FM 7059 WEINSTOCK A theory of turbulent transport Appendices D, E, F,

APPENDIX ~~C~~ D

To derive (2.13) it is convenient to first Fourier expand $\underset{\sim}{u}$ and $p^{(s)}$ in $\langle u_3 \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)} \rangle$ and $\langle \underset{\sim}{u}\underset{\sim}{u} \nabla p^{(s)}/\partial x_3 \rangle$. This derivation is limited to a weakly inhomogeneous turbulence for which $\langle \underset{\sim}{u}\underset{\sim}{u} \rangle$ is proportional to $\exp(2\underset{\sim}{a}\cdot\underset{\sim}{x})$ where $\underset{\sim}{a}$ is constant and directed along x_3; (i.e., $\underset{\sim}{a}\cdot\underset{\sim}{x} \equiv ax_3$). A simple model for this $\underset{\sim}{u}$ is

$$\underset{\sim}{u} = \underset{\sim}{u}^{o} \exp(ax_3) \tag{C.1}$$

where $\underset{\sim}{u}^o$ is statistically homogeneous; i.e., $\partial \langle \underset{\sim}{u}^o \underset{\sim}{u}^o \rangle / \partial x_3 = 0$. The Fourier expansion of $\underset{\sim}{u}$ is given by

$$\underset{\sim}{u} = (2\pi)^{-3} \int d\underset{\sim}{k} \underset{\sim}{u}^{o}_{\underset{\sim}{k}} \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x} + ax_3), \quad (\text{real } \underset{\sim}{k}) \tag{C.2}$$

where $\underset{\sim}{k}$ is real, and $\underset{\sim}{u}^o_{\underset{\sim}{k}}$ is the Fourier transform of $\underset{\sim}{u}^o \equiv \underset{\sim}{u} \exp(-ax_3)$.

We also need the Fourier expansion of $p^{(s)}$, which, since it is given by

$$\nabla^2 p^{(s)} = -\underset{\sim}{\nabla}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\underset{\sim}{u}), \tag{C.3}$$

is proportional to $\exp(2\underset{\sim}{a}\cdot\underset{\sim}{x})$ when (C.2) is substituted for $\underset{\sim}{u}$.

Therefore, we may define a homogeneous pressure field p^o by

$$p^{(s)} = p^o \exp(2\underset{\sim}{a}\cdot\underset{\sim}{x}) \tag{C.4}$$

and Fourier expand $p^{(s)}$ as

$$p^{(s)} = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} p^o_{\underset{\sim}{k}} \exp(i\cdot\underset{\sim}{k}\cdot\underset{\sim}{x} + 2\underset{\sim}{a}\cdot\underset{\sim}{x})$$

where $\underset{\sim}{k}$ is real and $p^o_{\underset{\sim}{k}}$ is the Fourier transform of $p^o \equiv p^{(s)}\exp(-2\underset{\sim}{a}\cdot\underset{\sim}{x})$.

The value of $p^o_{\underset{\sim}{k}}$ is determined from (C.3) by multiplying both sides with $\exp(-2\underset{\sim}{a}\cdot\underset{\sim}{x})$ and then taking the Fourier transform of both sides utilizing (C.2) and (C.4): The result is

$$p^o_{\underset{\sim}{k}} = - \int \frac{d\underset{\sim}{k}_a}{(2\pi)^3} \frac{(i\underset{\sim}{k}+2\underset{\sim}{a})\cdot(u^o_{\underset{\sim}{k}_a} u^o_{\underset{\sim}{k}_b})'\cdot(i\underset{\sim}{k}_a+\underset{\sim}{a})}{(i\underset{\sim}{k}+2\underset{\sim}{a})\cdot(i\underset{\sim}{k}+2\underset{\sim}{a})} \tag{C.5}$$

$$\underset{\sim}{k}_a + \underset{\sim}{k}_b \equiv \underset{\sim}{k}, \quad \text{(real)}$$

where the prime on $(u^o_{\underset{\sim}{k}_a} u^o_{\underset{\sim}{k}_b})'$ denotes that we exclude $\underset{\sim}{k}_a + \underset{\sim}{k}_b = 0$. (We caution that the symbol a is used for both the scale length of inhomogeneity and as an index in $\underset{\sim}{k}_a$, and ask the readers indulgence for this poor notation.) This equation together with (C.4) gives the Fourier expansion of $p^{(s)}$.

The Fourier expansion of $\langle u_3 \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)}\rangle$ can now be written by substitution of (C.2), (C.4), and (C.5) to yield

$$\langle u_3 \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)}\rangle = - \int \frac{d\underset{\sim}{k}_a d\underset{\sim}{k}_b d\underset{\sim}{k}_c d\underset{\sim}{k}_d}{(2\pi)^{12}} \frac{\langle w_{\underset{\sim}{k}_c} \underset{\sim}{u}_{\underset{\sim}{k}_d} (\underset{\sim}{u}_{\underset{\sim}{k}_a} \underset{\sim}{u}_{\underset{\sim}{k}_b})'\rangle : (i\underset{\sim}{k}_a+\underset{\sim}{a})(i\underset{\sim}{k}+2\underset{\sim}{a})^2}{(i\underset{\sim}{k}+2\underset{\sim}{a})\cdot(i\underset{\sim}{k}+2\underset{\sim}{a})}$$

$$\times \exp[-i(\underset{\sim}{k}+\underset{\sim}{k}_c+\underset{\sim}{k}_d)\cdot\underset{\sim}{x}] \tag{C.6}$$

$$\underset{\sim}{u}_{\underset{\sim}{k}_a} \equiv \underset{\sim}{u}^o_{\underset{\sim}{k}_a} \exp(\underset{\sim}{a}\cdot\underset{\sim}{x}) \text{ as defined by (C.1),}$$

where $w_{\underset{\sim}{k}_c}$ is the vertical component of $\underset{\sim}{u}_{\underset{\sim}{k}_c}$ and we have used the convenient vector notation $\langle w\underset{\sim}{u}(\underset{\sim}{u}\underset{\sim}{u})'\rangle : (i\underset{\sim}{k}_a + \underset{\sim}{a})(i\underset{\sim}{k} + 2\underset{\sim}{a}) = \langle w\underset{\sim}{u}(i\underset{\sim}{k} + 2\underset{\sim}{a})\cdot(\underset{\sim}{u}\underset{\sim}{u})'\cdot(i\underset{\sim}{k}_a + \underset{\sim}{a})\rangle$

The correlation in (C.6) is evaluated by cumulant expansion

$$\langle w_{\underset{\sim}{k}_c} \underset{\sim}{u}_{\underset{\sim}{k}_d} (\underset{\sim}{u}_{\underset{\sim}{k}_a} \underset{\sim}{u}_{\underset{\sim}{k}_b})'\rangle = \langle w_{\underset{\sim}{k}_c} \underset{\sim}{u}_{\underset{\sim}{k}_a}\rangle\langle \underset{\sim}{u}_{\underset{\sim}{k}_d} \underset{\sim}{u}_{\underset{\sim}{k}_b}\rangle$$

$$\langle w_{\underset{\sim}{k}_c} \underset{\sim}{u}_{\underset{\sim}{k}_b}\rangle\langle \underset{\sim}{u}_{\underset{\sim}{k}_d} \underset{\sim}{u}_{\underset{\sim}{k}_a}\rangle + Q^+, \tag{C.7}$$

where Q^+ is the forth-order cumulant. Our basic approximation is to neglect Q^+ as well as all other forth-order cumulants. Substitution of (C.7) in (C.6) and use of the quasi-homogeneous condition (B.5)--justified since $u^o_{\underset{\sim}{k}}$ is statistically homogeneous -- we have

$$\langle u_3 \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)}\rangle \sim -2 \int \frac{d\underset{\sim}{k}_a d\underset{\sim}{k}_b}{(2\pi)^6} \frac{\underset{\sim}{x}_3 (i\underset{\sim}{k}_a+\underset{\sim}{a}) : \underset{\approx}{S}(\underset{\sim}{k}_b)\underset{\approx}{S}(\underset{\sim}{k}_a)\cdot(i\underset{\sim}{k}+2\underset{\sim}{a})^2}{-k^2 + 4i\underset{\sim}{a}\cdot\underset{\sim}{k} + 4a^2} \tag{C.8}$$

$$\underset{\approx}{S}(\underset{\sim}{k}) \equiv V^{-1}\langle u^o_{\underset{\sim}{k}}(u^o_{\underset{\sim}{k}})^*\rangle \exp(2\underset{\sim}{a}\cdot\underset{\sim}{x})$$

where the asterisk denotes the complex conjugate and we have used the vector notation $\langle w_{\underset{\sim}{k}_c} u^*_{\underset{\sim}{k}_a}\rangle \equiv \underset{\sim}{x}_3\cdot\langle \underset{\sim}{u}_{\underset{\sim}{k}_c} \underset{\sim}{u}^*_{\underset{\sim}{k}_a}\rangle$.

For small inhomogeneity ($a \ll k_o$), the integrand of (C.8) is expanded in powers of $\underset{\sim}{a}$, retaining terms to first order:

$$\langle u_3 \underset{\sim}{u}\underset{\sim}{\nabla} p^{(s)}\rangle = -2 \int \frac{d\underset{\sim}{k}_a d\underset{\sim}{k}_b}{(2\Pi)^6} [\underset{\sim}{a}\cdot\underset{\approx}{S}(\underset{\sim}{k}_b)\cdot\hat{\underset{\sim}{x}}_3 \underset{\approx}{S}(\underset{\sim}{k}_a)\cdot\underset{\sim}{k}_b\underset{\sim}{k} + \underset{\sim}{a}\cdot\underset{\approx}{S}(\underset{\sim}{k}_a)\underset{\sim}{k}_a\cdot\underset{\approx}{S}(\underset{\sim}{k}_b)\cdot\hat{\underset{\sim}{x}}_3\underset{\sim}{k}$$
$$+ 2\left(\underset{\sim}{a} - \frac{2\underset{\sim}{a}\cdot\underset{\sim}{k}^2}{k^2}\right)\underset{\sim}{k}_b\cdot\underset{\approx}{S}(\underset{\sim}{k}_a)\underset{\sim}{k}_a\cdot\underset{\approx}{S}(\underset{\sim}{k}_b)\cdot\hat{\underset{\sim}{x}}_3]k^{-2}, \qquad (C.9)$$

where we have used the incompressibility condition $(i\underset{\sim}{k}_a + a)\cdot\underset{\approx}{S}(\underset{\sim}{k}_a)=0$ [i.e., $(i\underset{\sim}{k}_a+\underset{\sim}{a})\cdot u_{\underset{\sim}{k}_a} = 0$ which follows from the Fourier transform of $\underset{\sim}{\nabla}\cdot\underset{\sim}{u}\exp(-\underset{\sim}{a}\cdot\underset{\sim}{x}) = 0$], and the zeroth order term vanishes because it is an odd power of $\underset{\sim}{k}$ and of $\underset{\sim}{k}$. This vanishing is as should be since $\langle u_3 \underset{\sim}{u}\underset{\sim}{\nabla}\underset{\sim}{p}^{(s)}\rangle$ is zero for a homogeneous turbulence (i.e., for a = 0). The integrations in (C.9) are straight-forwardly evaluated in the small anisotropy limit when $\underset{\approx}{S}(k_1)$ is given by

$$\underset{\approx}{S}(\underset{\sim}{k}_a) = 2\Pi^2\left(\underset{\approx}{I} - \frac{\underset{\sim}{k}_a\underset{\sim}{k}_a}{k_a^2}\right)\frac{E(k_a)}{k_a^2}, \qquad \text{isotropic } \underset{\approx}{S} \qquad (C.10)$$

where $E(k_a)$ is the scalar energy spectrum normalized by $\int dk_a E(k_a) = (3/2)v_o^2$. Substitution of (C.10) in (C.9) allows us to readily calculate the $\underset{\sim}{x}_3\underset{\sim}{x}_3$ $\underset{\sim}{x}_1\underset{\sim}{x}_1$ and $\underset{\sim}{x}_1\underset{\sim}{x}_3$ components of the pressure correlation. These are found to

$$\langle u_3^2 \partial p^{(s)}/\partial x_3\rangle = \frac{1}{15}\left(\frac{3}{2}v_o{}^2\right)^2 a$$

$$\langle u_3 u_1 \partial p^{(s)}/\partial x_1) = \frac{2}{225}\left(\frac{3}{2}v_o{}^2\right)^2 a \qquad (C.11)$$

$$\langle u_3 u_1 \partial p^{(s)}/\partial x_3\rangle = 0$$

These components are seen to be very small.

Before comparison of (C.11) with $\underset{\approx}{A}^{o}$ we must have the components of $\langle \underset{\sim}{u}\underset{\sim}{u}\partial p^{(s)}/\partial x_3\rangle$ to confirm (2.13). An expression for $\langle \underset{\sim}{u}\underset{\sim}{u}\partial p^{(s)}/\partial x_3\rangle$ is obtained in the same manner as was done for (C.8) by substitution of the Fourier expansions (C.2) and (C.5): The result is

$$\langle \underset{\sim}{u}\underset{\sim}{u}\frac{\partial p^{(s)}}{\partial x_3}\rangle = -2\int \frac{d\underset{\sim}{k}_a d\underset{\sim}{k}_b}{(2\Pi)^6}\;\frac{(i\underset{\sim}{k}_a+\underset{\sim}{a})\cdot\underset{\approx}{S}(\underset{\sim}{k}_b)\underset{\approx}{S}(\underset{\sim}{k}_a)\cdot(i\underset{\sim}{k}+2\underset{\sim}{a})^2(ik_3+a)}{-k^2+4i\underset{\sim}{a}\cdot\underset{\sim}{k}+4a^2} \qquad (C.12)$$

Evaluation of this integral in the limits of small a and small anisotropy yields the $\hat{\underset{\sim}{x}}_3\underset{\sim}{x}_3$, $\underset{\sim}{x}_1\underset{\sim}{x}_1$, and $\underset{\sim}{x}_1\underset{\sim}{x}_3$ components of (C.12) as

$$\langle u_3u_3\partial p^{(s)}/\partial x_3\rangle = \frac{1}{15}\left(\frac{3}{2}v_o^2\right)^2 a$$

$$\langle u_1u_1\partial p^{(s)}/\partial x_3\rangle = -\frac{8}{225}\left(\frac{3}{2}v_o^2\right)^2 a \qquad (C.13)$$

$$\langle u_1u_3\partial p^{(s)}/\partial x_3\rangle = 0.$$

For comparison, the components of $\underset{\approx}{A}^{o}$ are also evaluated in the limit of small anistropy to obtain, with (2.8),

$$A^{o}_{33} = \frac{24}{9}\left(\frac{3}{2}v_o^2\right)^2 a$$

$$A^{o}_{11} = \frac{8}{9}\left(\frac{3}{2}v_o^2\right)^2 a \qquad (C.14)$$

$$A^{o}_{13} = 0$$

Upon comparison, it is seen that the diagonal components of $\underset{\approx}{\Pi}^{(s)}$ $(1+T_r)\langle u_3\underset{\sim}{u}\underset{\sim}{\nabla}p^{(s)}\rangle + \langle\underset{\sim}{u}\underset{\sim}{u}\partial p^{(s)}/\partial x_3\rangle$ are much smaller in magnitude than the corresponding diagonal components of $\underset{\approx}{A}^{o}$, in conformity with (2.13). (The off-diagonalelements are all zero). Additionally, $\underset{\approx}{\Pi}^{(s)}$ is not proportional to $\underset{\approx}{A}^{o}$ since $\Pi^{(s)}_{11}/\Pi^{(s)}_{33} \neq A^{o}_{11}/A^{o}_{33}$.

PART 1

To help evaluate $\underset{\approx}{\Pi}^{(U)}$ we express $p^{(U)}$ as a Fourier integral, valid for periodic boundary conditions or for an asymptotically large system:

$$p^{(U)} = 2i \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} \frac{k_1 u_3^+(\underset{\sim}{k})}{k^2} \frac{\partial U_O}{\partial x_3} \exp\,(i\underset{\sim}{k}\cdot\underset{\sim}{R}), \qquad (D.1)$$

where $u_3^+(\underset{\sim}{k})$ is the Fourier transform of $u_3(\underset{\sim}{x},t)$, and we have limited ourselves to a unidirectional mean flow $\underset{\sim}{U} = [U_O(x_3),0,0]$ which may vary only with x_3. The spatial inhomogeneity factor $\exp(\underset{\sim}{a}\cdot\underset{\sim}{x})$ is ignored in (D.1) since it would only contribute to a higher than first order dependence of $\underset{\approx}{\Pi}^{(U)}$ on a; i.e., the triple-moment in $\underset{\approx}{\Pi}^{(U)}$ is, itself, first order in a. Substitution of (D.1) into (2.11), the definition of $\underset{\approx}{\Pi}^{(U)}$, we have

$$\underset{\approx}{\Pi}^{(U)} = -2\frac{\partial U_O}{\partial x_3} \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} [(1+T_r)\langle u_3 \underset{\sim}{u} u_3^+(\underset{\sim}{k})\rangle \frac{\underset{\sim}{k}k_{1}}{k^2} + \langle \underset{\sim}{u}\underset{\sim}{u} u_3^+(\underset{\sim}{k})\rangle \frac{k_1 k_3}{k^2}]\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}), \quad (D.2)$$

where the quantities u_3 and $\underset{\sim}{u}$ without a super plus are not Fourier transformed.

Our goal is to determine the 13 off - diagonal element and the order of magnitude of the diagonal elements of $\underset{\approx}{\Pi}^{(U)}$ in the limit of small forcing, small $|(\partial U_O/\partial x_3)|$ and $|\partial\theta_O/\partial x_3|$, and small anisotropy. Each element is considered separately: $\Pi_{13}^{(U)}$, $\Pi_{11}^{(U)}$ and $\Pi_{33}^{(U)}$, in that order

The 13 component is expressed by (D.2) as

$$\Pi_{13}^{(U)} = -\frac{\partial U_O}{\partial x_3} \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} [2\langle u_3^2 u_3^+(\underset{\sim}{k})\rangle \frac{k_1^2}{k^2} + 4\langle u_3 u_1 u_3^+(\underset{\sim}{k})\rangle \frac{k_1 k_3}{k^2}]. \quad (D.3)$$

To evaluate this integral we need the triple-moments $\langle u_1^2 u_3^+(\underset{\sim}{k})\rangle$ and $\langle u_3 u_1 u^+(k)\rangle$; later in this appendix we will also need $\langle u_3^2 u_3^+(\underset{\sim}{k})\rangle$. These moments can be obtained with the aid (2.10). To lowest order in $\partial U_O/\partial x_3$ and/or g/θ_O -- and neglect of $\underset{\approx}{Q}^{(4)}$ -- (2.10) gives us

$$3\langle u_3 u_1^2\rangle = -\tau_o A^o_{11}$$
$$3\langle u_3^3\rangle = -\tau_o A^o_{33} \qquad (D.4)$$

$$3\langle u^2 u_1\rangle = -\tau_o\left(A^o_{13} + \Pi^{(U)}_{13} + \langle u_3^3\rangle \frac{\partial U_o}{\partial x_3}\right). \qquad (D.5)$$

The first order terms $\langle u_3^3\rangle \partial U_o/\partial x_3$ and $\Pi^{(U)}_{13}$ are included in (D.5) since A^o_{13} is, itself, first order; i.e., $A^o_{13} \propto \langle u_1 u_3\rangle \propto \partial U_o/\partial x_3$. The g/Θ_o terms have been neglected in (D.5) as second order in the forcing since these terms have the form $\langle u_3 u_1 \theta\rangle g/\Theta_o$ and $\langle u_3 u_1 \theta\rangle$ is of order in $\partial U_o/\partial x_3$ by virtue of the fact it contains u_1. The $\Pi^{(s)}_{13}$ and $\Pi^{(\theta)}_{13}$ are also second order and therefore neglected.

What is required for (D.3) are the Fourier components $\langle u_3 u_j u_3^+(\underset{\sim}{k})\rangle$ where $j = 1,3$. These components can be derived in a formal procedure by taking the Fourier transformof (2.3) and substituting it for $u_3^+(\underset{\sim}{k})$ in $\langle u_3 u_j u_3^+(\underset{\sim}{k})\rangle$ -- a lengthy procedure.Instead, we use a simpler, but more heuristic, method based on (D.4) and (D.5) which gives the same result. In this method, Fourier expansions of A^o_{33} and A^o_{13} in the right side of (D.4) are written, after substitution of our inhomogeneity relation $\partial\langle \underset{\sim}{u}\underset{\sim}{u}\rangle/\partial x_3 = 2a\langle \underset{\sim}{u}\underset{\sim}{u}\rangle$ into (2.8), as follows:

$$A^o_{j3} = 3a \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} \langle u^2\rangle\langle u_j u_3^+(\underset{\sim}{k})\rangle \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}) \qquad (D.6)$$

(where $\langle u_3^2\rangle$ and u_j are not Fourier expanded). In addition, in the left side of (D.4) is Fourier expanded as $\langle u_3 \underset{\sim}{u}\underset{\sim}{u}\rangle = (2\Pi)^{-3} \int d\underset{\sim}{k}\langle \underset{\sim}{u}\underset{\sim}{u}u_3^+(\underset{\sim}{k})\rangle \exp(ik\cdot x)$. The Fourier components of each side of (D.4) are then equated to yield

$$3\langle u_1^2 u_3^+(\underset{\sim}{k})\rangle = -\tau_o A^+_{11}(\underset{\sim}{k})$$
$$3\langle u_3^2 u_3^+(\underset{\sim}{k})\rangle = -\tau_o A^+_{33}(\underset{\sim}{k}), \qquad (D.7)$$

where

$$A^{+}_{11}(\underset{\sim}{k}) \equiv 2a\langle u_1^2\rangle\langle u_3 u_3^{+}(\underset{\sim}{k})\rangle + 4a\langle u_1 u_3\rangle\langle u_1 u_3^{+}(\underset{\sim}{k})\rangle,$$
$$A^{+}_{j}(k) \equiv 6a\langle u_3^2\rangle\langle u_j u_3^{+}(\underset{\sim}{k})\rangle. \tag{D.8}$$

It may be heuristic to equate the Fourier components of both sides of (D.4), but we have also derived the same (D.7) in more rigorous, and lengthy fashion. Similarly, Fourier expansion of both sides of (D.5), with use of (D.3) for $\Pi^{(U)}_{12}$, we have

$$3\langle u_3 u_1 u_3^{+}(\underset{\sim}{k})\rangle = -\tau_O\Big[A^{+}_{13}(\underset{\sim}{k}) - \frac{\partial U_O}{\partial x_3}\Big\{\langle u_3^2 u_3^{+}(\underset{\sim}{k})\rangle\Big(\frac{2k_1^2}{k^2} - 1\Big) + 4\langle u_3 u_1 u_3^{+}\rangle\frac{k_1 k_3}{k^2}\Big\}\Big], \tag{D.9}$$

Eqn's. (D.7) and (D.8) give $\langle u_3^2 u_3^{+}(k)\rangle$ and $\langle u_1^2 u_3^{+}(\underset{\sim}{k})\rangle$ in terms of second-moments. To obtain $\langle u_3 u_1 u_3^{+}(\underset{\sim}{k})\rangle$ in terms of second-moments, (D.7) is substituted in (D.9) to yield

$$3\langle u_3 u_1 u_3^{+}(\underset{\sim}{k})\rangle = -\tau_O\Big[A^{+}_{13}(\underset{\sim}{k}) + \frac{1}{3}\tau_O\frac{\partial U_O}{\partial x_3}A^{+}_{33}\Big(\frac{2k_1^2}{k^2} - 1\Big)\Big]$$
$$\times\Big(1 - \frac{4}{3}\tau_O\frac{\partial U_O}{\partial x_3}\frac{k_1 k_3}{k^2}\Big)^{-1}. \tag{D.10}$$

Evaluation of $\Pi^{(U)}_{13}$ can now be made by substitution of (D.7) and (D.10) in (D.3) which, to lowest order in $\partial U_O/\partial x_3$, yields

$$\Pi^{(U)}_{13} = \tau_O\frac{\partial U_O}{\partial x_3}\int\frac{d\underset{\sim}{k}}{(2\Pi)^3}\Big[\Big(\frac{2k_1^2}{3k^2} + \frac{4}{9}\tau_O\frac{\partial U_O}{\partial x_3}\Big\{\frac{2k_1^2}{k^2} - 1\Big\}\frac{k_1 k_3}{k^2}\Big)A^{+}_{33}(\underset{\sim}{k})$$
$$+ \tau_O\frac{\partial U_O}{\partial x_3}\Big(\frac{4}{3}\frac{k_1 k_3}{k^2}\Big)^2 A^{+}_{13}(\underset{\sim}{k})\Big]. \tag{D.11}$$

The quantity $A^{+}_{13}(\underset{\sim}{k})$ is expressed in terms of spectra as

$$A^{+}_{j3}(\underset{\sim}{k}) = 6a\langle u_3^2\rangle S_{j3}(\underset{\sim}{k}). \tag{D.12}$$

This expression obtained from (D.8), (B.5) and use of

$$u_j u_3^+(\underset{\sim}{k})\rangle = (2\Pi)^{-3} \int d\underset{\sim}{k}_a \langle u_j^+(\underset{\sim}{k}_a) u_3^+(\underset{\sim}{k})\rangle \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}) = S_{j3}(\underset{\sim}{k})\exp(-i\underset{\sim}{k}\cdot\underset{\sim}{x}).$$

For $S_{j3}(\underset{\sim}{k})$ we use the isotropic expression (C.10). Finally, by substitution of (D.12) and (C.10), the integrations in (D.11) are readily performed to yield

$$\Pi_{13}^{(U)} = \tau_O \frac{\partial U_O}{\partial x_3} (6a\langle u_3^2\rangle \frac{2}{15} \langle u_3^2\rangle, \tag{D.13}$$

where we used $\int dkE(k) \approx (3/2)\langle u_3^2\rangle$ near isotropy. This relation is expressed in terms of A_{33}^O by substitution of (2.8):

$$\Pi_{13}^{(U)} = \frac{2}{15} \tau_O \frac{\partial U_O}{\partial x_3} A_{33}^O, \tag{D.14}$$

the desired result. This expression gives $\Pi_{13}^{(U)}$ in terms of second-moments to first order in $\partial U_O/\partial x_3$ and $\partial \Theta_O/\partial x_3$ for small anisotropy. To judge the importance of this term we note that its magnitude is less than that of A_{13}^O, the principal term of the 13 component of (2.15), but only by a factor of about 1/3 to 1/2; i.e., for the magnitude of $A_{13}^O = 6a\langle u_3^2\rangle\langle u_1 u_3\rangle$ we use $\langle u_1 u_3\rangle \sim -(\tau_O/3)(\partial U_O/\partial x_3)\langle u_2^3\rangle$ (e.g., Tennekes and Lumley, 1972) to get $A_{13}^O \sim -(1/3) \tau_O(\partial U_O/\partial x_3)A_{33}^O$.

In comparison, $\Pi_{13}^{(U)}$ is not negligible.

As for the element $\Pi_{11}^{(U)}$, its order of magnitude is estimated by substitution of (D.7) and (D.10) into the 11 component of (D.2)

$$\Pi_{11}^{(U)} = \frac{2\tau_O}{3} \frac{\partial U_O}{\partial x_3} \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} \Big[2\Big\{A_{13}^+(\underset{\sim}{k}) + \frac{\tau_O}{3} \frac{\partial U_O}{\partial x_3} (A_{33}^+(\underset{\sim}{k})\Big\} \frac{k_1^2}{k^2} + A_{11}^+(\underset{\sim}{k})\frac{k_1 k_3}{k^2}\Big]. \tag{D.15}$$

For asympotically small isotropy the A_{13}^{+} integral term is asumptotically small since, with (D.12) and (C.10), $A_{13}^{+} \propto S_{13}(\underset{\sim}{k})$ and $\int d\underset{\sim}{k} S_{13}(\underset{\sim}{k}) k^2/k^2$ is zero when the isotropic S_{13} is used. Similarly, for the A_{11}^{+} integral term. The remaining A_{33}^{+} term does not vanish at isotropy, but is second order in $\tau_O \partial U_O/\partial x_3$. Therefore

$$\Pi_{11}^{(U)} = O[(\tau_O \partial U_O/\partial x_3)^2], \qquad (D.16)$$

in the limit of small anisotropy and small $(\tau_O \partial U_O/\partial x_3)^2$.

The last element we consider is $\Pi_{33}^{(U)}$. It is given by substitution of (D.7) into (D.2)

$$\Pi_{33}^{(U)} = 2\tau_O \frac{\partial U_O}{\partial x_3} \int \frac{d\underset{\sim}{k}}{(2\Pi)^3} A_{33}^{+}(\underset{\sim}{k}) \frac{k_1 k_3}{k^2} . \qquad (D.17)$$

This integral, too, is small near isotropy since $A_{33}^{+} \propto S_{33}(\underset{\sim}{k})$. However, we can say that $\int d\underset{\sim}{k} S_{33}(\underset{\sim}{k}) k_1 k_3/k^2$ is of order $\partial U_O/\partial x_3$ since a small shear magnitude will cause a deviation from isotropy (proportional to the shear). Therefore

$$\Pi_{33}^{(U)} = O[(\tau_O \partial U_O/\partial x_3)^2], \qquad (D.18)$$

in our limit of weak shear and small anisotropy.

To end this part of the appendix, we note that (D.14), (D.16) and (D.18) can be combined in the tensorial form

$$\underset{\approx}{\Pi}^{(U)} = \frac{2}{15} \tau_O A_{33}^{O} (\underset{\sim\sim}{\nabla U} + \underset{\sim\sim}{\nabla U}^T) + \delta_O, \qquad (D.19)$$

where δ_O denotes terms of order $(\partial U_O/\partial x_3)^2$ and $(\partial U_O/\partial x_3)(\partial \Theta_O/\partial x_3)$ neglected in (D.4) and other places, $\underset{\sim\sim}{\nabla U}^T$ denotes the transpose of $\underset{\sim\sim}{\nabla U}$, we have used $\Pi_{31}^{(U)} = \Pi_{13}^{(U)}$. The form used in (2.14') is obtained by substitution of (D.4) into (D.19):

$$\underline{\underline{\Pi}}^{(U)} = -\frac{2}{5}\langle u_3^3\rangle(\nabla U + \nabla U^T) + \delta_o, \qquad (D.20)$$

as we set out to prove.

PART 2

To derive (2.23) we substitute (D.1) into $\langle u\nabla p^{(U)}\theta\rangle$ and obtain

$$(1+T_r)\langle u\nabla p^{(U)}\Theta\rangle = -2\frac{\partial U_o}{\partial x_3}\int\frac{dk}{(2\Pi)^3}\langle u\theta u^+(k)\rangle\frac{kk_1}{k^2}\exp(ik\cdot x). \qquad (D.21)$$

To lowest order in mean gradients, $\langle u_j\theta u_3\rangle$ is given by (2.26) as

$$\langle u_j\theta u_3\rangle = -(\tau_o/3)A_{j3}^{(\theta)}. \qquad (D.22)$$

Substitution of (2.19) for $A_{13}^{(\theta)}$ and use of $\alpha\langle uu\rangle/\alpha x_3 = 2a\langle uu\rangle$, we afterwards Fourier expand both sides of (D.22) and heuristically equate Fourier components to obtain

$$\langle u_j\Theta u_3^+(k)\rangle = -(2/3)a\tau_o[2\langle u_3\theta\rangle\langle u_j u_3^+(k)\rangle + \langle\theta u_j\rangle\langle u_3u_3^+(k)\rangle]. \qquad (D.23)$$

This could be established by a more rigorous but much more complex derivation. Substitution of (D.23) and

$$\langle u_j u_3^+(k)\rangle = \delta_{j3}(k)\exp(-ik\cdot x) \qquad (D.24)$$

into the 13 component of (D.21) we have

$$\langle\theta(u_1\frac{\partial p^{(U)}}{\partial x_3} + u_3\frac{\partial p^{(U)}}{\partial x_1})\rangle \equiv \frac{4a\tau_o}{3}\frac{\partial U_o}{\partial x_3}\int\frac{dk}{(2\Pi)^3}[2\langle u_3\theta\rangle S_{13}(k)\frac{k_1k_3}{k^2}$$

$$+ \langle u_1\theta\rangle S_{33}(k)\frac{k_1k_3}{k^2} + 3\langle u_3\theta\rangle S_{33}(k)\frac{k_1^2}{k^2}]. \qquad (D.25)$$

In the asymptotic limit of small anisotropy, the integral is readily evaluated to be

$$\langle\theta(u_1\frac{\partial p^{(U)}}{\partial x_3} + u_3\frac{\partial p^{(U)}}{\partial x_1})\rangle = \frac{4}{5}a\tau_o\frac{\partial U_o}{\partial x_3}\langle u_3\theta\rangle\langle u_3^2\rangle \qquad (D.26)$$

$$= -\frac{2}{5}\frac{\partial U_O}{\partial x_3}\langle u_3^2\theta\rangle,$$

the desired expression for 13 element.

The 11 element of (D.21) is evaluated by substitution of (D.23) and (D.24):

$$2\langle u_1 \frac{\partial p^{(U)}}{\partial x_1}\theta\rangle = \frac{4a\tau_O}{3}\frac{\partial U_O}{\partial x_3}\int \frac{d\underset{\sim}{k}}{(2\Pi)^3}\left[\langle u_3\theta\rangle S_{13}(\underset{\sim}{k}) + \langle u_1\theta\rangle S_{33}(\underset{\sim}{k})\right]\frac{k_1^2}{k^2} . \quad (D.27)$$

The second term in the integrand is second order since $\langle u_1\theta\rangle$ is of the order $(\partial U_O/\partial x_3)(\partial\Theta_O/\partial x_3)$ in the asymptotic limits of small anisotropy and small mean gradients. The first term in the integrand integrates to zero near isotropy since $S_{13}k_1^2$ is an odd function of k_1 and k_3.

Similarly, the 33 component of $\langle \underset{\sim}{u}\underset{\sim}{\nabla} p^{(U)}\theta\rangle$ is also found to be asymptotically small in the limit of small anisotropy and small mean gradients. Hence, in those limits, the diagonal elements of $\langle \underset{\sim}{u}\underset{\sim}{\nabla} p^{(U)}\theta\rangle$ are negligible and, we have seen, the off-diagonal 13 element is given by (D.26). These diagonal and off-diagonal elements can be combined in the tensorial form

$$(1 + T_r)\langle \underset{\sim}{u}\underset{\sim}{\nabla} p^{(U)}\theta\rangle = -\frac{2}{5}\langle u_3^2\theta\rangle(\underset{\sim}{\nabla}\underset{\sim}{U} + \underset{\sim}{\nabla}\underset{\sim}{U}^T), \quad (D.28)$$

for our case of $\underset{\sim}{\nabla}\underset{\sim}{U} = \hat{\underset{\sim}{x}}_3\hat{\underset{\sim}{x}}_1\partial U_O/\partial x_3$. This equation is (2.23) as we wished to prove.

To derive (2.23'), (D.1) is substituted into the Fourier expansion of

$$\langle \underset{\sim}{\nabla} p^{(U)}\theta^2\rangle = -2\frac{\partial U_O}{\partial x_3}\int \frac{d\underset{\sim}{k}}{(2\Pi)^3}\langle\theta^2 u_3^+(\underset{\sim}{k})\rangle\frac{\underset{\sim}{k}k_1}{k^2}\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}). \quad (D.29)$$

The integral is evaluated by use of (2.27) in the limit of small mean gradients:

$$3\langle u_3\theta^2\rangle = -\tau_O A_3^{(\theta^2)}. \quad (D.30)$$

As for (D.7), the Fourier expansions of both sides of (D.30) combined with use of $\partial\langle u_3\theta\rangle/\partial x_3 = 2a\langle u_3\theta\rangle$ yields

$$\langle\theta^2 u_3^+(\underset{\sim}{k})\rangle = -\frac{2}{3}a\tau_o[\langle\theta^2\rangle\langle u_3 u_3^+(\underset{\sim}{k})\rangle + 2\langle u_3\theta\rangle\langle\theta u_3^+(\underset{\sim}{k})\rangle]. \quad (D.31)$$

Substitution of (D.31) and (D.24) in (D.29) we have

$$\langle\underset{\sim}{\nabla}p^{(U)}\theta^2\rangle = \frac{4a\tau_o}{3}\frac{\partial U_o}{\partial x_3}\int\frac{d\underset{\sim}{k}}{(2\Pi)^3}[\langle\theta^2\rangle S_{33}(\underset{\sim}{k}) + 2\langle u_3\theta\rangle R_{33}(\underset{\sim}{k})]\frac{\underset{\sim}{k}k_1}{k^2}, \quad (D.32)$$

where $R_{33}(\underset{\sim}{k}) \equiv \langle\theta^+(\underset{\sim}{k})u_3^+(\underset{\sim}{k})\rangle V^{-1}$ and $\theta^+(\underset{\sim}{k})$ is the Fourier transform of $\theta(\underset{\sim}{x})$.

The integration can be done in the small anisotropy limit and use of $R_{33}(\underset{\sim}{k}) \sim S_{33}(\underset{\sim}{k})\ \langle\theta u_3\rangle/\langle u_3^2\rangle$ in that limit. The result is

$$\langle\frac{\partial p^{(U)}}{\partial x_j}\theta^2\rangle = (\frac{4a\tau_o}{3})\frac{\partial U_o}{\partial x_3}(\frac{2}{5}\langle\theta_3^2\rangle\langle u^2\rangle + \frac{4}{5}\langle u_3\theta\rangle^2)\delta_{ij}, \quad (D.32)$$

where δ_{ij} is the Kronecker delta.

After substitution of (2.20), this equation can be written in the vector form

$$\langle\underset{\sim}{\nabla}p^{(U)}\theta^2\rangle = -\frac{4}{5}\langle\theta^2\underset{\sim}{u}\rangle\cdot\underset{\approx}{\nabla U},$$

which is (2.23').

APPENDIX E F

PART 1

The purpose of this appendix is to derive (2.16), (2.17) and (2.18). To derive (2.16) we substitute into $\langle \mathbf{u}\mathbf{u}\theta \rangle$ an expression for θ obtained from the fluctuating part of the thermodynamic equation. That equation is

$$\frac{\partial \theta}{\partial t} - \sigma \nabla^2 \theta = -(\mathbf{u}\cdot\nabla\theta)' - \mathbf{u}\cdot\nabla\Theta_0 - \mathbf{U}\cdot\nabla\theta, \qquad (E.1)$$

where $(\mathbf{u}\cdot\nabla\theta)' \equiv \mathbf{u}\cdot\nabla\theta - \langle \mathbf{u}\cdot\nabla\theta \rangle$, σ is the thermal conductivity, and the conductivity term has been placed on the left side for later convenience. Eq'n. (E.1) is formally integrated in the same way as was the Navier-Stokes equation in Sec. 2 with the result

$$\theta(t) = G_\sigma(t)\theta(0) - \int_0^t dt_1 G_\sigma(t-t_1)[(\mathbf{u}\cdot\nabla\theta)' + \mathbf{u}\cdot\nabla\Theta_0 + \mathbf{U}\cdot\nabla\theta]_{t_1} \qquad (E.2)$$

$$G_\sigma(t) \equiv \exp[-(t-t_1)\nu\nabla^2]$$

where the subscript t_1 in the integrand is to remind us that the terms in square brackets are all to be evaluated at time t_1; e.g., $\mathbf{u} = \mathbf{u}(t_1)$. Substitution of (E.2) into $\langle \mathbf{u}\mathbf{u}\theta \rangle$ we have

$$\langle \mathbf{u}\mathbf{u}\theta \rangle = (I.V.)_1 - \int_0^t dt_1 \langle \mathbf{u}(t)\mathbf{u}(t) G_\sigma[(\mathbf{u}\cdot\nabla\theta)' - \mathbf{u}\cdot\nabla\theta_0 + \mathbf{U}\cdot\nabla\theta]t_1 \qquad (E.3)$$

where (I.V.) denotes the initial value term $\langle \mathbf{u}(t)\mathbf{u}(t)G_\sigma(t)\theta(0) \rangle$. A needed additional expression for $\langle \mathbf{u}\mathbf{u}\theta \rangle$ is obtained by substitution of (2.3) for $\mathbf{u}$ in $\langle \mathbf{u}\mathbf{u}\theta \rangle$ to obtain

$$\langle \mathbf{u}\mathbf{u}\theta \rangle = (I.V.)_2 - \int_0^t dt_1 \langle \mathbf{u}(t)\theta(t) G_\nu[\mathbf{u}\cdot\nabla\mathbf{u})' + \frac{\nabla p}{\rho_0} + \mathbf{u}\cdot\nabla\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{u} + \frac{g\theta}{\Theta_0}]_{t_1} \rangle \qquad (E.4)$$

where $(I.V.)_2 \equiv \langle \underset{\sim}{u}(t)\theta(t)G_\nu(t)\underset{\sim}{u}(0)\rangle$. We next add (E.4) and the transpose of (E.4) to (E.3). The result is

$$3\langle \underset{\sim}{u}\underset{\sim}{u}\theta\rangle = I.V. - \int_0^t dt_1[\underset{\approx}{B}(t-t_1) + (1 + T_r)\{\langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)\frac{\underset{\sim}{\nabla}p(t_1)}{\rho_0}\rangle$$

$$+\langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)\theta(t_1)\frac{g}{\Theta_0} + \langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)\underset{\sim}{u}(t)\rangle\cdot\underset{\sim\sim}{\nabla U}\}$$

$$+ \langle \underset{\sim}{u}(t)\underset{\sim}{u}(t)G_\sigma(t-t_1)\underset{\sim}{u}(t_1)\rangle\cdot\underset{\sim\sim}{\nabla U}] \qquad (E.5)$$

$$\underset{\approx}{B}(t-t_1) \equiv (1 + T_r)\langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)[\underset{\sim}{u}(t_1)\cdot\underset{\sim\sim}{\nabla}\mathring{\underset{\sim}{u}}(t_1)]'\rangle$$

$$+ \langle \underset{\sim}{u}(t)\underset{\sim}{u}(t)G_\nu(t-t_1)[\underset{\sim}{u}(t_1)\cdot\underset{\sim}{\nabla}\theta(t_1)]'\rangle \qquad (E.5)$$

where $I.V. \equiv (I.V.)_1 + (1 + T_r)(I.V.)_2$ is henceforth ignored as small for $t > \tau_0$ and, as for (2.5), the terms containing $\underset{\sim}{U}\cdot\underset{\sim}{\nabla}$ collectively vanish since their sum in (E.5) is of the form $\underset{\sim}{U}\cdot\underset{\sim\sim}{\nabla}\langle uuG_\nu\theta\rangle$ when $\nu k_0^2 \ll k_0 v_0$.

The quantity $\int dt_1\underset{\approx}{B}(t-t_1)$ is evaluated in the same manner as the evaluation of $\int dt_1\underset{\approx}{A}(t-t_1)$ in App. B; i.e., cumulant expansion of the fourth-moments in $\underset{\approx}{B}(t-t_1)$ in terms of second-moments, followed by time integration of second-moments as in (B.9) but with the velocity spectrum $\underset{\approx}{S}(\underset{\sim}{k}_b;t,t_1)$ replaced by the cross-spectrum $\langle \underset{\sim}{u}^*_{\underset{\sim}{k}_b}(t)\theta_{\underset{\sim}{k}_b}(t_1)\rangle V^{-1}$. A simplifying approximation made is $\sigma \approx \nu$. The result of this evaluation of $\underset{\approx}{B}$ is

$$\int_0^t dt_1\underset{\approx}{B}(t-t_1) = \tau_0\underset{\approx}{A}^\theta + \int_0^t dt_1\underset{\approx}{Q}^{(\theta)}(t-t_1) \qquad (E.6)$$

where $\underset{\approx}{A}^\theta$ is defined by (2.19) and $\underset{\approx}{Q}^{(\theta)}(t-t_1)$ is defined in the paragraph following(2.21).

The remaining time integrations in (E.5) can all be expressed as

$$\int_0^t dt_1\langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)\frac{\underset{\sim}{\nabla}p(t_1)}{\rho_0}\rangle = \tau_1\langle u(t)\theta(t)\frac{\underset{\sim}{\nabla}p(t)}{\rho_0}\rangle \qquad (E.7)$$

$$\int_0^t dt_1 \langle \underset{\sim}{u}(t)\theta(t)G_\nu(t-t_1)\theta(t_1)\rangle = \tau_2\langle u(t)\theta^2(t)\rangle \qquad \text{(E.8)}$$

$$\int_0^t dt_1 \langle \underset{\sim}{u}(t)\underset{\sim}{u}(t)G_\sigma(t-t_1)\underset{\sim}{u}(t_1)\rangle = \tau_3\langle \underset{\sim}{u}(t)\underset{\sim}{u}(t)\underset{\sim}{u}(t)\rangle \qquad \text{(E.9)}$$

where τ_1, τ_2, τ_3 are to be determined. This determination is similar to the evaluation of $\int dt_1 \underset{\approx}{A}$ and $\int dt_1 \langle u_3 \underset{\sim}{u} G_\nu u\rangle$ in App. B; i.e., we express the integrands in terms of (two-time) fourth-moments, expand the fourth-moments in terms of products of second-moments $S(k;t,t_1)$ and fourth-cumulants, neglect the fourth-cumulants, and, finally, evaluate the time integrals by use of (B.8). The result is given by

$$\tau_i \approx \begin{cases} t, & t \ll \tau_{\underset{\sim}{k}} \\ \tau_0, & t \gg \tau_{\underset{\sim}{k}} \end{cases} \qquad (i = 1,2,3) \qquad \text{(E.10)}$$

Finally, substitution of (E.6) to (E.10) in (E.5) yields (2.16) as we set out to prove.

As for (2.17) or (2.18) their derivations are similar to that of (2.16), i.e., for (2.17) we substitute (E.2) for θ into $\langle \underset{\sim}{u}\theta\theta\rangle$ followed by substitution of (2.3) for $\underset{\sim}{u}$ and proceed with same steps as (E.3) to (E.10) but with $\underset{\sim}{u}(t)$ replaced by $\theta(t)$ in the appropriate places. Similarly for (2.18), except that we begin with substitution of (E.2) into $\langle\theta^3\rangle$.

PART 2

To derive (2.22), we substitute (C.5) for $p^{(S)}$ into the Fourier expansion of $\langle \underset{\sim}{u}\underset{\sim}{\nabla} p^{(s)}\theta\rangle$ to obtain

$$\langle \underset{\sim}{u}\underset{\sim}{\nabla} p^{(s)}\theta\rangle = -\int \frac{d\underset{\sim}{k}_a d\underset{\sim}{k}_b d\underset{\sim}{k}_c d\underset{\sim}{k}_d}{(2\Pi)^{12}} \frac{\langle \theta_{\underset{\sim}{k}_c} \underset{\sim}{u}_{\underset{\sim}{k}_d} (\underset{\sim}{u}_{\underset{\sim}{k}_a} \underset{\sim}{u}_{\underset{\sim}{k}_b})'\rangle : (i\underset{\sim}{k}_a + \underset{\sim}{a})(i\underset{\sim}{k} + 2\underset{\sim}{a})^2}{(i\underset{\sim}{k} + 2\underset{\sim}{a})\cdot(i\underset{\sim}{k} + 2a)}$$

$$\times \exp\left[-i(\underset{\sim}{k} + \underset{\sim}{k}_c + \underset{\sim}{k}_d)\cdot\underset{\sim}{x}\right], \qquad \text{(E.11)}$$

where $k \equiv k_a + k_b$. Upon comparison, it can be seen that the right side of (E.11) is exactly the same as the right side of (C.6) when $w_{\underset{\sim}{k}_c}$ is replaced by $\Theta_{\underset{\sim}{k}_c}$. For this reason, (E.11) is converted to (C.9) when u_3 in the left side of (C.9) is replaced by Θ and $\underset{\sim}{x}_3 \cdot \underset{\approx}{S}(\underset{\sim}{k}_b)$ in the right side is replaced by $\underset{\sim}{x}_3 \cdot \underset{\approx}{R}(\underset{\sim}{k}_b)$, where $\underset{\approx}{R}$ is defined by $\underset{\approx}{R}(\underset{\sim}{k}_b) \equiv V^{-1}\langle \hat{\underset{\sim}{x}}_3 \theta_{\underset{\sim}{k}_b} u^*_{\underset{\sim}{k}_b} \rangle$ (by its definition $\underset{\approx}{R}$ is obtained by replacing $\underset{\sim}{u}_{\underset{\sim}{k}_b}$ in $\underset{\approx}{S}(\underset{\sim}{k}_b)$ by $\hat{\underset{\sim}{x}}_3 \Theta_{\underset{\sim}{k}_b}$). With these replacements, the elements of (E.11) are the same as (C.11) with u_3 on the left side of (C.11) replaced by Θ and (v_o^2) on the right side replaced by $\langle \theta u_3 \rangle$ -- assuming near isotropy for $\underset{\approx}{R}$ as well as $\underset{\approx}{S}$ to derive (C.11). As a result, the 33, 11 and 13 components of (E.11) are implied by (C.11) to be as follows:

$$\langle u_3 \theta \partial p^{(s)}/\partial x_3 \rangle = \frac{1}{15} (\frac{3}{2} v_o^2)(\frac{3}{2} \langle u_3 \theta \rangle) a$$

$$\langle \theta u_1 \partial p^{(s)}/\partial x_1 \rangle = \frac{2}{225} (\frac{3}{2} v_o^2)(\frac{3}{2} \langle u_3 \theta \rangle a \qquad \text{(E.12)}$$

$$\langle \theta u_1 \partial p^{(s)}/\partial x_3 \rangle = 0.$$

These relations can be combined in the tensor form of (2.22)

$$(1 + t_r)\langle \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)} \theta \rangle = \underset{\approx}{f} \cdot \underset{\approx}{A}^{(\theta)}, \qquad \text{(near isotropy)} \qquad \text{(E.13)}$$

where f_{ij} are numerical constants determined by (E.12).

The derivation of (2.22') is similar to the derivation of (E.13). Thus, we substitute (C.5) for $p^{(s)}$ into the Fourier expansion of $\langle \underset{\sim}{\nabla} p^{(s)} \theta^2 \rangle$ and note that it would be the same as the Fourier expansion of the $\underset{\sim}{x}_3$ component of $\langle \underset{\sim}{u} \underset{\sim}{\nabla} p^{(s)} u_3 \rangle$ given by (C.6) if $w_{\underset{\sim}{k}_a}$ and $\hat{\underset{\sim}{x}}_3 \cdot \underset{\sim}{u}_{\underset{\sim}{k}_b}$ were replaced by $\theta_{\underset{\sim}{k}_a}$ and $\theta_{\underset{\sim}{k}_b}$, respectively.

Therefore, $\langle \nabla p^{(s)} \theta^2 \rangle$ is given by the x_3 component of (C.9) with $u_3 u_3$ on the left side replaced by Θ^2 and $x_3 \cdot SS \cdot x_3$ on right side replaced by $x_3 \cdot SR \cdot x_3$, where R is defined after (E.11).

It follows that the elements of $\langle \nabla p^{(s)} \theta^2 \rangle$ are given by (C.11) with u_3 appropriately replaced by Θ. These elements are

$$\langle \Theta^2 \partial p^{(s)} \partial x_3 \rangle = \frac{1}{15} (\frac{3}{2} \langle u_3 \theta \rangle^2 a \qquad \text{(E.14)}$$

$$\langle \Theta^2 \partial p^{(s)} \partial x_1 \rangle = 0$$

in the limit of small anisotropy. This expression can be written in the form given by (2.22'), since, for isotropy, $\langle u_3^2 \rangle \langle \Theta^2 \rangle \sim \langle u_3 \Theta \rangle^2$ and $A_3^{(\Theta^2)} \sim 6 \langle u_3 \Theta \rangle^2 a$, using $\partial \langle u_3 \Theta \rangle \partial x_3 = 2a \langle u_3 \Theta \rangle$. The essential point of (2.22') and (E.14) is that $\langle \Theta^2 \nabla p^{(s)} \rangle$ is negligibly small.