

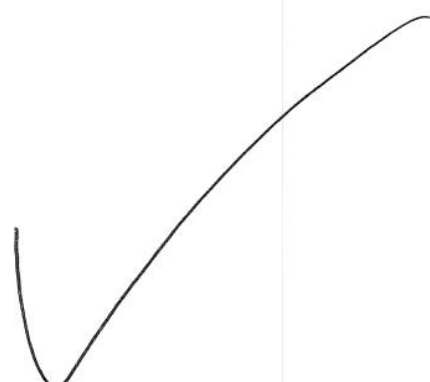
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Subject: addendum of an article
Status:

Dear Dr. Drath,

in an article published in the Journal of Fluid Mechanics (1989, vol 209, pp. 617-637), by S.T. Vuong and S.S. Sadhal, on the "Growth and translation of a liquid-vapour compound drop in a second liquid", it is mentioned that an addendum containing various integral expressions can be obtained from the editors of the Journal of Fluid Mechanics. Therefore, I would like to ask you if this addendum can still be obtained, and if so, what I have to do in order to obtain it.

Best regards
jan fransaer

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Karen. Please could you send him
a copy of this.

Linda.

Growth and translation of a liquid-vapour
compound drop in a second liquid.

Part 1: fluid mechanics.

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APPENDIX

A Integral Expansions

In this Appendix, we have given the various integral expansions that are needed for the solution.

A.1 Expansion for the uniform stream

The uniform stream is represented by the stream function

$$\psi(\xi, \eta) = \frac{1}{2}r^2 = \frac{1}{2} \frac{\sinh^2 \xi}{(\cosh \xi - \cos \eta)^2}, \quad (\text{A.1})$$

or in terms of the dependent variable $\Phi(\xi, \eta)$, we have

$$\Phi(\xi, \eta) = (\cosh \xi - \cos \eta)^{\frac{3}{2}} \psi = \frac{1}{2} \frac{\sinh^2 \xi}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}}. \quad (\text{A.2})$$

Thus we would require an expansion of the right hand side of (A.2) in the form

$$\frac{1}{2} \frac{\sinh^2 \xi}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} = \int_0^\infty G(\eta, \lambda) \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda \quad (\text{A.3})$$

However, the denominator of $\Phi(\xi, \eta)$ in equation (A.2) does not go to zero fast enough as $\xi \rightarrow \infty$ to allow a proper expansion. Instead, the usual procedure (as formulated by Payne & Pell (1960)) is to expand a function that goes to zero faster as $\xi \rightarrow \infty$ but behaves as a uniform stream in the far field. The details of that derivation have been given by Payne & Pell (1960). The appropriate function for expansion is

$$\frac{1}{2} \left[\frac{1}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} - \frac{1}{(\cosh \xi + \cos \eta)^{\frac{1}{2}}} \right] = \int_0^\infty F(\eta, \lambda) P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda, \quad (\text{A.4})$$

where only the first term on the left arises from the uniform stream. The function $F(\eta, \lambda)$ is found to be:

$$F(\eta, \lambda) = \frac{-\cos \eta [\cosh \lambda(\pi - |\eta|) + \cosh \lambda \eta]}{\sqrt{2}(1 + \lambda^2) \cosh \lambda \pi} + \frac{\lambda \sin |\eta| [-\sinh \lambda(\pi - |\eta|) + \sinh \lambda |\eta|]}{\sqrt{2}(1 + \lambda^2) \cosh \lambda \pi}, \quad (\text{A.5})$$

$(-\pi < \eta < \pi).$

The derivatives of $F(\eta, \xi)$ with respect to η are:

$$F'(\eta, \lambda) = \frac{\sin \eta [\cosh \lambda(\pi - |\eta|) + \cosh \lambda \eta]}{\sqrt{2} \cosh \lambda \pi} \quad (\text{A.6})$$

$$F''(\eta, \lambda) = \frac{\cos \eta [\cosh \lambda(\pi - |\eta|) + \cosh \lambda \eta]}{\sqrt{2} \cosh \lambda \pi} + \frac{\lambda \sin |\eta| [-\sinh \lambda(\pi - |\eta|) + \sinh \lambda |\eta|]}{\sqrt{2} \cosh \lambda \pi} \quad (\text{A.7})$$

A.2 Expansion of a constant function

Here we begin with the orthogonality relation for the Mehler-Fock transform:

$$\frac{\delta(\xi - \xi')}{\sinh \xi} = \int_0^\infty \lambda \tanh \lambda \pi P_{-\frac{1}{2}+i\lambda}(\cosh \xi') P_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda. \quad (\text{A.8})$$

Multiplying both sides by $\sinh \xi d\xi$ and integrating from 0 to ξ gives:

$$\int_0^\xi \delta(\xi - \xi') d\xi = \int_0^\infty \lambda \tanh \lambda \pi P_{-\frac{1}{2}+i\lambda}(\cosh \xi') \int_0^\xi P_{-\frac{1}{2}+i\lambda}(\cosh \xi) \sinh \xi d\xi d\lambda. \quad (\text{A.9})$$

Since by integrating Legendre's equation, we have

$$\int_0^\xi P_{-\frac{1}{2}+i\lambda}(\cosh \xi) \sinh \xi d\xi = -\frac{1}{\lambda^2 + \frac{1}{4}} \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda, \quad (\text{A.10})$$

equation (A.9), upon integrating the δ -function, can be written as

$$H(\xi - \xi') = -\int_0^\infty \lambda \tanh \lambda \pi P_{-\frac{1}{2}+i\lambda}(\cosh \xi') \frac{1}{\lambda^2 + \frac{1}{4}} \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda, \quad (\text{A.11})$$

where $H(\xi - \xi')$ is the Heaviside step function. Letting $\xi' = 0$ and maintaining $\xi > 0$:

$$H(\xi) = 1 = -\int_0^\infty \frac{\lambda \tanh \lambda \pi}{\lambda^2 + \frac{1}{4}} P_{-\frac{1}{2}+i\lambda}(1) \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda, \quad (\xi > 0). \quad (\text{A.12})$$

Since $P_{-\frac{1}{2}+i\lambda}(1) = 1$, we may write

$$1 = -\int_0^\infty \frac{\lambda \tanh \lambda \pi}{\lambda^2 + \frac{1}{4}} \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda, \quad \xi > 0. \quad (\text{A.13})$$

A.3 Expansion of the term $(\cosh \xi - \cos \eta)^{-\frac{1}{2}}$

Using the integral definition given in Hobson (1931, pg. 451):

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} = \sqrt{2} \int_0^\infty \frac{\cosh \lambda(\pi - \eta)}{\cosh \lambda \pi} P_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda, \quad 0 < \eta < 2\pi \quad (\text{A.14})$$

and differentiating with respect to η , we obtain

$$\frac{\sin \eta}{2(\cosh \xi - \cos \eta)^{\frac{3}{2}}} = \sqrt{2} \int_0^\infty \frac{\lambda \sinh \lambda(\pi - \eta)}{\cosh \lambda \pi} P_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda. \quad (\text{A.15})$$

Now, by multiplying both sides by $\sinh \xi d\xi$, and integrating from 0 to ξ it is not difficult to see that

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} - \frac{1}{(1 - \cos \eta)^{\frac{1}{2}}} = \sqrt{2} \int_0^\infty \frac{\lambda}{(\lambda^2 + \frac{1}{4}) \sin \eta \cosh \lambda \pi} \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda. \quad (\text{A.16})$$

Since the second term on the left is constant in ξ , we may replace it by the constant expansion (A.13) multiplied by the appropriate factor. Thus, we have

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} = \int_0^\infty \frac{\lambda}{\lambda^2 + \frac{1}{4}} \left[\frac{\sqrt{2} \sinh \lambda(\pi - \eta)}{\sin \eta \cosh \lambda \pi} - \frac{\tanh \lambda \pi}{(1 - \cos \eta)^{\frac{1}{2}}} \right] \sinh^2 \xi P_{-\frac{1}{2}+i\lambda}'(\cosh \xi) d\lambda \quad 0 < \eta < 2\pi \quad (\text{A.17})$$

or

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{1}{2}}} = \int_0^\infty \frac{\lambda}{\lambda^2 + \frac{1}{4}} \left[\frac{\sqrt{2} \sinh \lambda(\pi - |\eta|)}{\sin |\eta| \cosh \lambda \pi} - \frac{\tanh \lambda \pi}{(1 - \cos \eta)^{\frac{1}{2}}} \right] \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda$$

$$(-\pi < \eta < \pi) \quad (\text{A.18})$$

A.4 Expansion of the term $(\cosh \xi - \cos \eta)^{-\frac{3}{2}}$

Differentiating the above expression (A.18) with respect to η gives:

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{3}{2}}} = \int_0^\infty \frac{\lambda}{\lambda^2 + \frac{1}{4}} \left[\frac{2\sqrt{2}}{\sin^3 |\eta| \cosh \lambda \pi} \times \right. \\ \left. \{ \lambda \sin |\eta| \cosh \lambda(\pi - |\eta|) + \cos \eta \sinh \lambda(\pi - |\eta|) \} - \frac{\tanh \lambda \pi}{(1 - \cos \eta)^{\frac{3}{2}}} \right] \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda$$

$$-\pi < \eta < \pi \quad (\text{A.19})$$

A.5 Expansion of the term $(\cosh \xi - \cos \eta)^{-\frac{5}{2}}$

Differentiation of the above expression (A.19) with respect to η yields:

$$\frac{1}{(\cosh \xi - \cos \eta)^{\frac{5}{2}}} = \int_0^\infty \frac{\lambda}{\lambda^2 + \frac{1}{4}} \left[\frac{4\sqrt{2}}{3 \sin^5 |\eta| \cosh \lambda \pi} \{ 3 \cos \eta \sinh \lambda(\pi - |\eta|) \right. \\ \left. + (1 + \lambda^2) \sin^2 |\eta| \sinh \lambda(\pi - |\eta|) + 3 \lambda \sin |\eta| \cosh \lambda(\pi - |\eta|) \} - \frac{\tanh \lambda \pi}{(1 - \cos \eta)^{\frac{5}{2}}} \right] \\ \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda, \quad -\pi < \eta < \pi \quad (\text{A.20})$$

A.6 Expansion of the term $(\cosh \xi - \cos \eta)^{\frac{1}{2}}$

By multiplying both sides of (A.14) with $\sinh \xi d\xi$ and integrating from 0 to ξ one obtains:

$$(\cosh \xi - \cos \eta)^{\frac{1}{2}} - (1 - \cos \eta)^{\frac{1}{2}} = - \int_0^\infty \frac{1}{\sqrt{2}} \frac{1}{\lambda^2 + \frac{1}{4}} \frac{\cosh \lambda(\pi - \eta)}{\cosh \lambda \pi} \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda. \quad (\text{A.21})$$

Again, representing the second term on the left with expansion of a constant function in ξ , gives the following:

$$(\cosh \xi - \cos \eta)^{\frac{1}{2}} = - \int_0^\infty \frac{1}{\lambda^2 + \frac{1}{4}} \left[\lambda \tanh \lambda \pi (1 - \cos \eta)^{\frac{1}{2}} \right. \\ \left. + \frac{1}{\sqrt{2}} \frac{\cosh \lambda(\pi - \eta)}{\cosh \lambda \pi} \right] \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda$$

$$(0 < \eta < 2\pi) \quad (\text{A.22})$$

or

$$(\cosh \xi - \cos \eta)^{\frac{1}{2}} = - \int_0^\infty \frac{1}{\lambda^2 + \frac{1}{4}} \left[\lambda \tanh \lambda \pi (1 - \cos \eta)^{\frac{1}{2}} \right. \\ \left. + \frac{1}{\sqrt{2}} \frac{\cosh \lambda(\pi - |\eta|)}{\cosh \lambda \pi} \right] \sinh^2 \xi P'_{-\frac{1}{2}+i\lambda}(\cosh \xi) d\lambda$$

$$(-\pi < \eta < \pi). \quad (\text{A.23})$$

References

- [1] PAYNE, L.E. & PELL, W.W. 1960. The Stokes flow problem for a class of axially symmetric bodies. *J. Fluid Mech.* **7**: 529-549.
- [2] HOBSON, E.W. 1931 *The theory of spherical and ellipsoidal harmonics*. Cambridge University Press.