To JFH.

Kida & Hunt

Appendix A

The coefficients $F_q(t)$, $F_{11}(t)$, $F_{12}(t)$, $F_{21}(t)$ and $F_{22}(t)$ in (3.29) and (3.30) are given by

$$F_q(t) = \left\langle \frac{1}{2D} \sum_{k=1}^{3} \left(1 - X_k{}^2\right) \mathrm{e}^{2\alpha_k t} \right\rangle_{|\boldsymbol{X}|=1}, \tag{A.1}$$

$$\begin{aligned} F_{11}(t) = \frac{1}{F_q(t)} \Bigg\langle \frac{1}{4D^2} \sum_{(1,2,3)} \Big[&\left(1 - X_1{}^2\right)^2 X_1{}^2 + \mathrm{e}^{3\alpha_1 t}\left(1 + X_1{}^2\right) X_2{}^2 X_3{}^2 \\ &+ \mathrm{e}^{-3\alpha_1 t}\left(X_1{}^2 + X_2{}^2 X_3{}^2\right) X_1{}^2 \\ &+ 2\,\mathrm{e}^{(\alpha_1-\alpha_2)t}\left(1 - X_1{}^2\right) X_2{}^2 X_3{}^2 \\ &+ \mathrm{e}^{2(\alpha_1-\alpha_2)t}\left(X_2{}^2 - X_3{}^2 X_1{}^2\right) X_2{}^2 \Big] \Bigg\rangle_{|\boldsymbol{X}|=1}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} F_{12}(t) = \frac{1}{F_q(t)} \Bigg\langle \frac{1}{8D^2} \sum_{(1,2,3)} \Big[&\left(1 - X_1{}^2\right)\left(3 - 7X_1{}^2\right) X_1{}^2 - \mathrm{e}^{3\alpha_1 t}\left(5 - 7X_1{}^2\right) X_2{}^2 X_3{}^2 \\ &+ \mathrm{e}^{-3\alpha_1 t}\left(X_1{}^2 + 7X_2{}^2 X_3{}^2\right) X_1{}^2 \\ &+ 2\,\mathrm{e}^{(\alpha_1-\alpha_2)t}\left(1 - 7X_1{}^2\right) X_2{}^2 X_3{}^2 \\ &+ \mathrm{e}^{2(\alpha_1-\alpha_2)t}\left(4X_3{}^2 - X_2{}^2 - 7X_3{}^2 X_1{}^2\right) X_2{}^2 \Big] \Bigg\rangle_{|\boldsymbol{X}|=1}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} F_{21}(t) = -\frac{1}{F_q(t)} \Bigg\langle \frac{1}{4D^2} \sum_{(1,2,3)} \Big[&\frac{1}{2}\left(1 - X_1{}^2\right)^2 X_1{}^2 + \mathrm{e}^{3\alpha_1 t}\left(1 + X_1{}^2\right) X_2{}^2 X_3{}^2 \\ &- 2\,\mathrm{e}^{(\alpha_1-\alpha_2)t}\left(1 - X_1{}^2\right) X_1{}^2 X_2{}^2 \end{aligned}$$

$$- e^{2(\alpha_1-\alpha_2)t}\left(X_3{}^2 - X_1{}^2 X_2{}^2\right) X_2{}^2\Big]\Bigg\rangle_{|\boldsymbol{X}|=1}, \tag{A.4}$$

$$F_{22}(t) = -\frac{1}{F_q(t)} \Bigg\langle \frac{1}{8\,D^2} \sum_{(1,2,3)} \Big[\frac{1}{2}\left(1 - X_1{}^2\right)\left(3 - 7X_1{}^2\right) X_1{}^2 - e^{3\alpha_1 t}\left(5 - 7X_1{}^2\right) X_2{}^2 X_3{}^2$$

$$- 14\, e^{(\alpha_1-\alpha_2)t}\left(1 - X_1{}^2\right) X_1{}^2 X_2{}^2$$

$$- e^{2(\alpha_1-\alpha_2)t}\left(4X_2{}^2 - X_3{}^2 - 7X_1{}^2 X_2{}^2\right) X_2{}^2\Big]\Bigg\rangle_{|\boldsymbol{X}|=1}, \tag{A.5}$$

where

$$D = \sum_{k=1}^{3} e^{-2\alpha_k t}\, X_k{}^2 \tag{A.6}$$

and $\sum_{(1,2,3)}$ means that the summation should be carried out over all permutations of (1, 2, 3). $\langle\ \ \rangle_{|\boldsymbol{X}|=1}$ denotes an average over a unit sphere in $\boldsymbol{X}$-space.

Appendix B

As shown in §4.1, the rate-of-strain tensor $\underline{\underline{\alpha}}$ can in general be divided into its irrotational and rotational components;

$$\underline{\underline{\alpha}} = \underline{\underline{\alpha}}^{(I)} + \underline{\underline{\alpha}}^{(R)}, \tag{B.1}$$

where $\underline{\underline{\alpha}}^{(I)}$ is symmetric and traceless, and $\underline{\underline{\alpha}}^{(R)}$ is antisymmetric, i.e.

$$\alpha_{ij}{}^{(I)} = \alpha_{ji}{}^{(I)}, \qquad \alpha_{aa}{}^{(I)} = 0 \qquad \text{and} \qquad \alpha_{ij}{}^{(R)} = -\alpha_{ji}{}^{(R)}. \tag{B.2}$$

Under the assumption that there is no correlation between $\underline{\underline{\alpha}}^{(I)}$ amd $\underline{\underline{\alpha}}^{(R)}$, i.e.

$$\left\langle \alpha_{ij}{}^{(I)} \alpha_{kl}{}^{(R)} \right\rangle = 0 \tag{B.3}$$

the ensemble averages of the second order moments of $\underline{\underline{\alpha}}$ is expressed as the sum of those of the two components as

$$\left\langle \alpha_{ij}\alpha_{kl} \right\rangle = \left\langle \alpha_{ij}{}^{(I)} \alpha_{kl}{}^{(I)} \right\rangle + \left\langle \alpha_{ij}{}^{(R)} \alpha_{kl}{}^{(R)} \right\rangle. \tag{B.4}$$

For a (pseudo-) isotropic turbulence the second order moments of any tensor T_{ij} can in general be written as

$$\left\langle T_{ij} T_{kl} \right\rangle = A\, \delta_{ij}\delta_{kl} + B\, \delta_{ik}\delta_{jl} + C\, \delta_{il}\delta_{jk}, \tag{B.5}$$

where A, B and C are constants.

It follows from (B.2) and (B.5) that

$$\left\langle \alpha_{ij}{}^{(I)} \alpha_{kl}{}^{(I)} \right\rangle = -\frac{1}{5}\, {\alpha_0}^2 \delta_{ij}\delta_{kl} + \frac{3}{10}\, {\alpha_0}^2 \delta_{ik}\delta_{jl} + \frac{3}{10}\, {\alpha_0}^2 \delta_{il}\delta_{jk} \tag{B.6}$$

and

$$\left\langle \alpha_{ij}{}^{(R)} \alpha_{kl}{}^{(R)} \right\rangle = \frac{{\Omega_0}^2}{3} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right), \tag{B.7}$$

where $\underline{\underline{\alpha}}^{(I)}$ and $\underline{\underline{\alpha}}^{(R)}$ have been normalized respectively as

$$\left\langle \alpha_{ab}{}^{(I)} \alpha_{ab}{}^{(I)} \right\rangle = 3{\alpha_0}^2 \tag{B.8}$$

and

$$\left\langle \alpha_{ab}{}^{(R)} \alpha_{ab}{}^{(R)} \right\rangle = 2\Omega_0{}^2. \tag{B.9}$$

Substitution of (B.6) and (B.7) into (B.4) then gives us that

$$\left\langle \alpha_{ij}\alpha_{kl} \right\rangle = -\frac{1}{5}\,\alpha_0{}^2 \delta_{ij}\delta_{kl} + \left(\frac{3}{10}\,\alpha_0{}^2 + \frac{\Omega_0{}^2}{3} \right) \delta_{ik}\delta_{jl} + \left(\frac{3}{10}\,\alpha_0{}^2 - \frac{\Omega_0{}^2}{3} \right) \delta_{il}\delta_{jk}. \tag{B.10}$$

If the turbulence is isotropic in the strict sense, or if it has no helicity, we have the relation

$$\left\langle \alpha_{11}{}^2 \right\rangle = \frac{1}{2} \left\langle \alpha_{21}{}^2 \right\rangle \tag{B.11}$$

(see Batchelor 1953) or, from (B.10),

$$\frac{\alpha_0{}^2}{2} = \frac{\Omega_0{}^2}{3}. \tag{B.12}$$

Then we find

$$\left\langle \alpha_{ij}\alpha_{kl} \right\rangle = -\frac{1}{5}\,\alpha_0{}^2 \left(\delta_{ij}\delta_{kl} - 4\,\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right). \tag{B.13}$$

Incidentally in this case it is seen from (B.8), (B.9) and (B.12) that the norms of the irrotational and rotational components of $\underline{\underline{\alpha}}$ are equal;

$$\left\langle \alpha_{ij}{}^{(I)} \alpha_{kl}{}^{(I)} \right\rangle = \left\langle \alpha_{ij}{}^{(R)} \alpha_{kl}{}^{(R)} \right\rangle = 2 \left\langle \alpha_{ij}\alpha_{kl} \right\rangle. \tag{B.14}$$

For the reference to §5.3 we give here (without proof) the explicit form of the second order moments of an axisymmetric incompressible straining tensor $\underline{\underline{\alpha}}$. If we choose the symmetric axis in the direction of the x_3 axis, the second order moments of $\underline{\underline{\alpha}}$ is shown to be written as

$$\begin{aligned}
\left\langle \alpha_{ij}\alpha_{kl} \right\rangle &= \left(\tfrac{1}{2}\left\langle \alpha_{33}{}^2 \right\rangle - \left\langle \alpha_{11}{}^2 \right\rangle \right) \delta_{ij}\delta_{kl} + \left\langle \alpha_{12}{}^2 \right\rangle \delta_{ik}\delta_{jl} \\
&+ \left(2 \left\langle \alpha_{11}{}^2 \right\rangle - \left\langle \alpha_{12}{}^2 \right\rangle - \tfrac{1}{2} \left\langle \alpha_{33}{}^2 \right\rangle \right) \delta_{il}\delta_{jk} \\
&+ \left(\left\langle \alpha_{11}{}^2 \right\rangle - \left\langle \alpha_{33}{}^2 \right\rangle \right) \left(\delta_{ij}\lambda_k\lambda_l + \delta_{kl}\lambda_i\lambda_j \right) \\
&+ \left(\left\langle \alpha_{13}\alpha_{31} \right\rangle - 2 \left\langle \alpha_{11}{}^2 \right\rangle + \left\langle \alpha_{12}{}^2 \right\rangle + \tfrac{1}{2}\left\langle \alpha_{33}{}^2 \right\rangle \right) \left(\delta_{il}\lambda_j\lambda_k + \delta_{jk}\lambda_i\lambda_l \right) \\
&+ \left(\left\langle \alpha_{13}{}^2 \right\rangle - \left\langle \alpha_{12}{}^2 \right\rangle \right) \delta_{ik}\lambda_j\lambda_l + \left(\left\langle \alpha_{31}{}^2 \right\rangle - \left\langle \alpha_{12}{}^2 \right\rangle \right) \delta_{jl}\lambda_i\lambda_k \\
&+ \left(\left\langle \left\langle \alpha_{11}{}^2 \right\rangle + 2 \left\langle \alpha_{33}{}^2 \right\rangle - 2 \left\langle \alpha_{13}\alpha_{31} \right\rangle - \left\langle \alpha_{31}{}^2 \right\rangle - \left\langle \alpha_{13}{}^2 \right\rangle \right) \lambda_i\lambda_j\lambda_k\lambda_l,
\end{aligned} \tag{B.15}$$

where

$$\lambda_i = \delta_{i3} \tag{B.16}$$

is the unit vector in the direction of the x_3-axis.

Appendix C

The initial development of the velocity moment tensor $B_{ij}(t)$ of a small-scale isotropic turbulence distorted by an anisotropic strain field $\underline{\underline{\alpha}}$ with zero mean is calculated as follows.

The velocity moment tensor is calculated by integrating the energy spectrum tensor (2.27) with respect to $\boldsymbol{\kappa}$:

$$B_{ij}(t) = \int \left\langle \left(\frac{1}{\chi^2}(\delta_{ij} - \delta_{ia}\delta_{jb}) - \frac{\chi_i\chi_j}{\chi^4}\,\delta_{ab} \right) T_{ac}T_{bd} \right\rangle \Omega_{cd}(\boldsymbol{\kappa}, 0)\, d\boldsymbol{\kappa}. \tag{C.1}$$

In order to examine the small time behaviour of $B_{ij}(t)$, we first expand the deformation tensor (2.18b) in the power series of t as

$$S_{ij} = \delta_{ij} + \alpha_{ij}\,t + \frac{1}{2}\left(\underline{\underline{\alpha}}^2\right)_{ij} t^2 + \cdots, \tag{C.2}$$

the inverse of which is

$$\widetilde{S}_{ij}(t) = \delta_{ij} - \alpha_{ij}\,t + \frac{1}{2}\left(\underline{\underline{\alpha}}^2\right)_{ij} t^2 + \cdots. \tag{C.3}$$

Then the deformation of the wavenumber $\chi(t)$ is written in terms of $\boldsymbol{\kappa}$ as

$$\chi_i = \kappa_p \widetilde{S}_{pi} = \kappa_i - \kappa_p\alpha_{pi}\,t + \frac{1}{2}\,\kappa_p\left(\underline{\underline{\alpha}}^2\right)_{pi} t^2 - \cdots. \tag{C.4}$$

Thence,

$$\frac{1}{\chi^2} = \frac{1}{\kappa^2}\left[1 + \frac{2\kappa_p\kappa_q}{\kappa^2}\,t - \frac{\kappa_p\kappa_q}{\kappa^2}\left(\left(\underline{\underline{\alpha}}^2\right)_{pq} + \alpha_{pr}\alpha_{qr} + \frac{4\kappa_r\kappa_s}{\kappa^2}\,\alpha_{pq}\alpha_{rs}\right) t^2 + \cdots\right], \tag{C.5}$$

$$\frac{\chi_i\chi_k}{\chi^2} = \frac{1}{\kappa^2}\left\{\kappa_i\kappa_k + \left(\frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^2}\alpha_{pq} - \kappa_i\kappa_p\alpha_{pk} - \kappa_k\kappa_p\alpha_{pi}\right) t + \cdots\right\}, \tag{C.6}$$

and

$$\begin{aligned}\frac{\chi_i\chi_k}{\chi^4} = \frac{1}{\kappa^4}\Bigg\{\kappa_i\kappa_k &+ \left(\frac{4\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^2}\alpha_{pq} - \kappa_i\kappa_p\alpha_{pk} - \kappa_k\kappa_p\alpha_{pi}\right) t \\ &+ \Bigg[\frac{1}{2}\kappa_i\kappa_p\left(\underline{\underline{\alpha}}^2\right)_{pk} + \frac{1}{2}\kappa_k\kappa_p\left(\underline{\underline{\alpha}}^2\right)_{pi} + \kappa_p\kappa_q\alpha_{pi}\alpha_{qk} \\ &\quad - \frac{4\kappa_i\kappa_p\kappa_q\kappa_r}{\kappa^2}\alpha_{pq}\alpha_{rk} - \frac{4\kappa_k\kappa_p\kappa_q\kappa_r}{\kappa^2}\alpha_{pq}\alpha_{ri} \\ &\quad - \frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^2}\left(\left(\underline{\underline{\alpha}}^2\right)_{pq} + \alpha_{pr}\alpha_{qr}\right) \\ &\quad + \frac{12\kappa_i\kappa_k\kappa_p\kappa_q\kappa_r\kappa_s}{\kappa^4}\alpha_{pq}\alpha_{rs}\Bigg] t^2 + \cdots\Bigg\}.\end{aligned} \tag{C.7}$$

Substituting (C.6) into the the definition in (2.15a), the viscous term being omitted, we obtain the power series of the deformation tensor for the rate of change of vorticity as

$$\beta_{ij}(t) = \beta_{ij}{}^{(0)} + \beta_{ij}{}^{(1)}\, t + \cdots, \tag{C.8}$$

where the coefficients are

$$\beta_{ij}{}^{(0)} = \alpha_{ji} + \frac{\kappa_i\kappa_k}{\kappa^2}\left(\alpha_{kj} - \alpha_{jk}\right), \tag{C.9a}$$

$$\beta_{ij}{}^{(1)} = \left[\frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^4}\,\alpha_{pq} - \frac{\kappa_i\kappa_p}{\kappa^2}\,\alpha_{pk} - \frac{\kappa_k\kappa_p}{\kappa^2}\,\alpha_{pi}\right]\left(\alpha_{kj} - \alpha_{jk}\right), \tag{C.9b}$$

Then the vorticity deformation tensor (2.19a) can be expanded as

$$\begin{aligned}\underline{\underline{T}}(t) &= \underline{\underline{I}} + \int_0^t \underline{\underline{\beta}}(t')\, dt' + \int_0^t dt_1 \int_0^{t_1} dt_2\, \underline{\underline{\beta}}(t_1)\cdot\underline{\underline{\beta}}(t_2) + \cdots \\ &\equiv \underline{\underline{I}} + \underline{\underline{T}}^{(1)} t + \frac{1}{2}\underline{\underline{T}}^{(2)} t^2 + \cdots,\end{aligned} \tag{C.10}$$

where the coefficients are

$$\underline{\underline{T}}^{(1)} = \underline{\underline{\beta}}^{(0)}, \tag{C.11a}$$

$$\underline{\underline{T}}^{(2)} = \underline{\underline{\beta}}^{(1)} + {\underline{\underline{\beta}}^{(0)}}^2, \tag{C.11b}$$

By substituting into the small time expansions of $\chi(t)$ and $\underline{\underline{T}}(t)$, (C.4) and (C.10), and the relation between the energy and the vorticity spectrum tensors for isotropic turbulence,

$$\Omega_{ij}(\boldsymbol{\kappa},0) = \kappa^2 \Phi_{ij}(\boldsymbol{\kappa},0), \tag{C.12}$$

we obtain

$$\begin{aligned}
B_{ij} = \int \Bigg[& \delta_{ij}\delta_{cd} - \frac{\kappa_i\kappa_j}{\kappa^2}\delta_{cd} - \delta_{ic}\delta_{jd} \\
& + \Bigg\langle \left(\frac{4\kappa_p\kappa_q\kappa_r\kappa_s\alpha_{pq}\alpha_{rs}}{\kappa^4} - \frac{\kappa_p\kappa_q\left(\left(\underline{\underline{\alpha}}^2\right)_{pq} + \alpha_{pr}\alpha_{qr}\right)}{\kappa^2} \right)\left(\delta_{ij}\delta_{cd} - \delta_{ic}\delta_{jd}\right) \\
& - \Bigg(\frac{12\kappa_i\kappa_j\kappa_p\kappa_q\kappa_r\kappa_s\alpha_{pq}\alpha_{rs}}{\kappa^6} - \frac{2\kappa_i\kappa_j\kappa_p\kappa_q\left(\left(\underline{\underline{\alpha}}^2\right)_{pq} + \alpha_{pr}\alpha_{qr}\right)}{\kappa^4} \\
& \qquad - \frac{4\kappa_p\kappa_q\alpha_{pq}\left(\kappa_i\kappa_r\alpha_{rj} + \kappa_j\kappa_r\alpha_{ri}\right)}{\kappa^4} \\
& \qquad + \frac{\kappa_i\kappa_p\left(\underline{\underline{\alpha}}^2\right)_{pj}}{2\kappa^2} + \frac{\kappa_p\kappa_q\alpha_{pi}\alpha_{qj}}{\kappa^2} + \frac{\kappa_j\kappa_p\left(\underline{\underline{\alpha}}^2\right)_{pi}}{2\kappa^2} \Bigg)\delta_{cd} \\
& + \frac{2\kappa_p\kappa_q\alpha_{pq}}{\kappa^2}\left(T^{(1)}_{dc}\delta_{ij} + T^{(1)}_{cd}\delta_{ij} - T^{(1)}_{ic}\delta_{jd} + T^{(1)}_{jd}\delta_{ic}\right) \\
& - \left(\frac{4\kappa_i\kappa_j\kappa_p\kappa_q\alpha_{pq}}{\kappa^4} - \frac{\kappa_i\kappa_p\alpha_{pj}}{\kappa^2} - \frac{\kappa_j\kappa_p\alpha_{pi}}{\kappa^2} \right)\left(T^{(1)}_{dc} + T^{(1)}_{cd}\right) \\
& + \left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)\left(T^{(2)}_{dc} + T^{(2)}_{cd} + T^{(1)}_{ac}T^{(1)}_{ad}\right) \\
& - \delta_{jd}T^{(2)}_{ic} - \delta_{ic}T^{(2)}_{jd} - T^{(1)}_{ic}T^{(1)}_{jd} \Bigg\rangle t^2 + \cdots \Bigg] \Phi_{cd}(\boldsymbol{\kappa},0)\, d\boldsymbol{\kappa}.
\end{aligned} \tag{C.13}$$

Note that terms proportional to t, which are linear in $\underline{\underline{\alpha}}$, are dropped out after the ensemble average because $\langle\underline{\underline{\alpha}}\rangle = 0$.

Substituting the coefficients of $\underline{\underline{T}}$, (C.11) with (C.9), into (C.13) and using identities for the weighted integrals of $\Phi(\boldsymbol{\kappa},0)$ for isotropic turbulence, (D.2) $\sim$ (D.5), we obtain

$$\begin{aligned}
B_{ij}(t) = \frac{2}{3}\int E(k,0)dk\Big\{\delta_{ij} + \frac{t^2}{35}\Big[& 8\left\langle\underline{\underline{\alpha}}^2\right\rangle_{aa}\delta_{ij} + 8\left\langle\alpha_{ab}\alpha_{ab}\right\rangle\delta_{ij} - 5\left\langle\underline{\underline{\alpha}}^2\right\rangle_{ij} \\
& - 5\left\langle\underline{\underline{\alpha}}^2\right\rangle_{ji} - 12\left\langle\alpha_{ai}\alpha_{aj}\right\rangle + 2\left\langle\alpha_{ia}\alpha_{ja}\right\rangle\Big]\Big\}.
\end{aligned} \tag{C.14}$$

Appendix D

We give here several identities for the weighted integrals for isotropic turbulence.

For isotropic turbulence the energy spectral tensor $\Phi_{ij}(\boldsymbol{\kappa})$ is expressed in terms of a single scalar function $E(\kappa)$ as

$$\Phi_{ij}(\boldsymbol{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^4}\left(\kappa^2\delta_{ij} - \kappa_i\kappa_j\right), \tag{D.1}$$

where $\kappa = |\boldsymbol{\kappa}|$.

Then it is easy to show the following identies for the weighted integrals:

$$\int \Phi_{ab}(\boldsymbol{\kappa})\, d\boldsymbol{\kappa} = \frac{2}{3}\,\delta_{ab} \int E(\kappa)\, d\kappa, \tag{D.2}$$

$$\int \frac{\kappa_a\kappa_b}{\kappa^2}\,\Phi_{cd}(\boldsymbol{\kappa})\, d\boldsymbol{\kappa} = \frac{1}{15}\left(4\,\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) \int E(\kappa)\, d\kappa, \tag{D.3}$$

$$\begin{aligned}\int \frac{\kappa_a\kappa_b\kappa_c\kappa_d}{\kappa^4}\,\Phi_{ef}(\boldsymbol{\kappa})\, d\boldsymbol{\kappa} = \frac{1}{105}\Big[&6\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right)\delta_{ef} \\ &- \delta_{ab}\delta_{ce}\delta_{df} - \delta_{ab}\delta_{cf}\delta_{de} - \delta_{ac}\delta_{be}\delta_{df} \\ &- \delta_{ac}\delta_{bf}\delta_{de} - \delta_{ad}\delta_{be}\delta_{cf} - \delta_{ad}\delta_{bf}\delta_{ce} \\ &- \delta_{ae}\delta_{bc}\delta_{df} - \delta_{ae}\delta_{bd}\delta_{cf} - \delta_{ae}\delta_{bf}\delta_{cd} \\ &- \delta_{af}\delta_{bc}\delta_{de} - \delta_{af}\delta_{bd}\delta_{ce} - \delta_{af}\delta_{be}\delta_{cd}\Big] \int E(\kappa)\, d\kappa,\end{aligned} \tag{D.4}$$

$$\begin{aligned}\int \frac{\kappa_a\kappa_b\kappa_c\kappa_d\kappa_e\kappa_f}{\kappa^6}\,\Phi_{ll}(\boldsymbol{\kappa})\, d\boldsymbol{\kappa} = \frac{2}{105}\Big(&\delta_{ab}\delta_{cd}\delta_{ef} + \delta_{ac}\delta_{bd}\delta_{ef} + \delta_{ad}\delta_{bc}\delta_{ef} \\ &+ \delta_{ab}\delta_{ce}\delta_{df} + \delta_{ab}\delta_{cf}\delta_{de} + \delta_{ac}\delta_{be}\delta_{df} \\ &+ \delta_{ac}\delta_{bf}\delta_{de} + \delta_{ad}\delta_{be}\delta_{cf} + \delta_{ad}\delta_{bf}\delta_{ce} \\ &+ \delta_{ae}\delta_{bc}\delta_{df} + \delta_{ae}\delta_{bd}\delta_{cf} + \delta_{ae}\delta_{bf}\delta_{cd} \\ &+ \delta_{af}\delta_{bc}\delta_{de} + \delta_{af}\delta_{bd}\delta_{ce} + \delta_{af}\delta_{be}\delta_{cd}\Big] \int E(\kappa)\, d\kappa.\end{aligned} \tag{D.5}$$

Appendix E

Here we describe a detailed calculation for the initial evolution of the vorticity moment tensor $W_{ij}(t)$ distorted by an isotropic random rotational straining motion and by an irrotational mean straining motion.

Let $\underline{\underline{A}}^*$ be the sum of a large-scale irrotational mean strain $\underline{\underline{A}}$ and a large-scale random rotational strain $\underline{\underline{\alpha}}$:

$$\underline{\underline{A}}^* = \underline{\underline{A}} + \underline{\underline{\alpha}}. \tag{E.1}$$

We choose a co-ordinate system in which $\underline{\underline{A}}$ is diagonalized, so that

$$A_{ij} = A^{(i)}\,\delta_{ij}. \tag{E.2}$$

The deformation tensor (2.18b) is now written as

$$\underline{\underline{S}}(t) = \exp\left[\int_0^t \underline{\underline{A}}^*\,dt'\right] = \exp[\underline{\underline{A}}^* t], \tag{E.3}$$

which is expanded for small t as

$$S_{ij} = \delta_{ij} + A^*_{ij}t + \frac{1}{2}\left(\underline{\underline{A}}^{*2}\right)_{ij}t^2 + \cdots. \tag{E.4}$$

Its inverse is

$$\widetilde{S}_{ij}(t) = \delta_{ij} - A^*_{ij}t + \frac{1}{2}\left(\underline{\underline{A}}^{*2}\right)_{ij}t^2 + \cdots, \tag{E.5}$$

which gives the deformation of the wavenumber $\chi(t)$ in terms of $\boldsymbol{\kappa}$ as

$$\chi_i = \kappa_p \widetilde{S}_{pi} = \kappa_i - \kappa_p A^*_{pi}\,t + \frac{1}{2}\kappa_p\left(\underline{\underline{A}}^{*2}\right)_{pi}t^2 - \cdots. \tag{E.6}$$

Thence,

$$\frac{\chi_i\chi_k}{\chi^2} = \frac{1}{\kappa^2}\left\{\kappa_i\kappa_k + \left(\frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^2}A^*_{pq} - \kappa_i\kappa_p A^*_{pk} - \kappa_k\kappa_p A^*_{pi}\right)t\right.$$
$$+\left[\frac{1}{2}\kappa_i\kappa_p\left(\underline{\underline{A}}^{*2}\right)_{pk} + \frac{1}{2}\kappa_k\kappa_p\left(\underline{\underline{A}}^{*2}\right)_{pi} + \kappa_p\kappa_q A^*_{pi}A^*_{qk}\right.$$

$$
\begin{aligned}
&- \frac{2\kappa_i\kappa_p\kappa_q\kappa_r}{\kappa^2} A^*_{pq}A^*_{rk} - \frac{2\kappa_k\kappa_p\kappa_q\kappa_r}{\kappa^2} A^*_{pq}A^*_{ri} \\
&- \frac{\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^2}\left(\left(\underline{\underline{A}}^{*2}\right)_{pq} + A^*_{pr}A^*_{qr} \right) \\
&+ \frac{4\kappa_i\kappa_k\kappa_p\kappa_q\kappa_r\kappa_s}{\kappa^4} A^*_{pq}A^*_{rs} \Bigg] t^2 + \cdots \Bigg\}.
\end{aligned} \tag{E.7}
$$

From the definition in (2.15a), the viscous term being omitted, the deformation tensor for the rate of change of vorticity is

$$
\beta_{ij}(t) = A^*_{ji} + \frac{\chi_i\chi_k}{\chi^2}(\alpha_{kj} - \alpha_{jk}). \tag{E.8}
$$

This can be expanded as

$$
\beta_{ij}(t) = \beta_{ij}{}^{(0)} + \beta_{ij}{}^{(1)}\, t + \frac{1}{2}\beta_{ij}{}^{(2)}\, t^2 + \cdots, \tag{E.9}
$$

where the coefficients are

$$
\beta_{ij}{}^{(0)} = A^*_{ji} + \frac{\kappa_i\kappa_k}{\kappa^2}(\alpha_{kj} - \alpha_{jk}), \tag{E.10a}
$$

$$
\beta_{ij}{}^{(1)} = \left[\frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^4} A^*_{pq} - \frac{\kappa_i\kappa_p}{\kappa^2} A^*_{pk} - \frac{\kappa_k\kappa_p}{\kappa^2} A^*_{pi} \right] (\alpha_{kj} - \alpha_{jk}), \tag{E.10b}
$$

$$
\begin{aligned}
\beta_{ij}{}^{(2)} = \Bigg\{ &\frac{\kappa_i\kappa_p}{\kappa^2}\left(\underline{\underline{A}}^{*2}\right)_{pk} + \frac{\kappa_k\kappa_p}{\kappa^2}\left(\underline{\underline{A}}^{*2}\right)_{pi} + \frac{2\kappa_p\kappa_q}{\kappa^2} A^*_{pi}A^*_{qk} \\
&- \frac{4\kappa_i\kappa_p\kappa_q\kappa_r}{\kappa^4} A^*_{pq}A^*_{rk} - \frac{4\kappa_k\kappa_p\kappa_q\kappa_r}{\kappa^4} A^*_{pq}A^*_{ri} \\
&- \frac{2\kappa_i\kappa_k\kappa_p\kappa_q}{\kappa^4}\left(\left(\underline{\underline{A}}^{*2}\right)_{pq} + A^*_{pr}A^*_{qr} \right) \\
&+ \frac{8\kappa_i\kappa_k\kappa_p\kappa_q\kappa_r\kappa_s}{\kappa^6} A^*_{pq}A^*_{rs} \Bigg\} (\alpha_{kj} - \alpha_{jk}).
\end{aligned} \tag{E.10c}
$$

Thence the vorticity deformation tensor

$$
\underline{\underline{T}}(t) = \exp\left[\int_0^t \underline{\underline{\beta}}(t')\, dt' \right] \tag{E.11}
$$

can be expanded as

$$
\begin{aligned}
\underline{\underline{T}}(t) &= \underline{\underline{I}} + \int_0^t \underline{\underline{\beta}}(t')\,dt' + \int_0^t dt_1 \int_0^{t_1} dt_2\, \underline{\underline{\beta}}(t_1)\cdot\underline{\underline{\beta}}(t_2) \\
&\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, \underline{\underline{\beta}}(t_1)\cdot\underline{\underline{\beta}}(t_2)\cdot\underline{\underline{\beta}}(t_3) + \cdots \\
&\equiv \underline{\underline{I}} + \underline{\underline{T}}^{(1)} t + \frac{1}{2!}\underline{\underline{T}}^{(2)} t^2 + \frac{1}{3!}\underline{\underline{T}}^{(3)} t^3 + \cdots,
\end{aligned} \tag{E.12}
$$

where the coefficients are

$$
\underline{\underline{T}}^{(1)} = \underline{\underline{\beta}}^{(0)}, \tag{E.13a}
$$

$$
\underline{\underline{T}}^{(2)} = \underline{\underline{\beta}}^{(1)} + \underline{\underline{\beta}}^{(0)^2}, \tag{E.13b}
$$

$$
\underline{\underline{T}}^{(3)} = \underline{\underline{\beta}}^{(2)} + \underline{\underline{\beta}}^{(0)}\cdot\underline{\underline{\beta}}^{(1)} + 2\,\underline{\underline{\beta}}^{(1)}\cdot\underline{\underline{\beta}}^{(0)} + \underline{\underline{\beta}}^{(0)^3}. \tag{E.13c}
$$

By substituting the small time expansion of $\underline{\underline{T}}(t)$, (E.12), into the definition of the vorticity moment tensor, (6.3), we obtain

$$
\begin{aligned}
W_{ij}(t) = \int \Big\langle & \delta_{ip}\delta_{jq} + \Big(\delta_{ip}{T_{jq}}^{(1)} + {T_{ip}}^{(1)}\delta_{jq}\Big)\, t \\
& + \frac{1}{2!}\Big(\delta_{ip}{T_{jq}}^{(2)} + 2{T_{ip}}^{(1)}{T_{jq}}^{(1)} + {T_{ip}}^{(2)}\delta_{jq}\Big)\, t^2 \\
& + \frac{1}{3!}\Big(\delta_{ip}{T_{jq}}^{(3)} + 3{T_{ip}}^{(1)}{T_{jq}}^{(2)} + 3{T_{ip}}^{(2)}{T_{jq}}^{(1)} + {T_{ip}}^{(3)}\delta_{jq}\Big)\, t^3 \\
& + \cdots \Big\rangle\, \Omega_{pq}(\boldsymbol{\kappa}, 0)\, d\boldsymbol{\kappa}.
\end{aligned} \tag{E.14}
$$

For isotropic Gaussian straining, satisfying (6.4) and (B.10), the moments of the coefficients of $\underline{\underline{\beta}}$ are calculated to be

$$
\Big\langle {\beta_{ij}}^{(0)} \Big\rangle = A_{ji} = A^{(i)}\delta_{ij}, \tag{E.15a}
$$

$$
\Big\langle {\beta_{ij}}^{(1)} \Big\rangle = -\frac{2}{3}\,{\Omega_0}^2\delta_{ij} + 2\,{\Omega_0}^2\frac{\kappa_i\kappa_j}{\kappa^2}, \tag{E.15b}
$$

$$
\begin{aligned}
\Big\langle {\beta_{ij}}^{(2)} \Big\rangle = \frac{2}{3}\,{\Omega_0}^2\Big[& A^{(i)}\delta_{ij} - \big(7A^{(i)} + 6A^{(j)}\big)\frac{\kappa_i\kappa_j}{\kappa^2} \\
& - A^{(p)}\frac{\kappa_p\kappa_p}{\kappa^2}\Big(\delta_{ij} - \frac{16\kappa_i\kappa_j}{\kappa^2}\Big)\Big],
\end{aligned} \tag{E.15c}
$$

$$
\Big\langle \Big(\underline{\underline{\beta}}^{(0)^2}\Big)_{ij} \Big\rangle = \big(A^{(i)}\big)^2\delta_{ij} + {\alpha_0}^2\delta_{ij} + \frac{2}{3}\,{\Omega_0}^2\,\frac{\kappa_i\kappa_j}{\kappa^2}, \tag{E.15d}
$$

$$\left\langle\left(\underline{\underline{\beta}}^{(0)^3}\right)_{ij}\right\rangle = \left(A^{(i)}\right)^3\delta_{ij} + \frac{21}{10}{\alpha_0}^2 A^{(i)}\delta_{ij} + \frac{2}{3}{\Omega_0}^2\Big[\frac{1}{2}A^{(i)}\delta_{ij} + \left(A^{(i)} + A^{(j)}\right)\frac{\kappa_i\kappa_j}{\kappa^2} + A^{(p)}\frac{\kappa_p\kappa_p}{\kappa^2}\Big(\delta_{ij} - \frac{2\kappa_i\kappa_j}{\kappa^2}\Big)\Big], \qquad \text{(E.15e)}$$

$$\left\langle{\beta_{ip}}^{(0)}{\beta_{jq}}^{(0)}\right\rangle = A^{(i)}A^{(j)}\delta_{ip}\delta_{jq} + {\alpha_0}^2\Big(-\frac{1}{5}\delta_{ip}\delta_{jq} + \frac{3}{10}\delta_{ij}\delta_{pq} + \frac{3}{10}\delta_{iq}\delta_{jp}\Big) + \frac{2}{3}{\Omega_0}^2\left(\frac{1}{2}\delta_{ij}\delta_{pq} - \frac{1}{2}\delta_{iq}\delta_{jp} + \frac{\kappa_i\kappa_q}{\kappa^2}\delta_{jp} + \frac{\kappa_j\kappa_p}{\kappa^2}\delta_{iq} - \frac{2\kappa_i\kappa_j\kappa_p\kappa_q}{\kappa^4}\right), \qquad \text{(E.15f)}$$

$$\left\langle\left(\underline{\underline{\beta}}^{(1)}\cdot\underline{\underline{\beta}}^{(0)}\right)_{ij}\right\rangle = \frac{2}{3}{\Omega_0}^2\Big[-A^{(i)}\delta_{ij} - \left(2A^{(i)} - 3A^{(j)}\right)\frac{\kappa_i\kappa_j}{\kappa^2} + 2A^{(p)}\frac{\kappa_p\kappa_p\kappa_i\kappa_j}{\kappa^4})\Big], \qquad \text{(E.15g)}$$

$$\left\langle\left(\underline{\underline{\beta}}^{(0)}\cdot\underline{\underline{\beta}}^{(1)}\right)_{ij}\right\rangle = \frac{2}{3}{\Omega_0}^2\Big[-A^{(i)}\delta_{ij} + \left(4A^{(i)} - A^{(j)}\right)\frac{\kappa_i\kappa_j}{\kappa^2}\Big], \qquad \text{(E.15h)}$$

$$\left\langle{\beta_{ip}}^{(0)}{\beta_{jq}}^{(1)}\right\rangle = \frac{2}{3}{\Omega_0}^2\left(-A^{(i)}\delta_{ip}\delta_{jq} + 3A^{(i)}\frac{\kappa_j\kappa_q}{\kappa^2}\delta_{ip} + \left(A^{(i)} - A^{(j)}\right)\frac{\kappa_i\kappa_j}{\kappa^2}\delta_{pq} - \left(A^{(j)} + A^{(p)}\right)\frac{\kappa_j\kappa_p}{\kappa^2}\delta_{iq} + 2A^{(r)}\frac{\kappa_j\kappa_p\kappa_r\kappa_r}{\kappa^4}\delta_{iq} + 2\left(A^{(j)} + A^{(p)}\right)\frac{\kappa_i\kappa_j\kappa_p\kappa_q}{\kappa^4} - 4A^{(r)}\frac{\kappa_i\kappa_j\kappa_p\kappa_q\kappa_r\kappa_r}{\kappa^6}\right), \qquad \text{(E.15i)}$$

$$\left\langle{\beta_{ip}}^{(0)}\left(\underline{\underline{\beta}}^{(0)^2}\right)_{jq}\right\rangle = A^{(i)}\left(A^{(j)}\right)^2\delta_{ip}\delta_{jq} + {\alpha_0}^2\Big[\Big(A^{(i)} - \frac{2}{5}A^{(j)}\Big)\delta_{ip}\delta_{jq} + \frac{3}{10}\Big(A^{(i)} + A^{(p)}\Big)\delta_{ij}\delta_{pq} + \frac{3}{10}\Big(A^{(i)} + A^{(j)}\Big)\delta_{iq}\delta_{jp}\Big] + \frac{2}{3}{\Omega_0}^2\left(\frac{1}{2}\Big(A^{(i)} + A^{(p)}\Big)\delta_{ij}\delta_{pq} - \frac{1}{2}\Big(A^{(i)} + A^{(j)}\Big)\delta_{iq}\delta_{jp} + A^{(i)}\frac{\kappa_j\kappa_q}{\kappa^2}\delta_{ip} + \Big(A^{(j)} + A^{(q)}\Big)\frac{\kappa_i\kappa_q}{\kappa^2}\delta_{jp} + \Big(A^{(i)} + A^{(j)}\Big)\frac{\kappa_j\kappa_p}{\kappa^2}\delta_{iq} - 2\Big(A^{(j)} + A^{(q)}\Big)\frac{\kappa_i\kappa_j\kappa_p\kappa_q}{\kappa^4}\right). \qquad \text{(E.15j)}$$

Thence the moments of the coefficients of $\underline{\underline{T}}$ are given in terms of $\alpha_0{}^2$, $\Omega_0{}^2$ and the principal values of the mean straining matrix $A^{(i)}$:

$$\left\langle T_{ij}{}^{(1)} \right\rangle = \left\langle \beta_{ij}{}^{(0)} \right\rangle = A^{(i)} \delta_{ij}, \tag{E.16a}$$

$$\begin{aligned} \left\langle T_{ij}{}^{(2)} \right\rangle &= \left\langle \beta_{ij}{}^{(1)} \right\rangle + \left\langle \left(\underline{\underline{\beta}}^{(0)^2} \right)_{ij} \right\rangle \\ &= \left(A^{(i)}\right)^2 \delta_{ij} + \alpha_0{}^2 \delta_{ij} - \frac{2}{3} \Omega_0{}^2 \left(\delta_{ij} - \frac{4\kappa_i\kappa_j}{\kappa^2} \right), \end{aligned} \tag{E.16b}$$

$$\begin{aligned} \left\langle T_{ij}{}^{(3)} \right\rangle &= \left\langle \beta_{ij}{}^{(2)} \right\rangle + \left\langle \left(\underline{\underline{\beta}}^{(0)} \cdot \underline{\underline{\beta}}^{(1)} \right)_{ij} \right\rangle + 2 \left\langle \left(\underline{\underline{\beta}}^{(1)} \cdot \underline{\underline{\beta}}^{(0)} \right)_{ij} \right\rangle + \left\langle \left(\underline{\underline{\beta}}^{(0)} \right)^3_{ij} \right\rangle \\ &= \left(A^{(i)}\right)^3 \delta_{ij} + \frac{21}{10} \alpha_0{}^2 A^{(i)} \delta_{ij} \\ &\quad - \frac{2}{3} \Omega_0{}^2 \left(\frac{3}{2} A^{(i)} \delta_{ij} + 6\, A^{(i)} \frac{\kappa_i\kappa_j}{\kappa^2} - 18\, A^{(p)} \frac{\kappa_p\kappa_p\kappa_i\kappa_j}{\kappa^4} \right), \end{aligned} \tag{E.16c}$$

$$\begin{aligned} \left\langle T_{ip}{}^{(1)} T_{jq}{}^{(1)} \right\rangle &= \left\langle \beta_{ip}{}^{(0)} \beta_{jq}{}^{(0)} \right\rangle \\ &= A^{(i)} A^{(j)} \delta_{ip}\delta_{jq} + \alpha_0{}^2 \left(-\frac{1}{5} \delta_{ip}\delta_{jq} + \frac{3}{10} \delta_{ij}\delta_{pq} + \frac{3}{10} \delta_{iq}\delta_{jp} \right) \\ &\quad + \frac{2}{3} \Omega_0{}^2 \left(\frac{1}{2} \delta_{ij}\delta_{pq} - \frac{1}{2} \delta_{iq}\delta_{jp} + \frac{\kappa_i\kappa_q}{\kappa^2} \delta_{jp} \right. \\ &\qquad\qquad \left. + \frac{\kappa_j\kappa_p}{\kappa^2} \delta_{iq} - \frac{2\kappa_i\kappa_j\kappa_p\kappa_q}{\kappa^4} \right), \end{aligned} \tag{E.16d}$$

$$\begin{aligned} \left\langle T_{ip}{}^{(1)} T_{jq}{}^{(2)} \right\rangle &= \left\langle \beta_{ip}{}^{(0)} \beta_{jq}{}^{(1)} \right\rangle + \left\langle \beta_{ip}{}^{(0)} \left(\underline{\underline{\beta}}^{(0)^2} \right)_{jq} \right\rangle \\ &= A^{(i)} \left(A^{(j)}\right)^2 \delta_{ip}\delta_{jq} \\ &\quad + \alpha_0{}^2 \left[\left(A^{(i)} - \frac{2}{5} A^{(j)} \right) \delta_{ip}\delta_{jq} + \frac{3}{10} \left(A^{(i)} + A^{(p)} \right) \delta_{ij}\delta_{pq} \right. \\ &\qquad\qquad \left. + \frac{3}{10} \left(A^{(i)} + A^{(j)} \right) \delta_{iq}\delta_{jp} \right] \\ &\quad + \frac{2}{3} \Omega_0{}^2 \left[\frac{1}{2} \left(A^{(i)} + A^{(p)} \right) \delta_{ij}\delta_{pq} - \frac{1}{2} \left(A^{(i)} + A^{(j)} \right) \delta_{iq}\delta_{jp} \right. \\ &\qquad - A^{(i)} \delta_{ip}\delta_{jq} + 4A^{(i)} \frac{\kappa_j\kappa_q}{\kappa^2} \delta_{ip} + \left(A^{(j)} + A^{(q)} \right) \frac{\kappa_i\kappa_q}{\kappa^2} \delta_{jp} \\ &\qquad + \left(A^{(i)} - A^{(p)} \right) \frac{\kappa_j\kappa_p}{\kappa^2} \delta_{iq} + \left(A^{(i)} - A^{(j)} \right) \frac{\kappa_i\kappa_j}{\kappa^2} \delta_{pq} \\ &\qquad + 2A^{(r)} \frac{\kappa_j\kappa_p\kappa_r\kappa_r}{\kappa^4} \delta_{iq} + 2\left(A^{(p)} - A^{(q)} \right) \frac{\kappa_i\kappa_j\kappa_p\kappa_q}{\kappa^4} \\ &\qquad \left. - 4A^{(r)} \frac{\kappa_i\kappa_j\kappa_p\kappa_q\kappa_r\kappa_r}{\kappa^6} \right]. \end{aligned} \tag{E.16e}$$

Substituting into (E.14) these expressions and the isotropic form of the vorticity spectral tensor,

$$\Omega_{ij}(\boldsymbol{\kappa},0) = \frac{E(\kappa,0)}{4\pi\kappa^2}(\kappa^2\delta_{ij} - \kappa_i\kappa_j), \tag{E.17}$$

and using the identities for the weighted integrals for isotropic turbulence (D.2) $\sim$ (D.5), we finally obtain the expression (6.5).