]\right\}; \quad \text{(B 14)}$$

for $x_0 < x$:

$$F_{\mathrm{y}}^{\beta} = -2in\hat{\Omega}_1^{\beta}\sum_{j=1}^{3} A_j\Lambda_j^{\beta}\Gamma_{\mathrm{y}}\exp\left(-\lambda_j x^*\right) + 2in\,\mathrm{Im}\left\{K\sum_{j=1}^{3} A_j\Lambda_j^{\beta}\Phi_{\mathrm{y}}\exp\left(-\zeta_j x^*\right)\right\}, \quad \text{(B 15)}$$

where $\hat{\Omega}_1^{\beta}$, Γ_{y}, Φ_{y}, ζ_j and Λ_j^{β} are given by (5.2) & (5.3), (A 13), (A 14), (A 15) and (B 5),

$$\Psi_{\mathrm{w}} = \mu_j^3 - k\lambda_j, \qquad \Psi_{\mathrm{p}} = 2k\lambda_j, \quad \Psi_{\mathrm{T}} = \sigma\left(\mu_j^2 + ik\lambda_j\right), \quad \text{(B 16)}$$

$$Q_{\pm}^{\mathrm{v}}(x) = \frac{\cosh\left[\left(\lambda_j \pm k\right)x\right]}{\cosh\left[\left(\lambda_j \pm k\right)x_0\right]}, \qquad Q_{\pm}^{\mathrm{s}}(x) = \frac{\sinh\left[\left(\lambda_j \pm k\right)x\right]}{\sinh\left[\left(\lambda_j \pm k\right)x_0\right]}, \quad \text{(B 17)}$$

$$Z_j^{\mathrm{v}} = \frac{1}{2} + \frac{1}{2}\exp\left(-2\zeta_j x_0\right), \qquad Z_j^{\mathrm{s}} = \frac{1}{2} - \frac{1}{2}\exp\left(-2\zeta_j x_0\right), \quad \text{(B 18)}$$

and

$$K = \exp(-kx_0)\sinh(kx_0) = \frac{1}{2} - \frac{1}{2}\exp(-2kx_0). \quad \text{(B 19)}$$

Note that (B 15) reduces to (A 12) as $x_0 \to 0$, and (B 14) and (B 15) agree with (1.B 14) - (1.B 16) as $x_0 \to \infty$. Using and the continuity properties of the basic-state and zeroth-order solutions, it may be deduced from the definitions (A 8) - (A 10) that F_{y}^{β} must be continuous at $x = x_0$. Making use of the identity $2K\Lambda_j^{\beta} = \Lambda_j^{\beta} + K - Z_j^{\beta}$ it may be shown that the two versions of F_{y}^{β} given by (B 145) and (B 15) are in fact continuous at $x = x_0$.

B.3 First-order problem - homogeneous solution

As in Appendix A the general solutions of equations (B 8) – (B 12) may be expressed as the sum of homogeneous and particular solutions; see (A 16). The homogeneous solutions satisfying the symmetry and decay conditions are:
For $0 < x < x_0$:

$$\left\{w_{\mathrm{h}}^{\beta}, p_{\mathrm{h}}^{\beta}, T_{\mathrm{h}}^{\beta}\right\} = i\sum_{j=1}^{3}\left\{\mu_j^3, \mu_j, \mu_j^2\right\}\tilde{C}_j^{\beta}Q_j^{\beta}(x), \quad \text{(B 20)}$$

$$\left.\begin{aligned} v_{\mathrm{h}}^{\beta} &= \mathrm{i}a\widetilde{C}_4^{\beta}Q_{\mathrm{a}}^{\beta}(x) - \mathrm{i}mn\sum_{j=1}^{3}\widetilde{C}_j^{\beta}Q_j^{\beta}(x), \\ u_{\mathrm{h}}^{\beta} &= -\mathrm{i}m\widetilde{C}_4^{\beta}R_{\mathrm{a}}^{\beta}(x) + \mathrm{i}n\sum_{j=1}^{3}\widetilde{C}_j^{\beta}\lambda_j R_j^{\beta}(x). \end{aligned}\right\} \qquad \text{(B 21)}$$

For $x_0 < x$:

$$\left\{w_{\mathrm{h}}^{\beta}, p_{\mathrm{h}}^{\beta}, T_{\mathrm{h}}^{\beta}\right\} = \mathrm{i}\sum_{j=1}^{3}\left\{\mu_j^3, \mu_j, \mu_j^2\right\}C_j^{\beta}\exp\left(-\lambda_j x^*\right), \qquad \text{(B 22)}$$

$$\left\{u_{\mathrm{h}}^{\beta}, v_{\mathrm{h}}^{\beta}\right\} = \mathrm{i}\{m, a\}C_4^{\beta}\exp\left(-ax^*\right) + \mathrm{i}\sum_{j=1}^{3}\left\{-n\lambda_j, -mn\right\}C_j^{\beta}\exp\left(-\lambda_j x^*\right), \qquad \text{(B 23)}$$

where the functions Q_j^{β} and R_j^{β} are defined by (B 6) and (B 7),

$$Q_{\mathrm{a}}^{\mathrm{v}}(x) = \frac{\cosh(ax)}{\cosh(ax_0)}, \qquad Q_{\mathrm{a}}^{\mathrm{s}}(x) = \frac{\sinh(ax)}{\sinh(ax_0)}, \qquad \text{(B 24)}$$

and

$$R_{\mathrm{a}}^{\mathrm{v}}(x) = \frac{\sinh(ax)}{\cosh(ax_0)} \qquad R_{\mathrm{a}}^{\mathrm{s}}(x) = \frac{\cosh(ax)}{\sinh(ax_0)}. \qquad \text{(B 25)}$$

The coefficients C_j^{β}, $\widetilde{C}_j^{\beta}$, C_4^{β}, and $\widetilde{C}_4^{\beta}$ are to be determined by satisfying conditions (3.17*b*) and (B 13), once the particular solutions are found. Note that (B 22) is identical to (1.B 11) with the upper sign.

B.4 First-order problem - particular solution

The particular solutions of equations (B 8) – (B 10) may be expressed as follows. For $0 < x < x_0$:

$$\begin{aligned} y_{\mathrm{p}}^{\beta} = &-2\mathrm{i}n\hat{\Omega}_1^{\beta}\sum_{j=1}^{3}A_j\Lambda_j^{\beta}\left[y_{\mathrm{a}j}xR_j^{\beta}(x) - \tilde{y}_{\mathrm{a}j}Q_j^{\beta}(x)\right] \\ &-\mathrm{i}n\,\mathrm{Im}\left\{\sum_{j=1}^{3}A_j\left[Z_j^{\beta}y_{\mathrm{b}j}Q_+^{\beta}(x) + \left(\Lambda_j^{\beta} - K\right)y_{\mathrm{c}j}Q_-^{\beta}(x)\right]\right\} \end{aligned} \qquad \text{(B 26)}$$

For $x_0 < x$:

$$y_p^\beta = 2in\hat{\Omega}_1^\beta \sum_{j=1}^{3} A_j \Lambda_j^\beta \left[y_{aj} x^* + \tilde{y}_{aj}\right] \exp\left(-\lambda_j x^*\right)$$
$$+ 2in \,\mathrm{Im}\left\{K \sum_{j=1}^{3} A_j \Lambda_j^\beta y_{bj} \exp\left(-\zeta_j x^*\right)\right\} \qquad \text{(B 27)}$$

where $y = w, p$, or T, and $x*$ and ζ_j are defined by (A 3) and (A 15). The forms of (B 26) and (B 27) have been chosen so that the coefficients with subscripts a, b, and c are the same for both varicose and sinuous modes; thus these variables lack the superscript β. (B 27) may be compared with (A 18) in the limit $x_0 \to 0$ and with (1.B 17) in the limit $x_0 \to \infty$.

The terms in (B 26) and (B 27) containing the variable $\tilde{y}_{aj}$ are functionally identical to the homogeneous solutions. Thus one of each of these variables (for $y = w, p$, or T) may be set to zero arbitrarily. We choose

$$\tilde{p}_{aj} = 0. \qquad \text{(B 28)}$$

The remaining variables with subscript a must satisfy

$$\mu_j p_{aj} - T_{aj} = \mu_j w_{aj} + T_{aj} + n^2 p_{aj} = \mu_j T_{aj} - w_{aj} = 0, \qquad \text{(B 29)}$$

and

$$\left.\begin{aligned} 2\lambda_j p_{aj} + \tilde{T}_{aj} &= 0, \\ 2\lambda_j w_{aj} - \mu_j \tilde{w}_{aj} - \tilde{T}_{aj} &= \mu_j^3, \\ 2\lambda_j T_{aj} - \mu_j \tilde{T}_{aj} + \tilde{w}_{aj} &= \sigma\mu_j^2. \end{aligned}\right\} \qquad \text{(B 30)}$$

The solutions of (B 29) and (B 30) are:

$$\left.\begin{aligned} \{w_{aj}, p_{aj}, T_{aj}\} &= -(1+\sigma)A_j\{\mu_j^4, \mu_j^2, \mu_j^3\}, \\ \tilde{T}_{aj} = \frac{(1+\sigma)\mu_j^4}{3n^2 + 2\mu_j}, \quad \tilde{w}_{aj} &= \frac{\mu_j^2\left(n^2\sigma - 2n^2 - 2\mu_j\right)}{3n^2 + 2\mu_j}, \end{aligned}\right\} \qquad \text{(B 31)}$$

The variables with subscript b satisfy

$$\left.\begin{aligned} \gamma_j p_{bj} - T_{bj} &= -2k\lambda_j, \\ \gamma_j w_{bj} + n^2 p_{bj} + T_{bj} &= \mu_j^3 + k\lambda_j, \\ -w_{bj} \qquad + \gamma_j T_{bj} &= \sigma\left[\mu_j^2 - ik\lambda_j\right], \end{aligned}\right\} \qquad \text{(B 32)}$$

while those with subscript c satisfy

$$\left.\begin{aligned}\psi_j p_{cj} - T_{cj} &= 2k\lambda_j,\\ \psi_j w_{cj} + n^2 p_{cj} + T_{cj} &= \mu_j^3 - k\lambda_j,\\ -w_{cj} \qquad + \psi_j T_{cj} &= \sigma\left[\mu_j^2 + \mathrm{i}k\lambda_j\right],\end{aligned}\right\} \tag{B 33}$$

where

$$\gamma_j = (\lambda_j + k)^2 - a^2 \quad \text{and} \quad \psi_j = (\lambda_j - k)^2 - a^2. \tag{B 34}$$

Note that (B 32) is identical to both (1.B 18) and (A 22) and that (B 33) follows from (B 32) by changing the sign of each term containing k. The solutions of (B 32) and (B 33) are:

$$\left.\begin{aligned}w_{bj} &= \frac{1}{D_j}\left[\quad n^2\gamma_j 2k\lambda_j + \gamma_j^2\left(\mu_j^3 + k\lambda_j\right) - \left(\gamma_j + n^2\right)\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right],\\ p_{bj} &= \frac{1}{D_j}\left[-\left(1+\gamma_j^2\right)2k\lambda_j + \left(\mu_j^3 + k\lambda_j\right) + \gamma_j\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right],\\ T_{bj} &= \frac{1}{D_j}\left[\quad n^2 2k\lambda_j + \gamma_j\left(\mu_j^3 + k\lambda_j\right) + \gamma_j^2\sigma\left(\mu_j^2 - \mathrm{i}k\lambda_j\right)\right],\end{aligned}\right\} \tag{B 35}$$

$$\left.\begin{aligned}w_{cj} &= \frac{1}{E_j}\left[-n^2\psi_j 2k\lambda_j + \psi_j^2\left(\mu_j^3 - k\lambda_j\right) - \left(\psi_j + n^2\right)\sigma\left(\mu_j^2 + \mathrm{i}k\lambda_j\right)\right],\\ p_{cj} &= \frac{1}{E_j}\left[\left(1+\psi_j^2\right)2k\lambda_j + \left(\mu_j^3 - k\lambda_j\right) + \psi_j\sigma\left(\mu_j^2 + \mathrm{i}k\lambda_j\right)\right],\\ T_{cj} &= \frac{1}{E_j}\left[-n^2 2k\lambda_j + \psi_j\left(\mu_j^3 - k\lambda_j\right) + \psi_j^2\sigma\left(\mu_j^2 + \mathrm{i}k\lambda_j\right)\right],\end{aligned}\right\} \tag{B 36}$$

where

$$D_j = \gamma_j^3 + \gamma_j + n^2 \quad \text{and} \quad E_j = \psi_j^3 + \psi_j + n^2. \tag{B 37}$$

Now equation (B 11) for v_p^β may be expressed as follows.

For $0 < x < x_0$:

$$\left.\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - a^2\right)v_p^\beta = {} & 2\mathrm{i}mn^2\hat{\Omega}_1^\beta \sum_{j=1}^{3} A_j\Lambda_j^\beta\left[p_{aj}xR_j^\beta(x) + Q_j^\beta(x)\right]\\ & + \mathrm{i}mn^2\,\mathrm{Im}\left\{\sum_{j=1}^{3} A_j\left[Z_j^\beta\left(p_{bj}+1\right)Q_+^\beta(x) + \left(\Lambda_j^\beta - K\right)\left(p_{cj}+1\right)Q_-^\beta(x)\right]\right\};\end{aligned}\right\} \tag{B 38}$$

For $x_0 < x$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - a^2\right) v_{\mathrm{p}}^{\beta} = -2imn^2 \hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j \Lambda_j^{\beta} \left(p_{\mathrm{a}j} x^* - 1\right) \exp\left(-\lambda_j x^*\right)$$
$$- 2imn^2 \,\mathrm{Im}\left\{ K \sum_{j=1}^{3} A_j \Lambda_j^{\beta} \left(p_{\mathrm{b}j} + 1\right) \exp\left(-\zeta_j x^*\right)\right\}. \quad \text{(B 39)}$$

Compare (B 39) with (1.B 21) with the upper sign, in the limit $x_0 \to \infty$. The solutions of (B 38) and (B 39) may be expressed as (B 26) and (B 27) with

$$v_{\mathrm{a}j} = -mn \frac{p_{\mathrm{a}j}}{\mu_j}, \qquad \widetilde{v}_{\mathrm{a}j} = \frac{mn}{\mu_j}\left[1 - \frac{2\lambda_j}{\mu_j} p_{\mathrm{a}j}\right],$$
$$v_{\mathrm{b}j} = -mn \frac{\left(p_{\mathrm{b}j} + 1\right)}{\gamma_j}, \qquad v_{\mathrm{c}j} = -mn \frac{\left(p_{\mathrm{c}j} + 1\right)}{\psi_j}. \quad \text{(B 40)}$$

The particular solution of (B 12) may be expressed as follows.

For $0 < x < x_0$:

$$u_{\mathrm{p}}^{\beta} = 2in \hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j \Lambda_j^{\beta} \left[u_{\mathrm{a}j} x \, Q_j^{\beta}(x) - \widetilde{u}_{\mathrm{a}j} R_j^{\beta}(x)\right]$$
$$+ in \,\mathrm{Im}\left\{ \sum_{j=1}^{3} A_j \left[u_{\mathrm{b}j} Z_j^{\beta} R_{+}^{\beta}(x) + u_{\mathrm{c}j}\left(\Lambda_j^{\beta} - K\right) R_{-}^{\beta}(x)\right]\right\} \quad \text{(B 412)}$$

For $x_0 < x$:

$$u_{\mathrm{p}}^{\beta} = 2in \hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j \Lambda_j^{\beta} \left[u_{\mathrm{a}j} x^* + \widetilde{u}_{\mathrm{a}j}\right] \exp\left(-\lambda_j x^*\right)$$
$$+ 2in \,\mathrm{Im}\left\{ K \sum_{j=1}^{3} A_j \Lambda_j^{\beta} u_{\mathrm{b}j} \exp\left(-\zeta_j x^*\right)\right\} \quad \text{(B 42)}$$

where

$$R_{\pm}^{\mathrm{v}}(x) = \frac{\sinh\left[\left(\lambda_j \pm k\right)x\right]}{\cosh\left[\left(\lambda_j \pm k\right)x_0\right]}, \qquad R_{\pm}^{\mathrm{s}}(x) = \frac{\cosh\left[\left(\lambda_j \pm k\right)x\right]}{\sinh\left[\left(\lambda_j \pm k\right)x_0\right]}, \quad \text{(B 43)}$$

$$\left.\begin{aligned} u_{\mathrm{a}j} &= \frac{\left(m v_{\mathrm{a}j} + n w_{\mathrm{a}j}\right)}{\lambda_j}, \qquad u_{\mathrm{b}j} = \frac{\left(m v_{\mathrm{b}j} + n w_{\mathrm{b}j}\right)}{\lambda_j + k}, \\ \widetilde{u}_{\mathrm{a}j} &= \frac{\left(m \widetilde{v}_{\mathrm{a}j} + n \widetilde{w}_{\mathrm{a}j}\right)}{\lambda_j} + \frac{\left(m v_{\mathrm{a}j} + n w_{\mathrm{a}j}\right)}{\lambda_j^2}, \qquad u_{\mathrm{c}j} = \frac{\left(m v_{\mathrm{c}j} + n w_{\mathrm{c}j}\right)}{\lambda_j - k}. \end{aligned}\right\} \quad \text{(B 44)}$$

B.5 First-order problem - boundary conditions

Continuity conditions (3.17*b*) and (B 13) are satisfied provided the homogeneous parts of the solutions satisfy the following nonhomogeneous equations:

$$y_{\mathrm{h}}^{\beta}(x_0+) - y_{\mathrm{h}}^{\beta}(x_0-) = 2\mathrm{i}\hat{y}^{\beta} \tag{B 45}$$

for $y = u, v, w, p$, and T, and

$$\frac{\mathrm{d}y_{\mathrm{h}}^{\beta}}{\mathrm{d}x}(x_0+) - \frac{\mathrm{d}y_{\mathrm{h}}^{\beta}}{\mathrm{d}x}(x_0-) = 2\mathrm{i}\hat{\hat{y}}^{\beta} \tag{B 46}$$

for $y = v, w, p$ and T, where

$$\hat{y}^{\beta} = \frac{\mathrm{i}}{2}\left[y_{\mathrm{p}}^{\beta}(x_0+) - y_{\mathrm{p}}^{\beta}(x_0-)\right] \tag{B 47}$$

and

$$\hat{\hat{y}}^{\beta} = \frac{\mathrm{i}}{2}\left[\frac{\mathrm{d}y_{\mathrm{p}}^{\beta}}{\mathrm{d}x}(x_0+) - \frac{\mathrm{d}y_{\mathrm{p}}^{\beta}}{\mathrm{d}x}(x_0-)\right]. \tag{B 48}$$

Note that each of these hatted and double-hatted variables is real. Using (B 5), (B 7), (B 17), (B 26), (B 27), (B 31), (B 35), (B 41), (B 42) and (B 44), and noting that $\Lambda_j^{\beta} R_j^{\beta}(x_0) = 1 - \Lambda_j^{\beta}$ and $2K\Lambda_j^{\beta} = \Lambda_j^{\beta} + K - Z_j^{\beta}$, these variables may be expressed as

$$\hat{y}^{\beta} = -n x_0 \hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j\left(1 - \Lambda_j^{\beta}\right) y_{\mathrm{a}j} - \frac{n}{2}\mathrm{Im}\left\{\sum_{j=1}^{3} A_j\left[\left(\Lambda_j^{\beta} + K\right) y_{\mathrm{b}j} + \left(\Lambda_j^{\beta} - K\right) y_{\mathrm{c}j}\right]\right\} \tag{B 49}$$

for $y = v, w, p$ and T;

$$\hat{u}^{\beta} = n\hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j\left[x_0 \Lambda_j^{\beta} u_{\mathrm{a}j} - \tilde{u}_{\mathrm{a}j}\right] + \frac{n}{2}\mathrm{Im}\left\{\sum_{j=1}^{3} A_j\left(1 - K - \Lambda_j^{\beta}\right)\left[u_{\mathrm{b}j} + u_{\mathrm{c}j}\right]\right\} \tag{B 50}$$

and

$$\begin{aligned}\hat{\hat{y}}^{\beta} = {} & n\hat{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j\left[\lambda_j \tilde{y}_{\mathrm{a}j} - \left(1 + \lambda_j x_0 \Lambda_j^{\beta}\right) y_{\mathrm{a}j}\right] \\ & - \frac{n}{2}\mathrm{Im}\left\{\sum_{j=1}^{3} A_j\left(1 - K - \Lambda_j^{\beta}\right)\left[\left(\lambda_j + k\right) y_{\mathrm{b}j} + \left(\lambda_j - k\right) y_{\mathrm{c}j}\right]\right\}\end{aligned} \tag{B 51}$$

for $y = v, w, p$ and T.

Equations (B 45) and (B 46) yield nine conditions for the eight unknowns C_j^β, $\widetilde{C}_j^\beta$, C_4^β, and $\widetilde{C}_4^\beta$. The condition on the derivative of p is an 'extra' one which has been derived from the remaining conditions in § 3. It follows that only eight of the conditions are independent, and the system is not overdetermined. In analogy with (1.B 25) – (1.B 28), and using (B 20) – (B 23), these may be expressed as

$$\sum_{j=1}^{3} \left\{\mu_j^3, \mu_j, \mu_j^2\right\}\left[C_j^\beta - \widetilde{C}_j^\beta\right] = 2\left\{\widehat{w}^\beta, \widehat{p}^\beta, \widehat{T}^\beta\right\}, \tag{B 52}$$

$$-\sum_{j=1}^{3} \left\{\mu_j^3, \mu_j, \mu_j^2\right\}\lambda_j\left[C_j^\beta + \frac{\left(1-\Lambda_j^\beta\right)}{\Lambda_j^\beta}\widetilde{C}_j^\beta\right] = 2\left\{\widehat{\widehat{w}}^\beta, \widehat{\widehat{p}}^\beta, \widehat{\widehat{T}}^\beta\right\}, \tag{B 53}$$

$$a\left(C_4^\beta - \widetilde{C}_4^\beta\right) + mn\sum_{j=1}^{3}\left(\widetilde{C}_j^\beta - C_j^\beta\right) = 2\widehat{v}^\beta, \tag{B 54}$$

$$-a^2\left[C_4^\beta + \frac{\left(1-\Gamma_\mathrm{a}^\beta\right)}{\Gamma_\mathrm{a}^\beta}\widetilde{C}_4^\beta\right] + mn\sum_{j=1}^{3}\lambda_j\left[C_j^\beta + \frac{\left(1-\Lambda_j^\beta\right)}{\Lambda_j^\beta}\widetilde{C}_j^\beta\right] = 2\widehat{\widehat{v}}^\beta, \tag{B 55}$$

$$m\left[C_4^\beta + \frac{\left(1-\Gamma_\mathrm{a}^\beta\right)}{\Gamma_\mathrm{a}^\beta}\widetilde{C}_4^\beta\right] - n\sum_{j=1}^{3}\lambda_j\left[C_j^\beta + \frac{\left(1-\Lambda_j^\beta\right)}{\Lambda_j^\beta}\widetilde{C}_j^\beta\right] = 2\widehat{u}^\beta. \tag{B 56}$$

Making use of (1.B 29) and the fact that

$$n\widehat{\widehat{w}}^\beta + n\widehat{\widehat{p}}^\beta + m\widehat{\widehat{v}}^\beta + a^2\widehat{u}^\beta = 0,$$

it may be verified that the solutions of (B 52) – (B 56) are:

$$\left.\begin{aligned} \widetilde{C}_4^\beta &= 2\Gamma_\mathrm{a}^\beta\left(H_4^\beta - B_4^\beta\right), & \widetilde{C}_j^\beta &= 2\Lambda_j^\beta\left(H_j^\beta - B_j^\beta\right), \\ C_4^\beta &= 2\Gamma_\mathrm{a}^\beta\left(H_4^\beta - B_4^\beta\right) + 2B_4^\beta, & C_j^\beta &= 2\Lambda_j^\beta\left(H_j^\beta - B_j^\beta\right) + 2B_j^\beta \end{aligned}\right\} \tag{B 57}$$

where

$$
\left.
\begin{aligned}
H_4^\beta &= \frac{m}{a^2 n}\left(\widehat{\widehat{w}}^{\,\beta}+\widehat{\widehat{p}}^{\,\beta}\right)-\frac{1}{a^2}\widehat{\widehat{v}}^{\,\beta}, &\qquad H_j^\beta &= \frac{2A_j}{\mu_j^3}\left[\mu_j\widehat{\widehat{w}}^{\,\beta}-n^2\widehat{\widehat{p}}^{\,\beta}+\mu_j^2\widehat{\widehat{T}}^{\,\beta}\right],\\
B_4^\beta &= \frac{\widehat{v}^\beta}{a}-\frac{m}{an}\left(\widehat{w}^\beta+\widehat{p}^\beta\right), &\qquad B_j^\beta &= \frac{\left[-\mu_j\widehat{w}^\beta+n^2\widehat{p}^\beta-\mu_j^2\widehat{T}^\beta\right]}{\mu_j\left(3n^2+2\mu_j\right)},
\end{aligned}
\right\}
\tag{B 58}
$$

and

$$
\Gamma_{\mathrm{a}}^{\mathrm{v}} = \frac{1}{2}+\frac{1}{2}\exp\left(-2ax_0\right), \qquad \Gamma_{\mathrm{a}}^{\mathrm{s}} = \frac{1}{2}-\frac{1}{2}\exp\left(-2ax_0\right). \tag{B 59}
$$

It remains to develop an expression for Ω_2^β, using (3.17e), (3.19), (3.22), (A 16), (B 23) and (B 42) evaluated at $x^* = 0$:

$$
\begin{aligned}
\Omega_2^\beta = {} & -2m\left[\Gamma_{\mathrm{a}}^\beta\left(H_4^\beta-B_4^\beta\right)+B_4^\beta\right]+2n\sum_{j=1}^{3}\lambda_j\left[\left(\Lambda_j^\beta-1\right)\left(H_j^\beta-B_j^\beta\right)+B_j^\beta\right]\\
& -2n\,\widehat{\Omega}_1^\beta\sum_{j=1}^{3}A_j\Lambda_j^\beta\tilde{u}_{\mathrm{a}j}-2n\,\mathrm{Im}\left\{K\sum_{j=1}^{3}A_j\Lambda_j^\beta u_{\mathrm{b}j}\right\}
\end{aligned}
\tag{B 60}
$$

Here m and n are prescribed wavenumbers in the horizontal and vertical directions respectively, A_j and λ_j are given by (2.19) - (2.20), Ω_1^β is given by (5.1) - (5.3), Λ_j^β is given by (B 5), B_j^β, H_j^β, B_4^β and H_4^β are given by (B 57) - (B 58), with the hatted and double hatted variables given by (B 49) - (B 51) and the variables with subscripts a though c given by (B 31) - (B 33), (B 35), (B 36), (B 40) and (B 44).

There are two limit checks that can be made on this solution. First, as $x_0 \to \infty$, $\widehat{y}^\beta \to -\frac{n}{2}\mathrm{Im}\left\{\sum_{j=1}^{3}A_j y_{\mathrm{b}j}\right\}$ for $y = v, w, p, T$, which equals $\widehat{y}$ as defined by (1.B 28). Also, $\widehat{u}^\beta \to 0$, $\widehat{\widehat{y}}^{\,\beta} \to 0$ for $y = v, w, T$, $\widehat{\Omega}_1^\beta \to 0$, $\Lambda_j^\beta \to 1/2$, $\Gamma_{\mathrm{a}}^\beta \to 1/2$, $K \to 1/2$, $H_4^\beta \to 0$, $H_j^\beta \to 0$, and $\Omega_2^\beta = -2mB_4^\beta + 2n\sum_{j=1}^{3}\lambda_j B_j^\beta - n\,\mathrm{Im}\left\{\sum_{j=1}^{3}A_j u_{\mathrm{b}j}\right\}$ which agrees with (1.B 27). The second check is as $x_0 \to 0$. It is shown in appendix C that in this limit, solution (B 60) for Ω_2^β reduces to the thin-plume expressions (A 40) and (A 62).

Appendix C: Reconciliation of the general and thin plume solutions

The purpose of this appendix is to show that in the limit $x_0 \to 0$ the growth rate for the general case is equal to the thin-plume solution.

C 1. Reconciliation of Ω_1

Consider first the neutral growth rate Ω_1^β. We need to show in the limit $x_0 \to 0$ that (5.1) reduces to (4.7). Using the notation of (4.7) we may write (5.1) as

$$\tilde{\Omega}_1^\beta = \frac{1}{x_0}\left[\frac{1}{2}\sin(\sqrt{2}\,x_0)\exp(-\sqrt{2}\,x_0) + P\sum_{j=1}^{3}\lambda_j A_j \exp\left(-2\lambda_j x_0\right)\right], \tag{C 1}$$

where $P = +1$ for the varicose (even) mode (i.e. for $\beta = \mathrm{v}$) and $P = -1$ for the sinuous (odd) mode (i.e. for $\beta = \mathrm{s}$). As $x_0 \to 0$ this reduces to

$$\tilde{\Omega}_1^\beta = \frac{1}{\sqrt{2}} - 2P\sum_{j=1}^{3}\lambda_j^2 A_j\,, \tag{C 2}$$

which is equivalent to (4.8) and (4.9).

C 2. Reconciliation of Ω_2

Next we need to show that (B 60) in the limit $x_0 \to 0$ reduces to the expressions given by (A 45), (A 55) and (A 56) in the case $\beta = \mathrm{v}$ ($P = 1$) and to (A 64), (A 68) and (A 69) in the case $\beta = \mathrm{s}$ ($P = -1$). Using (B 58) and noting that

$$\hat{\Omega}_1^\beta = x_0\,\tilde{\Omega}_1^\beta\,, \tag{C 3}$$

(B 60) may be written as

$$\tilde{\Omega}_2^\beta = \frac{\Omega_2^\beta}{x_0^2} = \frac{2m^2\Gamma_\mathrm{a}^\beta}{a^2 n x_0^2}\hat{J}^\beta - \frac{2m^2\left(1-\Gamma_\mathrm{a}^\beta\right)}{a n x_0^2}\hat{J}^\beta - \frac{2n}{x_0}\tilde{\Omega}_1^\beta\sum_{j=1}^{3}A_j\Lambda_j^\beta\tilde{u}_{\mathrm{a}j}$$
$$- \frac{2n}{x_0^2}\,\mathrm{Im}\left[K\sum_{j=1}^{3}A_j\Lambda_j^\beta u_{\mathrm{b}j}\right] + \frac{2n}{x_0^2}\sum_{j=1}^{3}\lambda_j\left[\Lambda_j^\beta\left(H_j^\beta - B_j^\beta\right) + B_j^\beta\right] \tag{C 4}$$

where the parameter grouping J is defined by (A 42). The difficulty in reconciling the general and thin plume solutions stems from the apparently singular behavior of $\tilde{\Omega}_2^\beta$ as $x_0 \to 0$. In the remainder of this appendix we shall develop a systematic representation of $\tilde{\Omega}_2^\beta$

which is well behaved in this limit, and which agrees with the corrresponding expressions found in appendix A.

The first step in this process is to rewrite the last term in (C 4) involving the summation of B_j^β and H_j^β. Using the Taylor series expansion of Λ_j^β for small x_0 we may write

$$\Lambda_j^\beta = \frac{(1+P)}{2} - P x_0 \lambda_j + P x_0^2 \lambda_j^2 + O(x_0^3) . \tag{C 5}$$

Using this we may write

$$\begin{aligned} \frac{2n}{x_0^2}\sum_{j=1}^{3} \lambda_j\left[\Lambda_j^\beta\left(H_j^\beta - B_j^\beta\right) + B_j^\beta\right] &= \frac{n(1+P)}{x_0^2}\sum_{j=1}^{3} \lambda_j H_j^\beta + \frac{n(1-P)}{x_0^2}\sum_{j=1}^{3} \lambda_j B_j^\beta \\ &- \frac{2nP}{x_0}\sum_{j=1}^{3} \lambda_j^2\left(H_j^\beta - B_j^\beta\right) + 2nP\sum_{j=1}^{3} \lambda_j^3\left(H_j^\beta - B_j^\beta\right) + O(x_0). \end{aligned} \tag{C 6}$$

We may use (1.4.32), (1.B 29), (4.10) and (B 58) to evaluate the sums appearing in (C 6):

$$\sum_{j=1}^{3} \lambda_j H_j^\beta = \sum_{j=1}^{3} \frac{\left[\mu_j \hat{\hat{w}}^\beta - n^2 \hat{\hat{p}}^\beta + \mu_j^2 \hat{\hat{T}}^\beta\right]}{\mu_j(3n^2 + 2\mu_j)} = \frac{\left(\hat{\hat{w}}^\beta + \hat{\hat{p}}^\beta\right)}{n^2} = \frac{\hat{\hat{G}}^\beta}{a^2} - \frac{m^2 \hat{\hat{J}}^\beta}{a^2 n^2}, \tag{C 7}$$

$$\sum_{j=1}^{3} \lambda_j^2 H_j^\beta = \sum_{j=1}^{3} \frac{\lambda_j\left[\mu_j \hat{\hat{w}}^\beta - n^2 \hat{\hat{p}}^\beta + \mu_j^2 \hat{\hat{T}}^\beta\right]}{\mu_j(3n^2 + 2\mu_j)} = \hat{\hat{s}}^\beta, \tag{C 8}$$

$$\begin{aligned} \sum_{j=1}^{3} \lambda_j^3 H_j^\beta &= \sum_{j=1}^{3} (\mu_j + a^2)\frac{\left[\mu_j \hat{\hat{w}}^\beta - n^2 \hat{\hat{p}}^\beta + \mu_j^2 \hat{\hat{T}}^\beta\right]}{\mu_j(3n^2 + 2\mu_j)} \\ &= \frac{\left(a^2 \hat{\hat{w}}^\beta + m^2 \hat{\hat{p}}^\beta\right)}{n^2} = \hat{\hat{G}}^\beta - \hat{\hat{p}}^\beta - \frac{m^2 \hat{\hat{J}}^\beta}{n^2}, \end{aligned} \tag{C 9}$$

$$\sum_{j=1}^{3} \lambda_j B_j^\beta = \sum_{j=1}^{3} \frac{\lambda_j\left[-\mu_j \hat{w}^\beta + n^2 \hat{p}^\beta - \mu_j^2 \hat{T}^\beta\right]}{\mu_j(3n^2 + 2\mu_j)} = -\hat{s}^\beta, \tag{C 10}$$

$$\sum_{j=1}^{3} \lambda_j^2 B_j^\beta = \sum_{j=1}^{3} \left(m^2 - \mu_j^3\right) B_j^\beta = \sum_{j=1}^{3} \left(m^2 - \mu_j^3\right) \frac{\left[-\mu_j \widehat{w}^\beta + n^2 \widehat{p}^\beta - \mu_j^2 \widehat{T}^\beta\right]}{\mu_j \left(3n^2 + 2\mu_j\right)}$$

$$= -\frac{\left(a^2 \widehat{w}^\beta + m^2 \widehat{p}^\beta\right)}{n^2} = \widehat{p}^\beta - \widehat{G}^\beta + \frac{m^2}{n^2} \widehat{J}^\beta , \qquad \text{(C 11)}$$

$$\sum_{j=1}^{3} \lambda_j^3 B_j^\beta = \sum_{j=1}^{3} \left(m^2 - \mu_j^3\right) \frac{\lambda_j \left[-\mu_j \widehat{w}^\beta + n^2 \widehat{p}^\beta - \mu_j^2 \widehat{T}^\beta\right]}{\mu_j \left(3n^2 + 2\mu_j\right)}$$

$$= -m^2 \widehat{s}^\beta + M_4 \widehat{w}^\beta - n^2 M_3 \widehat{p}^\beta + M_5 \widehat{T}^\beta \qquad \text{(C 12)}$$

where J, s and G are defined by (A 42), (A 44) and (A 58), respectively. We will find the subsequent experssions are more compact if the versions involving G and J are used. Using (C 7) – (C 12), (C 4) may be expressed as

$$\widetilde{\Omega}_2^\beta = \frac{m^2}{a^2 n x_0^2} \left(2\,\Gamma_a^\beta - 1 - P\right) \widehat{\widehat{J}}^\beta - \frac{2m^2}{a n x_0^2} \left(1 - \Gamma_a^\beta - a P x_0\right) \widehat{J}^\beta - \frac{n(1-P)}{x_0^2} \widehat{s}^\beta + \frac{n(1+P)}{a^2 x_0^2} \widehat{\widehat{G}}^\beta$$

$$- \frac{2nP}{x_0} \left(\widehat{G}^\beta - \widehat{p}^\beta + \widehat{\widehat{s}}^\beta\right) - \frac{2n}{x_0} \widetilde{\Omega}_1^\beta \sum_{j=1}^{3} A_j \Lambda_j^\beta \widetilde{u}_{aj} - \frac{2n}{x_0^2} \operatorname{Im}\left[K \sum_{j=1}^{3} A_j \Lambda_j^\beta u_{bj}\right] \qquad \text{(C 13)}$$

$$+ 2nP \left(\widehat{\widehat{G}}^\beta - \widehat{\widehat{p}}^\beta - \frac{m^2 \widehat{\widehat{J}}^\beta}{n^2}\right) + 2nP\left(M_4 \widehat{w}^\beta - n^2 M_3 \widehat{p}^\beta + M_5 \widehat{T}^\beta\right) - 2m^2 nP\, \widehat{s}^\beta + O(x_0).$$

The next step in the reconciliation process is to divide all the hatted and double hatted variables into two parts. We start by writing (B 49) as

$$\widehat{y}^\beta = -n x_0^2 \widetilde{\Omega}_1^\beta \sum_{j=1}^{3} A_j \left(1 - \Lambda_j^\beta\right) y_{aj} - \frac{n}{2} \operatorname{Im}\left[\sum_{j=1}^{3} A_j \widehat{y}_j^{\beta \mathrm{I}}\right] \qquad \text{(C 14)}$$

for $y =$ v, w, p, G, J, and s with

$$\widehat{y}_j^{\beta \mathrm{I}} = \left(\Lambda_j^\beta + K\right) y_{bj} + \left(\Lambda_j^\beta - K\right) y_{cj} . \qquad \text{(C 15)}$$

Similarly (B 51) may be expressed as

$$\widehat{\widehat{y}}^\beta = n x_0\, \widetilde{\Omega}_1^\beta \sum_{j=1}^{3} A_j \left(\lambda_j \widetilde{y}_{aj} - y_{aj} - x_0 \lambda_j \Lambda_j^\beta y_{aj}\right) + \frac{n}{2} \operatorname{Im}\left[\sum_{j=1}^{3} A_j \left(\Lambda_j^\beta + K - 1\right) \widehat{\widehat{y}}_j^{\mathrm{I}}\right] \qquad \text{C 16)}$$

where

$$\hat{\hat{y}}_j^{\mathrm{I}} = \left(\lambda_j + k\right) y_{\mathrm{b}j} + \left(\lambda_j - k\right) y_{\mathrm{c}j} . \tag{C 17}$$

Note that $\hat{\hat{y}}_j^{\mathrm{I}}$ is independent of β.

Now, we may similarly divide $\widetilde{\Omega}_2^{\beta}$, and write (C 13) as

$$\widetilde{\Omega}_2^{\beta} = \widetilde{\Omega}_1^{\beta} \sum_{j=1}^{3} A_j\, Q_j^{\beta\Omega} + \mathrm{Im}\left[\sum_{j=1}^{3} A_j\, Q_j^{\beta\mathrm{I}}\right] \tag{C 18}$$

where

$$\begin{aligned} Q_j^{\beta\Omega} = {} & \frac{n^2(1+P)}{a^2 x_0}\left(\lambda_j \widetilde{G}_{\mathrm{a}j} - G_{\mathrm{a}j}\right) - \frac{m^2}{a^2 x_0}\left(2\Gamma_{\mathrm{a}}^{\beta} - 1 - P\right)\lambda_j - \frac{2n}{x_0}\Lambda_j^{\beta}\widetilde{u}_{\mathrm{a}j} \\ & - \frac{n^2(1+P)}{a^2}\lambda_j \Lambda_j^{\beta} G_{\mathrm{a}j} + n^2(1-P)\left(1 - \Lambda_j^{\beta}\right)s_{aj} - 2n^2 P\left(\lambda_j \widetilde{s}_{\mathrm{a}j} - s_{\mathrm{a}j}\right) + O(x_0) \end{aligned} \tag{C 19}$$

and

$$\begin{aligned} Q_j^{\beta\mathrm{I}} = {} & \frac{m^2}{2a^2 x_0^2}\left(2\Gamma_{\mathrm{a}}^{\beta} - 1 - P\right)\left(\Lambda_j^{\beta} + K - 1\right)\hat{\hat{J}}_j^{\mathrm{I}} + \frac{m^2}{a x_0^2}\left(1 - \Gamma_{\mathrm{a}}^{\beta} - aPx_0\right)\hat{J}_j^{\beta\mathrm{I}} + \frac{n^2(1-P)}{2x_0^2}\hat{s}_j^{\beta\mathrm{I}} \\ & + \frac{n^2(1+P)}{2a^2 x_0^2}\left(\Lambda_j^{\beta} + K - 1\right)\hat{\hat{G}}_j^{\mathrm{I}} + \frac{n^2 P}{x_0}\left(\hat{G}_j^{\beta\mathrm{I}} - \hat{p}_j^{\beta\mathrm{I}}\right) - \frac{n^2 P}{x_0}\left(\Lambda_j^{\beta} + K - 1\right)\hat{\hat{s}}_j^{\mathrm{I}} - \frac{2n}{x_0^2}K\Lambda_j^{\beta}u_{\mathrm{b}j} \\ & + n^2 P\left[\left(\Lambda_j^{\beta} + K - 1\right)\left(\hat{\hat{G}}_j^{\mathrm{I}} - \hat{\hat{p}}_j^{\mathrm{I}} - \frac{m^2 \hat{\hat{J}}_j^{\mathrm{I}}}{n^2}\right) - M_4 \hat{w}_j^{\beta\mathrm{I}} + n^2 M_3 \hat{p}_j^{\beta\mathrm{I}} - M_5 \hat{T}_j^{\beta\mathrm{I}} + m^2 \hat{s}_j^{\beta\mathrm{I}}\right] + O(x_0). \end{aligned} \tag{C 20}$$

(A 51) has been used to simplify (C 19).

The next step is to use (C 5) and the following similar expansions:

$$\Gamma_{\mathrm{a}}^{\beta} = \frac{(1+P)}{2} - aPx_0 + Pa^2 x_0^2 + O\left(x_0^3\right) \tag{C 21}$$

and

$$K = kx_0 - k^2 x_0^2 + O\left(x_0^3\right) \tag{C 22}$$

to eliminate Λ_j^{β}, $\Gamma_{\mathrm{a}}^{\beta}$ and K from (C 19) and (C 20). Using these and (C 15), we may write the varicose ($P = 1$) and sinuous ($P = -1$) versions of (C 19) - (C 20) correct to unit order in x_0 as follows.

$$\begin{aligned} Q_j^{\mathrm{v}\Omega} = {} & \frac{2}{x_0}\left[\frac{n^2}{a^2}\left(\lambda_j \widetilde{G}_{\mathrm{a}j} - G_{\mathrm{a}j}\right) - n\widetilde{u}_{\mathrm{a}j}\right] - \frac{2n^2}{a^2}\lambda_j G_{\mathrm{a}j} + \frac{m^2}{a}\lambda_j \\ & + 2n\lambda_j \widetilde{u}_{\mathrm{a}j} - 2n^2\left(\lambda_j \widetilde{s}_{\mathrm{a}j} - s_{\mathrm{a}j}\right) + O(x_0), \end{aligned} \tag{C 23}$$

$$Q_j^{\mathrm{vI}} = \frac{n^2}{x_0}\left[G_{\mathrm{b}j} - p_{\mathrm{b}j} + G_{\mathrm{c}j} - p_{\mathrm{c}j} - \frac{(\lambda_j - k)}{a^2}\hat{\hat{G}}_j^{\mathrm{I}} - \frac{2k}{n}u_{\mathrm{b}j}\right] + \frac{m^2}{a}(\lambda_j - k)\hat{\hat{J}}_j^{\mathrm{I}}$$
$$- am^2(J_{\mathrm{b}j} + J_{\mathrm{c}j}) + \frac{n^2}{a^2}(\lambda_j^2 - k^2)\hat{\hat{G}}_j^{\mathrm{I}} + 2kn(\lambda_j + k)u_{\mathrm{b}j}$$
$$+ n^2(\lambda_j - k)\left[\hat{\hat{s}}_j^{\mathrm{I}} - G_{\mathrm{b}j} + p_{\mathrm{b}j}\right] - n^2(\lambda_j + k)(G_{\mathrm{c}j} - p_{\mathrm{c}j}) - m^2n^2(s_{\mathrm{b}j} + s_{\mathrm{c}j}) \quad \text{(C 24)}$$
$$+ n^2\left[M_4(w_{\mathrm{b}j} + w_{\mathrm{c}j}) - n^2M_3(p_{\mathrm{b}j} + p_{\mathrm{c}j}) + M_5(T_{\mathrm{b}j} + T_{\mathrm{c}j})\right] + O(x_0),$$

$$Q_j^{\mathrm{s}\Omega} = \lambda_j\left(2n^2\tilde{s}_{\mathrm{a}j} - \frac{m^2}{a} - 2n\,\tilde{u}_{\mathrm{a}j}\right) + O(x_0), \quad \text{(C 25)}$$

$$Q_j^{\mathrm{sI}} = \frac{m^2}{a}(\lambda_j + k)\hat{\tilde{J}}_j^{\mathrm{I}} - n^2\left[(\lambda_j^2 + k^2)s_{\mathrm{b}j} + (\lambda_j^2 - k^2)s_{\mathrm{c}j}\right]$$
$$-\frac{m^2}{a}\left[(\lambda_j^2 + k^2)J_{\mathrm{b}j} + (\lambda_j^2 - k^2)J_{\mathrm{c}j}\right] - 2kn\lambda_j u_{\mathrm{b}j} + n^2(\lambda_j + k)\hat{\tilde{s}}_j^{\mathrm{I}} + O(x_0). \quad \text{(C 26)}$$

C 3. Reconciliation of varicose modes

Consider first expression (C 23) for $Q_j^{\mathrm{v}\Omega}$. This should be equal to Q_Ω^{v} given by (A 55). Using (A 58), (A 67) and (B 45) we see that (C 23) reduces to

$$Q_j^{\mathrm{v}\Omega} = \frac{2n^2\lambda_j}{a^2x_0} + \frac{m^2}{a}\lambda_j + \frac{2n^2\lambda_j}{\mu_j}\left(\lambda_j - \frac{2a^2p_{\mathrm{a}j}}{\mu_j}\right) - 2n^2\left(\lambda_j\tilde{s}_{\mathrm{a}j} - s_{\mathrm{a}j}\right) + O(x_0) \quad \text{(C 27)}$$

and in light of (A 52), the terms proportional to λ_j may be neglected and this may be further reduced to

$$Q_j^{\mathrm{v}\Omega} = 2n^2\left(\frac{\lambda_j^2}{\mu_j} - \frac{2a^2\lambda_j}{\mu_j^2}p_{\mathrm{a}j} + s_{\mathrm{a}j} - \lambda_j\tilde{s}_{\mathrm{a}j}\right) + O(x_0). \quad \text{(C 28)}$$

This is identical to (A 55).

Consider next expression (C 24) for Q_j^{vI}. This should be equal to $kQ_{\mathrm{I}}^{\mathrm{v}}$ where $Q_{\mathrm{I}}^{\mathrm{v}}$ is given by (A 56). This is the most complicated expression to evaluate. To clarify this process let us divide it into three parts:

$$Q_j^{\mathrm{vI}} = Q_j^{\mathrm{vA}} + Q_j^{\mathrm{vB}} + n^2Q_j^{\mathrm{vC}} + O(x_0) \quad \text{(C 29)}$$

where

$$Q_j^{\mathrm{vA}} = \frac{n^2}{x_0}\left[G_{\mathrm{b}j} - p_{\mathrm{b}j} + G_{\mathrm{c}j} - p_{\mathrm{c}j} - \frac{(\lambda_j - k)}{a^2}\hat{\hat{G}}_j^{\mathrm{I}} - \frac{2k}{n}u_{\mathrm{b}j}\right], \quad \text{(C 30)}$$

$$Q_j^{vB} = \frac{m^2}{a}\left(\lambda_j - k\right)\hat{\hat{J}}_j^{I} - am^2(J_{bj} + J_{cj}) + \frac{n^2}{a^2}\left(\lambda_j^2 - k^2\right)\hat{\hat{G}}_j^{I}$$
$$+ 2kn\left(\lambda_j + k\right)u_{bj} - n^2\left(\lambda_j - k\right)\left(G_{bj} - p_{bj}\right) - n^2\left(\lambda_j + k\right)\left(G_{cj} - p_{cj}\right) \qquad \text{(C 31)}$$

and

$$Q_j^{vC} = \left(\lambda_j - k\right)\hat{\hat{s}}_j^{I} - m^2(s_{bj} + s_{cj})$$
$$+ M_4(w_{bj} + w_{cj}) - n^2 M_3(p_{bj} + p_{cj}) + M_5(T_{bj} + T_{cj}). \qquad \text{(C 32)}$$

The first of these is singular in the limit $x_0 \to 0$ and must not contribute to $\widetilde{\Omega}_2^{\beta}$.

Using (B 45) and (C 17) to eliminate u_{bj}, $\hat{\hat{G}}_j^{I}$, $\hat{\hat{J}}_j^{I}$, and $\hat{\hat{s}}_j^{I}$, from (C 30) - (C 31), we have

$$Q_j^{vA} = -\frac{n^2}{a^2 x_0}\left[\frac{\left(\lambda_j - k\right)}{\left(\lambda_j + k\right)}\left(\gamma_j G_{bj} + a^2 p_{bj}\right) + \psi_j G_{cj} + a^2 p_{cj}\right], \qquad \text{(C 33)}$$

$$Q_j^{vB} = \frac{m^2}{a}\left(\gamma_j J_{bj} + \psi_j J_{cj}\right) - \frac{2km^2}{a}\left(\lambda_j + k\right)J_{bj} + \frac{n^2}{a^2}\left(\lambda_j - k\right)\left(\gamma_j G_{bj} + a^2 p_{bj}\right)$$
$$+ \frac{n^2}{a^2}\left(\lambda_j + k\right)\left(\psi_j G_{cj} + a^2 p_{cj}\right) + 2kn^2\left(G_{bj} - p_{bj}\right), \qquad \text{(C 34)}$$

$$Q_j^{vC} = M_3\left[\gamma_j p_{bj} - T_{bj} + \psi_j p_{cj} - T_{cj} - 2k\left(\lambda_j + k\right)p_{bj}\right]$$
$$+ M_2\left[\gamma_j T_{bj} - w_{bj} + \psi_j T_{cj} - w_{cj} - 2k\left(\lambda_j + k\right)T_{bj}\right] \qquad \text{(C 35)}$$
$$+ M_1\left[\gamma_j(w_{bj} + p_{bj}) + n^2 p_{bj} + \psi_j(w_{cj} + p_{cj}) + n^2 p_{cj} - 2k\left(\lambda_j + k\right)\left(w_{bj} + p_{bj}\right)\right]$$

where γ_j is given by (A 23) and ψ_j by (B 34). In writing (C 35) we have used the fact that

$$M_{q+3} = -M_{q+1} - n^2 M_q \qquad \text{(C 36)}$$

for any integer q, and have written

$$s = M_3 p + M_2 T + n^2 M_1(w + p). \qquad \text{(C 37)}$$

To evaluate (C 33) and (C 34), we must develop expressions for G_{cj} and J_{cj}. Combining (2.20), (2.21), (B 33), (B 40) and (A 58) we have that

$$G_{cj} = -\frac{\lambda_j\left(\lambda_j - k\right)}{\psi_j} - \frac{a^2}{\psi_j}p_{cj}, \qquad J_{cj} = 1 + \frac{k}{\psi_j}\left(\lambda_j - k\right). \qquad \text{(C 38)}$$

Using these (A 51) and (A 67) to eliminate G and J, (C 33) and (C 34) may be expressed as

$$Q_j^{vA} = \frac{2n^2}{a^2 x_0}\lambda_j(\lambda_j - k) \tag{C 39}$$

and

$$Q_j^{vB} = \frac{2m^2}{a}\left(\lambda_j^2 - a^2 - k\lambda_j + \frac{a^2k^2}{\gamma_j}\right) - \frac{2n^2}{a^2}\lambda_j\left(\lambda_j^2 - k^2\right) - \frac{2kn^2}{\gamma_j}\zeta_j\left(\lambda_j + \zeta_j p_{bj}\right) \tag{C 40}$$

where ζ_j is defined by (A 15). Using (4.10) and (A 52) we see that Q_j^{vA} does not contribute to $\tilde{\Omega}_2^{\beta}$ and may be ignored. Also (C 40) may be simplified to

$$Q_j^{vB} = \frac{2ak^2m^2}{\gamma_j} - \frac{2kn^2}{\gamma_j}\zeta_j\left(\lambda_j + \zeta_j p_{bj}\right). \tag{C 41}$$

Using (B 32), (B 33) and (C 37), (C 35) simplifies to

$$Q_j^{vC} = 2M_3\mu_j^3 + 2M_2\sigma\mu_j^3 - 2k\left(\lambda_j + k\right)s_{bj}. \tag{C 42}$$

Again taking note of (4.10) this may be simplified to

$$Q_j^{vC} = -2k\left(\lambda_j + k\right)s_{bj}. \tag{C 43}$$

Now we may recombine (C 41) and (C 43) to write

$$Q_j^{vI} = -2kn^2\zeta_j\left(\frac{\lambda_j + \zeta_j p_{bj}}{\gamma_j} + s_{bj}\right) + \frac{2ak^2m^2}{\gamma_j} + O(x_0). \tag{C 44}$$

This is identical to kQ_I^v where Q_I^v is given by (A 56).

C 4. Reconciliation of sinuous modes

Reconciliation of the sinuous mode is complicated by the fact that expression (A 64) is not of the same form as (C 18), having the extra term involving M_3^2. Combining (1.4.32) and (4.9) we have that

$$M_3 = \Omega_1^s - \frac{1}{\sqrt{2}} = 2\sum_{j=1}^{3}\lambda_j^2 A_j. \tag{C 45}$$

Using this and noting that $1/\sqrt{2} = \mathrm{Im}[k]$, we may write (A 64) as

$$\tilde{\Omega}_2^{s} = \tilde{\Omega}_1^{s}\sum_{j=1}^{3}\lambda_j A_j\left(Q_\Omega^{s} + \frac{2n^2}{a^2}\lambda_j\right) + \mathrm{Im}\left[k\sum_{j=1}^{3}\lambda_j A_j\left(Q_\mathrm{I}^{s} - \frac{2n^2}{a^2}\lambda_j\right)\right]. \quad \text{(C 46)}$$

Comparing this with (C 18) we see that $Q_j^{s\Omega}$ must be equivalent to $\lambda_j Q_\Omega^{s} + 2(n^2/a^2)\lambda_j^2$ and $Q_j^{s\mathrm{I}}$ must be equivalent to $k\lambda_j Q_\mathrm{I}^{s} - 2k(n^2/a^2)\lambda_j^2$ where $Q_j^{s\Omega}$, $Q_j^{s\mathrm{I}}$, Q_Ω^{s} and $Q_\mathrm{I}^{s}$are given by (C 25), (C 26), (A 68) and (A 69), respectively.

Consider first $Q_j^{s\Omega}$. Using (A 19), (A 58) and (B 44) to eliminate $\tilde{u}_{aj}$ from (C 25) and ignoring the term proportional to m^2, by virtue of (A 52), we have

$$Q_j^{s\Omega} = 2n^2\left[\lambda_j\tilde{s}_{aj} - \tilde{G}_{aj} - \frac{(G_{aj} - p_{aj})}{\lambda_j}\right] + O(x_0). \quad \text{(C 47)}$$

Next, using (A 67) to eliminate $\tilde{G}_{aj}$ and G_{aj}, we have, after some simplification and manipulation,

$$Q_j^{s\Omega} = 2\frac{n^2}{a^2}\lambda_j^2 + 2n^2\lambda_j\left[\frac{(\mu_j + 2a^2)}{\mu_j^2}p_{aj} - \frac{\lambda_j(\mu_j + a^2)}{a^2\mu_j} + \tilde{s}_{aj}\right] + O(x_0). \quad \text{(C 48)}$$

In this form it is clear that $Q_j^{s\Omega}$ is equivalent to $\lambda_j Q_\Omega^{s} + 2(n^2/a^2)\lambda_j^2$.

The final task is to show that $Q_j^{s\mathrm{I}}$ is equivalent to $k\lambda_j Q_\mathrm{I}^{s} - 2k(n^2/a^2)\lambda_j^2$. First, we shall use (C 18) to eliminate $\hat{\hat{J}}_j^{\mathrm{I}}$ and $\hat{\hat{s}}_j^{\mathrm{I}}$ from (C 26), giving

$$Q_j^{s\mathrm{I}} = 2k\lambda_j\left[\frac{m^2}{a}J_{bj} + n^2 s_{bj} - n u_{bj}\right] + O(x_0). \quad \text{(C 49)}$$

Next, using (A 58) and (B 44) to eliminate u_{bj} yields

$$Q_j^{s\mathrm{I}} = 2kn^2\lambda_j\left[s_{bj} - \frac{(G_{bj} - p_{bj})}{\lambda_j + k}\right] + \frac{2km^2\lambda_j}{a}J_{bj} + O(x_0). \quad \text{(C 50)}$$

EliminatingJ_{bj} and G_{bj} using (A 51) and (A 67), we have, after makng use of (A 52) and some manipulation,

$$Q_j^{s\mathrm{I}} = -2\frac{kn^2}{a^2}\lambda_j^2 + k\lambda_j\left[2n^2\left(\frac{\lambda_j}{a^2} + \frac{\lambda_j}{\gamma_j} + \frac{\zeta_j}{\gamma_j}p_{bj} + s_{bj}\right) - \frac{2km^2}{a\gamma_j}\zeta_j\right] + O(x_0). \quad \text{(C 51)}$$

This is the desired result.