SPECTRAL APPROACH TO NON-ISOTROPIC TURBULENCE SUBJECTED TO ROTATION

by

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APPENDIX

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NON LINEAR INTERACTIONS IN THE EQUATIONS GOVERNING THE EIGENMODES.

a) Fluctuating field.

The basic equation for $\xi_\varepsilon(\underline{k},t)$ writes as follows :

$$\left(\frac{\partial}{\partial t} + \nu k^2 - 2 i \varepsilon \Omega \mu_{\underline{k}}\right)\xi_\varepsilon = -\frac{i}{2} k_l N_i(-\varepsilon \underline{k}) P_{ij}(\underline{k}) \int_{\underline{p}+\underline{q}=\underline{k}} \hat{u}_l(\underline{p})\hat{u}_j(\underline{q})\, d^3\underline{p}$$

using the relation $\xi_\varepsilon(\underline{k},t) = \frac{1}{2}\underline{N}(-\varepsilon\underline{k})\cdot\underline{\hat{u}}(\underline{k},t)$; $\varepsilon = \pm 1$

Then, replacing $\hat{u}_l(\underline{p},t)$ and $\hat{u}_j(\underline{q},t)$ by corresponding expressions in term of eigenmodes, it comes :

$$\left(\frac{\partial}{\partial t} + \nu k^2 - 2 i \varepsilon \Omega \mu_{\underline{k}}\right)\xi_\varepsilon = -\frac{i}{2} \sum_{\substack{\varepsilon' \\ \varepsilon''}=\pm 1} [\underline{k}\cdot N(\varepsilon'\underline{p})][N(-\varepsilon\underline{k})\cdot N(\varepsilon''\underline{q})]\,\xi_{\varepsilon'}(\underline{p},t)\,\xi_{\varepsilon''}(\underline{q},t)\, d^3\underline{p}$$

In order to have the selection rule $\underline{k}+\underline{p}+\underline{q}=\underline{0}$ (more convenient for correlations) one changes $\underline{k}$ in $-\underline{k}$, so that :

$$\left(\frac{\partial}{\partial t} + \nu k^2 + 2 i \varepsilon \Omega \mu_{\underline{k}}\right)\xi^*_\varepsilon = \frac{i}{2} \sum_{\substack{\varepsilon' \\ \varepsilon''}=\pm 1} \int M_{\varepsilon\varepsilon'\varepsilon''}(\underline{k},\underline{p},\underline{q})\,\xi_{\varepsilon'}(\underline{p},t)\,\xi_{\varepsilon''}(\underline{q},t)\, d^3\underline{p}$$

with $M_{\varepsilon\varepsilon'\varepsilon''}(\underline{k},\underline{p},\underline{q}) = [\underline{k}\cdot\underline{N}(\varepsilon'\underline{p})][\underline{N}(\varepsilon\underline{k})\cdot\underline{N}(\varepsilon''\underline{q})]$; $\underline{k}+\underline{p}+\underline{q}=\underline{0}$

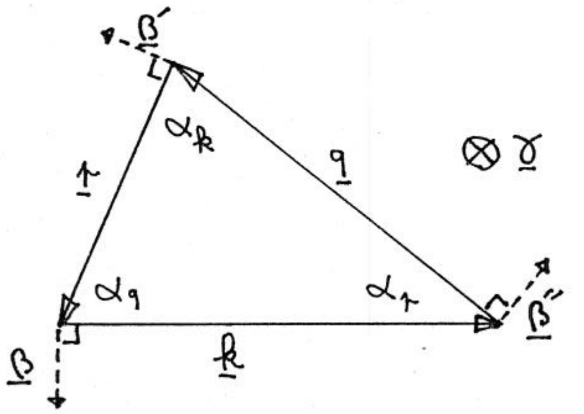
A better expression of the "influence matrix" $M_{\varepsilon\varepsilon'\varepsilon''}$ is found by means of the typical "triadic" local Craya's frames. Substituting $\underline{\gamma}$, unit vector normal of the triadic plane, to the polar direction $\underline{n}$, one obtained the following set of frames, relative to the three vectors $\underline{k}, \underline{p}, \underline{q}$:

$$\underline{\gamma}(\underline{k},\underline{t}) = \underline{k}\times\underline{t}/|\underline{k}\times\underline{t}| \quad ; \underline{\beta}(\underline{k},\underline{t}) = \underline{\gamma}\times\underline{k}/k$$

$$\underline{\gamma}(\underline{k},\underline{t}) \quad ; \underline{\beta}'(\underline{k},\underline{t}) = \underline{\gamma}\times\underline{t}/t$$

$$\underline{\gamma}(\underline{k},\underline{t}) \quad ; \underline{\beta}''(\underline{k},\underline{t}) = \underline{\gamma}\times\underline{q}/q$$

$\underline{\beta},\underline{\beta}',\underline{\beta}''$ substituted to $\hat{\underline{e}}(\underline{k}),\hat{\underline{e}}(\underline{t}),\hat{\underline{e}}(\underline{q})$ are now located in the triadic plane :



$\underline{k}+\underline{t}+\underline{q}=\underline{0}$

Introducing the corresponding complex formulation defined by :

$$\underline{W}(\varepsilon) = \underline{\beta} + I\varepsilon\underline{\gamma} \; ; \; \underline{W}'(\varepsilon') = \underline{\beta}' + I\varepsilon'\underline{\gamma} \; ; \; \underline{W}''(\varepsilon'') = \underline{\beta}'' + I\varepsilon''\underline{\gamma} \; ,$$

the different eigenvectors are linked by complex exponential, so that :

$$\underline{N}(\varepsilon\underline{k}) = \underline{W}(\varepsilon)\cdot e^{I\varepsilon\lambda} \; ; \; \underline{N}(\varepsilon'\underline{t}) = \underline{W}'(\varepsilon')\cdot e^{I\varepsilon'\lambda'} \; ; \; \underline{N}(\varepsilon''\underline{q}) = \underline{W}''(\varepsilon'')\cdot e^{I\varepsilon''\lambda''} \; ,$$

$$\varepsilon = \pm 1 \; , \; \varepsilon' = \pm 1 \; , \; \varepsilon'' = \pm 1 \; ,$$

where the angles $\lambda,\lambda',\lambda''$ characterize the rotation of the triadic plane respectively about $\underline{k},\underline{t},\underline{q}$. In counterpart, the calculation of any scalar product between two elements of the set $(\underline{k},\underline{t},\underline{q},\underline{W},\underline{W}',\underline{W}'')$ may be only expressed in terms of the moduli k,t,q:

$$M_{\varepsilon\varepsilon'\varepsilon''}(\underline{k},\underline{t},\underline{q}) = \exp\{I(\varepsilon\lambda+\varepsilon'\lambda'+\varepsilon''\lambda'')\}\left[\underline{k}\cdot\underline{W}'(\varepsilon')\right]\left[\underline{W}(\varepsilon)\cdot\underline{W}''(\varepsilon'')\right]$$

$$= \exp\{I(\varepsilon\lambda+\varepsilon'\lambda'+\varepsilon''\lambda'')\}\, m_{\varepsilon\varepsilon'\varepsilon''}(k,t,q)$$

with $m_{\varepsilon\varepsilon'\varepsilon''}(k,t,q) = -k\sin\alpha_q\left(-\cos\alpha_t - \varepsilon\varepsilon''\right)$

or

$$m_{\varepsilon\varepsilon'\varepsilon''}(k,t,q) = \frac{1}{2}C_{ktq}(\varepsilon k + \varepsilon' t + \varepsilon'' q)(\varepsilon k - \varepsilon' t + \varepsilon'' q)$$

using the geometric relations :

$$\frac{\sin\alpha_k}{k} = \frac{\sin\alpha_p}{p} = \frac{\sin\alpha_q}{q} = C_{kpq}$$
$$p^2 = (\underline{k}+\underline{q})^2 = k^2 + q^2 - 2kq\cos\alpha_p$$

b) Two point correlations.

By correlations of the eigenmodes, one can define :

$$\langle \xi^*_\varepsilon(\underline{p},t)\,\xi_{\varepsilon'}(\underline{k},t)\rangle = Z_{\varepsilon\varepsilon'}(\underline{k},t)\,\delta(\underline{k}-\underline{p})$$

with $Z_{\varepsilon\varepsilon'}(\underline{k},t) = \frac{1}{4} N_i(\varepsilon\underline{k})\,\hat{U}_{ij}(\underline{k},t)\,N_j(-\varepsilon'\underline{k})$

Then, considering that : $N_i(\underline{k})N_j(-\underline{k}) = P_{ij}(\underline{k}) - I\,\varepsilon_{ijl}\frac{k_l}{k}$, the set (e,Z,h) is simply identified.

$$Z_{11} = \frac{1}{2}\left(e + \frac{h}{k}\right)$$
$$Z_{-1-1} = \frac{1}{2}\left(e - \frac{h}{k}\right)$$
$$Z_{-11} = \frac{1}{2}Z$$

Note that the helicity ratio and the anisotropy ratio are directly given by :

$$\frac{h}{ke} = \frac{\langle |\xi_1|^2 - |\xi_{-1}|^2\rangle}{\langle |\xi_1|^2 + |\xi_{-1}|^2\rangle} \quad ; \quad \frac{Z}{e} = \frac{\langle \xi^*_{-1}\,\xi_1\rangle}{\langle |\xi_1|^2 + |\xi_{-1}|^2\rangle/2}$$

It would be possible to construct the rate equation governing the set (e,Z,h) in the EDQNM theory, by using only correlations of eigenmodes ξ_ε, even for third and fourth order correlations. Here the same result is obtained by a slightly different route. Starting from the expression of the quasi-normal term with the second order spectral tensor, we are only concerned with the integrand :

$$k_q N_q(\varepsilon''\underline{q})N_j(\varepsilon\underline{k})N_i(\varepsilon'\underline{p})\,N_l(-\varepsilon''\underline{q})N_m(-\varepsilon\underline{k})N_n(-\varepsilon'\underline{p})\ \times$$
$$\times\left\{P_{luv}(\underline{q})\,\hat{U}_{um}(\underline{k})\,\hat{U}_{vn}(\underline{p}) + P_{muv}(\underline{k})\,\hat{U}_{vn}(\underline{p})\,\hat{U}_{ul}(\underline{q}) + P_{nuv}(\underline{p})\,\hat{U}_{vl}(\underline{q})\,\hat{U}_{um}(\underline{k})\right\}$$

which becomes :

$$[\underline{k}\cdot\underline{W}''(\varepsilon'')][\underline{W}(\varepsilon)\cdot\underline{W}'(\varepsilon')]\,W''_l(-\varepsilon'')\,W_m(-\varepsilon)\,W'_n(-\varepsilon')\ \times\{\cdots\}$$

for the energy transfer $2T_{\lambda\lambda}$, and :

$$e^{-2I\lambda}(1+\varepsilon)\left[\underline{k}\cdot\underline{W}''(\varepsilon'')\right]\left[\underline{W}(\varepsilon)\cdot\underline{W}'(\varepsilon')\right]W''_{\ell}(-\varepsilon'')W_m(-\varepsilon)W'_n(-\varepsilon')\times\{\cdots\}$$

for the anisotropy transfer $(T_{\lambda\gamma}+T^{*}_{\lambda\gamma})N^{*}_{\lambda}N^{*}_{\gamma}$

Then, in the expression of $W''_{\ell}(-\varepsilon'')W_m(-\varepsilon)W'_n(-\varepsilon')\times\left\{P_{\ell uv}(q)\hat{U}_{um}(k)\hat{U}_{vn}(t)+\cdots\right.$,

$\underline{\underline{\hat{U}}}$ only appears under the following contracted form :

$$W_m(-\varepsilon)\hat{U}_{um}(\underline{k}) = e(\underline{k})\,W_u(-\varepsilon) + Z(\varepsilon\underline{k}).e^{2I\varepsilon\lambda}\,.\,W_u(\varepsilon)$$

and analogous expressions for $\underline{t}$ and $\underline{q}$.

As for the treatment of the non linear term in the equation relative to $\mathcal{Z}_{\varepsilon}$, geometric coefficients depending only on the moduli k,t,q are given by the scalar products including $\underline{k},\underline{t},\underline{q},\underline{W},\underline{W}',\underline{W}''$.

The detail ed formulations are given as follows :

$$\begin{aligned}T_{\lambda\lambda}+T^{*}_{\lambda\lambda} = \frac{1}{2}\sum_{\substack{\varepsilon\\ \varepsilon'=\pm1\\ \varepsilon''}}\iint_{\Delta k} dt\,dq\,.\frac{tq}{k}\int_0^{2\pi} d\lambda.\,C^2_{ktq}\cdot\theta^{\varepsilon\varepsilon'\varepsilon''}_{\underline{k}\underline{t}\underline{q}}\cdot\Big[A_1(\varepsilon k,\varepsilon' t,\varepsilon'' q)\times\\ \times\, e''(e'-e)-A_2(\varepsilon k,\varepsilon' t,\varepsilon'' q)\,e\,\mathcal{Z}''-A_3(\varepsilon k,\varepsilon' t,\varepsilon'' q)\,e''\mathcal{Z}\\ +A_5(\varepsilon k,\varepsilon' t,\varepsilon'' q)\,e'\mathcal{Z}''+A_4(\varepsilon k,\varepsilon' t,\varepsilon'' q)\,\mathcal{Z}''(\mathcal{Z}'-\mathcal{Z})\Big]\end{aligned}$$

$$\begin{aligned}(T_{\lambda\gamma}+T^{*}_{\lambda\gamma})N^{*}_{\lambda}N^{*}_{\gamma} = \sum_{\substack{\varepsilon=1\\ \varepsilon'=\pm1\\ \varepsilon''}}\iint_{\Delta k} dt\,dq\,\frac{tq}{k}\int_0^{2\pi} d\lambda\, e^{-2I\lambda}\,\theta^{\varepsilon\varepsilon'\varepsilon''}_{\underline{k}\underline{t}\underline{q}}\,C^2_{ktq}\cdot\Big[A_3(k,-\varepsilon' t,-\varepsilon'' q)\\ \times\, e''(e'-e)-A_4(k,-\varepsilon' t,-\varepsilon'' q)\,e\,\mathcal{Z}''-A_1(k,-\varepsilon' t,-\varepsilon'' q)\,e''\mathcal{Z}\\ +A_5(k,-\varepsilon' t,-\varepsilon'' q)\,e'\mathcal{Z}''+A_2(k,-\varepsilon' t,-\varepsilon'' q)\,\mathcal{Z}''(\mathcal{Z}'-\mathcal{Z})\Big]\end{aligned}$$

with

$$e = e(\underline{k},t)\,,\quad e' = e(\underline{r},t)\,,\quad e'' = e(\underline{q},t)\,,$$

$$z = Z(\varepsilon\underline{k},t)\,e^{2I\varepsilon\lambda}\,,\quad z' = Z(\varepsilon'\underline{r},t)\,e^{2I\varepsilon'\lambda'}\,,\quad z'' = Z(\varepsilon''\underline{q},t)\,e^{2I\varepsilon''\lambda''}$$

$$\theta^{\varepsilon\varepsilon'\varepsilon''}_{\underline{k}\underline{r}\underline{q}} = \theta_{krq}\Big/\left(1 - 2\Omega\theta_{krq}\left(\varepsilon\mu_{\underline{k}} + \varepsilon'\mu_{\underline{r}} + \varepsilon''\mu_{\underline{q}}\right)\right)$$

The angular coefficients, the expressions of which depend only on k, r, q are :

$$C_{krq} = \frac{\sin\alpha_k}{k} = \frac{\sin\alpha_r}{r} = \frac{\sin\alpha_q}{q}\;;\quad C^2_{krq} = \frac{(k+r+q)(-k+r+q)(k-r+q)(k+r-q)}{4k^2r^2q^2}$$

$$A_1(k,r,q) = -(r-q)(k-q)(k+r+q)^2$$

$$A_2(k,r,q) = -(r-q)(k+q)(k+r+q)(k+r-q)$$

$$A_3(k,r,q) = -(r-q)(-k-q)(k+r+q)(-k+r+q)$$

$$A_4(k,r,q) = -(r-q)(k-q)(k+r+q)(k+r-q)$$

$$A_5(k,r,q) = -(r^2-q^2)(k+r+q)(k+r-q)$$

The coefficients which depend on both k, r, q and on the orientation variables $\mu_{\underline{k}} = \cos\theta_k$ and λ are :

$$\mu_{\underline{r}} = \cos\theta_r = -\cos\alpha_q \cdot \underline{\cos\theta_k} - \sin\alpha_q\,\underline{\sin\theta_k\cos\lambda}$$

$$\mu_{\underline{q}} = \cos\theta_q = -\cos\alpha_r \cdot \underline{\cos\theta_k} + \sin\alpha_r\,\underline{\sin\theta_k\cos\lambda}$$

$$e(\underline{k}) = e(k,\mu_{\underline{k}})\,,\quad e(\underline{r}) = e(r,\mu_{\underline{r}})\,,\quad Z(\varepsilon\underline{k}) = Z(k,\varepsilon\mu_{\underline{k}})\;\ldots\text{ETC}$$

At last λ', λ'' are so that :

$$\sqrt{1-\nu_{\underline{k}}^2}\cdot \sin\lambda = \sqrt{1-\nu_{\underline{1}}^2}\,\sin\lambda' = \sqrt{1-\nu_{\underline{q}}^2}\,\sin\lambda''$$

$$\sqrt{1-\nu_{\underline{1}}^2}\,\cos\lambda' = \sin\alpha_1\cdot\nu_{\underline{k}} - \cos\alpha_1\sqrt{1-\nu_{\underline{k}}^2}\,\cos\lambda$$

$$\sqrt{1-\nu_{\underline{q}}^2}\,\cos\lambda'' = -\sin\alpha_q\cdot\nu_{\underline{k}} - \cos\alpha_q\sqrt{1-\nu_{\underline{k}}^2}\,\cos\lambda$$