APPENDICES FOR:

A strong-interaction theory for the motion of arbitrary three-dimensional clusters of spherical particles at low Reynolds number

by

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J. Fluid Mech (1988)

Appendix A Derivation of coordinate transformations (2.7), (2.8) and (2.9).

This appendix contains the coordinate transformations and transformations of the unit vectors needed to express the velocity disturbances of the j^{th} sphere in terms of a spherical coordinate system whose origin lies at the center of the k^{th} sphere.

We seek to relate the position of an arbitrary point in space relative to the spherical coordinates whose origin lies at the center of the j^{th} sphere (r_j, θ_j, ϕ_j) to the spherical coordinates originating from the center of the k^{th} sphere (r_k, θ_k, ϕ_k). The spherical coordinates (r_j, θ_j, ϕ_j) are first written in terms of a cartesian coordinate system (x_j, y_j, z_j) whose origin is also at the center of the j^{th} sphere as follows:

$$r_j = (x_j^2 + y_j^2 + z_j^2)^{1/2}$$

$$\theta_j = \tan^{-1} \left[\frac{(x_j^2 + y_j^2)^{1/2}}{z_j} \right] \qquad \text{(A-1)}$$

$$\phi_j = \tan^{-1} \left[\frac{y_j}{x_j} \right]$$

Then the cartesian coordinates (x_j, y_j, z_j) are related to a global cartesian coordinate system (X,Y,Z) via the relations

$$X = x_j + b_j$$
$$Y = y_j + c_j \qquad \text{(A-2)}$$
$$Z = z_j + d_j$$

where (b_j, c_j, d_j) is the location of the origin of the j^{th} sphere in the global system. In turn, the global coordinates (X,Y,Z) are related to a cartesian system (x_k, y_k, z_k) whose origin lies at the center of the k^{th} sphere yielding the expression

$$x_k = X - b_k$$
$$y_k = Y - c_k \qquad \text{(A-3)}$$
$$z_k = Z - d_k$$

where (b_k, c_k, d_k) is the origin of the k^{th} sphere in the global system (X,Y,Z). Then the cartesian coordinates (x_k, y_k, z_k) are related to a spherical coordinate system (r_k, θ_k, ϕ_k) having its origin at the center of the k^{th} sphere as follows:

$$x_k = r_k \sin\theta_k \cos\phi_k$$
$$y_k = r_k \sin\theta_k \sin\phi_k \qquad \text{(A-4)}$$
$$z_k = r_k \cos\theta_k$$

Finally combining (A-1) - (A-4) and simplifying yields (2.7) which is the desired result.

The unit vectors of a spherical coordinate system whose origin lies at the center of the j^{th} sphere $(\hat{e}_{r_j}, \hat{e}_{\theta_j}, \hat{e}_{\phi_j})$ are related to the unit vectors $(\hat{e}_{r_k}, \hat{e}_{\theta_k}, \hat{e}_{\phi_k})$ as follows. We first express the unit vectors $(\hat{e}_{r_j}, \hat{e}_{\theta_j}, \hat{e}_{\phi_j})$ in terms of cartesian unit vectors $(\hat{i}, \hat{j}, \hat{k})$ by the matrix equation:

$$\begin{bmatrix} \hat{e}_{r_j} \\ \hat{e}_{\theta_j} \\ \hat{e}_{\phi_j} \end{bmatrix} = \begin{bmatrix} \sin\theta_j \cos\phi_j & \sin\theta_j \sin\phi_j & \cos\theta_j \\ \cos\theta_j \cos\phi_j & \cos\theta_j \sin\phi_j & -\cos\theta_j \\ -\sin\phi_j & \cos\phi_j & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \qquad \text{(A-5)}$$

The subscript has been dropped from the cartesian unit vectors since their direction is independent of the origin of the coordinate system. Next, the cartesian unit vectors are related to the unit vectors of the spherical coordinate system whose origin lies at the k^{th} sphere via the relation:

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta_k \cos\phi_k & \cos\theta_k \cos\phi_k & -\sin\phi_k \\ \sin\theta_k \sin\phi_k & \cos\theta_k \sin\phi_k & \cos\phi_k \\ \cos\theta_k & -\sin\theta_k & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_{r_k} \\ \hat{e}_{\theta_k} \\ \hat{e}_{\phi_k} \end{bmatrix} \qquad \text{(A-6)}$$

Finally combining (A-5) and (A-6) and simplifying yields (2.8) and (2.9) which is the desired result.

Appendix B Explicit expressions of the primed coefficients in (2.11).

The coefficients of the unknown constants introduced in the collocation series and shown in (2.11) obtained in terms of the spherical coordinate system of the k^{th} sphere after using all the coordinate transformations of appendix A are:

For V_{r_k}:

$$A'_{jkmn} = r_j^{-(n+1)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk}\sin m\phi_j + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk}\cos m\phi_j\right] \quad \text{(B-1)}$$

$$B'_{jkmn} = r_j^{-(n+1)}\left[\sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{7jk}\sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{4jk}\cos m\phi_j\right] \quad \text{(B-2)}$$

$$C'_{jkmn} = r_j^{-(n+2)}\left[-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j\right.$$

$$\left. - \{(n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\} \cos m\phi_j\right] \quad \text{(B-3)}$$

$$D'_{jkmn} = r_j^{-(n+2)}\left[-\{(n+1) P_n^m(\xi_j) f_{1jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\}\sin m\phi_j\right.$$

$$\left. + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \cos m\phi_j\right] \quad \text{(B-4)}$$

$$E'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)}\left[m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{7jk} \sin m\phi_j\right.$$

$$+ \{(n+1) P_n^m(\xi_j) f_{1jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{4jk}\} \cos m\phi_j \quad \text{(B-5)}$$

$$F'_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [\{(n+1)\ P_n^m(\xi_j)\ f_{1jk} + \frac{(n-2)}{n}\ \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j}\ f_{4jk}\}\ \sin m\phi_j$$

$$- m\ \frac{(n-2)}{n}\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{7jk}\ \cos m\phi_j \qquad \text{(B-6)}$$

For V_{θ_k}:

$$A''_{jkmn} = r_j^{-(n+1)} [-m\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{5jk} \sin m\phi_j + \sin\theta_j\ \frac{\partial P_n^m(\xi_j)}{\partial\xi_j}\ f_{8jk} \cos m\phi_j] \qquad \text{(B-7)}$$

$$B''_{jkmn} = r_j^{-(n+1)} [\ \sin\theta_j\ \frac{\partial P_n^m(\xi_j)}{\partial\xi_j}\ f_{8jk} \sin m\phi_j + m\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{5jk} \cos m\phi_j] \qquad \text{(B-8)}$$

$$C''_{jkmn} = r_j^{-(n+2)} [\ -m\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{8jk}\ \sin m\phi_j$$

$$- \{(n+1)\ P_n^m(\xi_j)\ f_{2jk} + \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j}\ f_{5jk}\}\ \cos m\phi_j] \qquad \text{(B-9)}$$

$$D''_{jkmn} = r_j^{-(n+2)} [\ -\{(n+1)\ P_n^m(\xi_j)\ f_{2jk} + \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j}\ f_{5jk}\} \sin m\phi_j$$

$$+ \quad m\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{8jk}\ \cos m\phi_j] \qquad \text{(B-10)}$$

$$E''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [\ m\ \frac{(n-2)}{n}\ \frac{P_n^m(\xi_j)}{\sin\theta_j}\ f_{8jk}\ \sin m\phi_j$$

$$+ \{(n+1)\ P_n^m(\xi_j)\ f_{2jk} + \frac{(n-2)}{n}\ \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j}\ f_{5jk}\}\ \cos m\phi_j \qquad \text{(B-11)}$$

$$F''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [\{(n+1)\ P_n^m(\xi_j)\ f_{2jk} + \frac{(n-2)}{n}\ \sin\theta_j\ \frac{dP_n^m(\xi_j)}{d\xi_j}\ f_{5jk}\}\ \sin m\phi_j$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{8jk} \cos m\phi_j \qquad (B-12)$$

For V_{ϕ_k}:

$$A'''_{jkmn} = r_j^{-(n+1)} [-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \sin m\phi_j + \sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \cos m\phi_j] \qquad (B-13)$$

$$B'''_{jkmn} = r_j^{-(n+1)} [\sin\theta_j \frac{\partial P_n^m(\xi_j)}{\partial \xi_j} f_{9jk} \sin m\phi_j + m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{6jk} \cos m\phi_j] \qquad (B-14)$$

$$C'''_{jkmn} = r_j^{-(n+2)} [-m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j$$

$$- \{(n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}\} \cos m\phi_j] \qquad (B-15)$$

$$D'''_{jkmn} = r_j^{-(n+2)} [-\{(n+1) P_n^m(\xi_j) f_{3jk} + \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}\} \sin m\phi_j$$

$$+ \quad m \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j] \qquad (B-16)$$

$$E'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \sin m\phi_j$$

$$+ \{(n+1) P_n^m(\xi_j) f_{2jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}\} \cos m\phi_j \qquad (B-17)$$

$$F'''_{jkmn} = \frac{r_j^{-n}}{2\mu(2n-1)} [\{(n+1) P_n^m(\xi_j) f_{3jk} + \frac{(n-2)}{n} \sin\theta_j \frac{dP_n^m(\xi_j)}{d\xi_j} f_{6jk}\} \sin m\phi_j$$

$$- m \frac{(n-2)}{n} \frac{P_n^m(\xi_j)}{\sin\theta_j} f_{9jk} \cos m\phi_j \qquad (B-18)$$

where the functions f_{ijk}, i=1 to 9, j=1 to J, k=1 to J are given by (2.9) and the coordinates r_j, θ_j, ϕ_j can be written in terms of r_k, θ_k, ϕ_k using (2.7).

Appendix C Explicit expressions of the primed functions in (2.17).

The expressions for the Fourier coefficients in (2.17) for each of the velocity components are given below:

For the r_k component of velocity:

$$A'_0(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n} C'_{kk0n} + E_{kk0n} E'_{kk0n}) \tag{C-1}$$

$$A'_m(\theta_k) = \sum_{n=m}^{\infty} [(C_{kkmn} C'_{kkmn} + E_{kk0n} E'_{kk0n})]/\cos m\phi_k \tag{C-2}$$

$$B'_m(\theta_k) = \sum_{n=m}^{\infty} [(D_{kkmn} D'_{kkmn} + F_{kk0n} F'_{kk0n})]/\sin m\phi_k \tag{C-3}$$

and

$$\begin{aligned} F'(\phi_k) = {} & U_k f'_1(\phi_k) + V_k f'_2(\phi_k) + W_k f'_3(\phi_k) \\ & + (\Omega_x)_k f'_4(\phi_k) + (\Omega_y)_k f'_5(\phi_k) + (\Omega_z)_k f'_6(\phi_k) \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A'_{jkmn} A_{jkmn} + \ldots + F'_{jkmn} F_{jkmn}) \end{aligned} \tag{C-4}$$

where

$$f'_1(\phi_k) = \sin\theta_k \cos\phi_k \tag{C-5}$$

$$f'_2(\phi_k) = \sin\theta_k \sin\phi_k \tag{C-6}$$

$$f'_3(\phi_k) = \cos\theta_k \tag{C-7}$$

$$f'_4(\phi_k) = 0 \tag{C-8}$$

$$f'_5(\phi_k) = 0 \tag{C-9}$$

$$f'_6(\phi_k) = 0 \tag{C-10}$$

For the θ_k component of velocity:

$$A_0''(\theta_k) = \sum_{n=1}^{\infty} (C_{kk0n}\, C_{kk0n}'' + E_{kkon}\, E_{kkon}'') \tag{C-11}$$

$$A_m''(\theta_k) = \sum_{n=m}^{\infty} [(B_{kkmn} B_{kkmn}'' + C_{kkmn} C_{kkmn}'' + E_{kkon} E_{kkon}'')]/\cos m\phi_k \tag{C-12}$$

$$B_m''(\theta_k) = \sum_{n=m}^{\infty} [(A_{kkmn} A_{kkmn}'' + D_{kkmn} D_{kkmn}'' + F_{kkon} F_{kkon}'')]/\sin m\phi_k \tag{C-13}$$

and

$$\begin{aligned} F''(\phi_k) = {} & U_k\, f_1''(\phi_k) + V_k\, f_2''(\phi_k) + W_k\, f_3''(\phi_k) \\ & + (\Omega_x)_k\, f_4''(\phi_k) + (\Omega_y)_k\, f_5''(\phi_k) + (\Omega_z)_k\, f_6''(\phi_k) \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A_{jkmn}''\, A_{jkmn} + \ldots + F_{jkmn}''\, F_{jkmn}) \end{aligned} \tag{C-14}$$

where

$$f_1''(\phi_k) = \cos\theta_k \cos\phi_k \tag{C-15}$$

$$f_2''(\phi_k) = \cos\theta_k \sin\phi_k \tag{C-16}$$

$$f_3''(\phi_k) = -\sin\theta_k \tag{C-17}$$

$$f_4''(\phi_k) = -a_k \sin\phi_k \tag{C-18}$$

$$f_5''(\phi_k) = a_k \cos\phi_k \tag{C-19}$$

$$f_6''(\phi_k) = 0 \tag{C-20}$$

For the ϕ_k component of velocity:

$$A_0'''(\theta_k) = \sum_{n=1}^{\infty} (A_{kk0n}\ A_{kk0n}''') \tag{C-21}$$

$$A_m'''(\theta_k) = \sum_{n=m}^{\infty} [(A_{kkmn}A_{kkmn}''' + D_{kkmn}D_{kkmn}''' + F_{kkon}F_{kkon}''')]/\cos m\phi_k \tag{C-22}$$

$$B_m'''(\theta_k) = \sum_{n=m}^{\infty} [(B_{kkmn}B_{kkmn}''' + C_{kkmn}C_{kkmn}''' + E_{kkon}E_{kkon}''')]/\sin m\phi_k \tag{C-23}$$

and

$$\begin{aligned} F'''(\phi_k) = {} & U_k\ f_1'''(\phi_k) + V_k\ f_2'''(\phi_k) + W_k\ f_3'''(\phi_k) \\ & + (\Omega_x)_k\ f_4'''(\phi_k) + (\Omega_y)_k\ f_5'''(\phi_k) + (\Omega_z)_k\ f_6'''(\phi_k) \\ & - \sum_{\substack{j=1 \\ j\neq k}}^{J} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (A_{jkmn}'''\ A_{jkmn} + \ldots + F_{jkmn}'''\ F_{jkmn}) \end{aligned} \tag{C-24}$$

where

$$f_1'''(\phi_k) = -\sin\phi_k \tag{C-25}$$

$$f_2'''(\phi_k) = \cos\phi_k \tag{C-26}$$

$$f_3'''(\phi_k) = 0 \tag{C-27}$$

$$f_4'''(\phi_k) = -a_k\ \cos\theta_k\cos\phi_k \tag{C-28}$$

$$f_5'''(\phi_k) = -a_k\ \cos\theta_k\sin\phi_k \tag{C-29}$$

$$f_6'''(\phi_k) = a_k\ \sin\theta_k \tag{C-30}$$

Appendix D Collocation matrix equation for two spheres with two rings and two terms retained in the Fourier series.

$$
\left[\begin{array}{ccccccccc|cccccc|ccccccccc|cccccc}
\dot C_{1101} & \dot A_{1102} & \dot C_{1102} & \dot F_{1102} & \dot C_{1111} & \dot D_{1111} & \dot A_{1112} & \cdots & \dot F_{1112} & 0 & 0 & -\cos\phi_1 & 0 & 0 & 0 & \dot C_{2101} & \dot A_{2102} & \dot C_{2102} & \dot F_{2102} & \dot C_{2111} & \dot D_{2111} & \dot A_{2112} & \cdots & \dot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddot C_{1101} & \ddot A_{1102} & \ddot C_{1102} & \ddot F_{1102} & \ddot C_{1111} & \ddot D_{1111} & \ddot A_{1112} & \cdots & \ddot F_{1112} & 0 & 0 & \sin\phi_1 & 0 & 0 & 0 & \ddot C_{2101} & \ddot A_{2102} & \ddot C_{2102} & \ddot F_{2102} & \ddot C_{2111} & \ddot D_{2111} & \ddot A_{2112} & \cdots & \ddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dddot C_{1101} & \dddot A_{1102} & \dddot C_{1102} & \dddot F_{1102} & \dddot C_{1111} & \dddot D_{1111} & \dddot A_{1112} & \cdots & \dddot F_{1112} & 0 & 0 & 0 & 0 & 0 & -a_1\sin\phi_1 & \dddot C_{2101} & \dddot A_{2102} & \dddot C_{2102} & \dddot F_{2102} & \dddot C_{2111} & \dddot D_{2111} & \dddot A_{2112} & \cdots & \dddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot C_{1101} & \dot A_{1102} & \dot C_{1102} & \dot F_{1102} & \dot C_{1111} & \dot D_{1111} & \dot A_{1112} & \cdots & \dot F_{1112} & -\sin\phi_1 & 0 & 0 & 0 & 0 & 0 & \dot C_{2101} & \dot A_{2102} & \dot C_{2102} & \dot F_{2102} & \dot C_{2111} & \dot D_{2111} & \dot A_{2112} & \cdots & \dot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddot C_{1101} & \ddot A_{1102} & \ddot C_{1102} & \ddot F_{1102} & \ddot C_{1111} & \ddot D_{1111} & \ddot A_{1112} & \cdots & \ddot F_{1112} & -\cos\phi_1 & 0 & 0 & 0 & -a_1 & 0 & \ddot C_{2101} & \ddot A_{2102} & \ddot C_{2102} & \ddot F_{2102} & \ddot C_{2111} & \ddot D_{2111} & \ddot A_{2112} & \cdots & \ddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dddot C_{1101} & \dddot A_{1102} & \dddot C_{1102} & \dddot F_{1102} & \dddot C_{1111} & \dddot D_{1111} & \dddot A_{1112} & \cdots & \dddot F_{1112} & 0 & -1 & 0 & a_1\cos\phi_1 & 0 & 0 & \dddot C_{2101} & \dddot A_{2102} & \dddot C_{2102} & \dddot F_{2102} & \dddot C_{2111} & \dddot D_{2111} & \dddot A_{2112} & \cdots & \dddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot C_{1101} & \dot A_{1102} & \dot C_{1102} & \dot F_{1102} & \dot C_{1111} & \dot D_{1111} & \dot A_{1112} & \cdots & \dot F_{1112} & 0 & -\sin\phi_1 & 0 & 0 & 0 & 0 & \dot C_{2101} & \dot A_{2102} & \dot C_{2102} & \dot F_{2102} & \dot C_{2111} & \dot D_{2111} & \dot A_{2112} & \cdots & \dot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddot C_{1101} & \ddot A_{1102} & \ddot C_{1102} & \ddot F_{1102} & \ddot C_{1111} & \ddot D_{1111} & \ddot A_{1112} & \cdots & \ddot F_{1112} & 0 & -\cos\phi_1 & 0 & a_1 & 0 & 0 & \ddot C_{2101} & \ddot A_{2102} & \ddot C_{2102} & \ddot F_{2102} & \ddot C_{2111} & \ddot D_{2111} & \ddot A_{2112} & \cdots & \ddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dddot C_{1101} & \dddot A_{1102} & \dddot C_{1102} & \dddot F_{1102} & \dddot C_{1111} & \dddot D_{1111} & \dddot A_{1112} & \cdots & \dddot F_{1112} & 1 & 0 & 0 & 0 & a_1\cos\phi_1 & 0 & \dddot C_{2101} & \dddot A_{2102} & \dddot C_{2102} & \dddot F_{2102} & \dddot C_{2111} & \dddot D_{2111} & \dddot A_{2112} & \cdots & \dddot F_{2112} & 0 & 0 & 0 & 0 & 0 & 0 \\
\multicolumn{30}{c}{\vdots} \\
\multicolumn{30}{c}{\text{nine similar equations for sphere 1 and boundary ring at } \phi_k = \phi_2} \\
\multicolumn{30}{c}{\vdots} \\
\dot C_{1201} & \dot A_{1202} & \dot C_{1202} & \dot F_{1202} & \dot C_{1211} & \dot D_{1211} & \dot A_{1212} & \cdots & \dot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dot C_{2201} & \dot A_{2202} & \dot C_{2202} & \dot F_{2202} & \dot C_{2211} & \dot D_{2211} & \dot A_{2212} & \cdots & \dot F_{2212} & 0 & 0 & -\cos\phi_1 & 0 & 0 & 0 \\
\ddot C_{1201} & \ddot A_{1202} & \ddot C_{1202} & \ddot F_{1202} & \ddot C_{1211} & \ddot D_{1211} & \ddot A_{1212} & \cdots & \ddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \ddot C_{2201} & \ddot A_{2202} & \ddot C_{2202} & \ddot F_{2202} & \ddot C_{2211} & \ddot D_{2211} & \ddot A_{2212} & \cdots & \ddot F_{1112} & 0 & 0 & \sin\phi_1 & 0 & 0 & 0 \\
\dddot C_{1201} & \dddot A_{1202} & \dddot C_{1202} & \dddot F_{1202} & \dddot C_{1211} & \dddot D_{1211} & \dddot A_{1212} & \cdots & \dddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dddot C_{2201} & \dddot A_{2202} & \dddot C_{2202} & \dddot F_{2202} & \dddot C_{2211} & \dddot D_{2211} & \dddot A_{2212} & \cdots & \dddot F_{2212} & 0 & 0 & 0 & 0 & 0 & -a_2\sin\phi_1 \\
\dot C_{1201} & \dot A_{1202} & \dot C_{1202} & \dot F_{1202} & \dot C_{1211} & \dot D_{1211} & \dot A_{1212} & \cdots & \dot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dot C_{2201} & \dot A_{2202} & \dot C_{2202} & \dot F_{2202} & \dot C_{2211} & \dot D_{2211} & \dot A_{2212} & \cdots & \dot F_{2212} & -\sin\phi_1 & 0 & 0 & 0 & 0 & 0 \\
\ddot C_{1201} & \ddot A_{1202} & \ddot C_{1202} & \ddot F_{1202} & \ddot C_{1211} & \ddot D_{1211} & \ddot A_{1212} & \cdots & \ddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \ddot C_{2201} & \ddot A_{2202} & \ddot C_{2202} & \ddot F_{2202} & \ddot C_{2211} & \ddot D_{2211} & \ddot A_{2212} & \cdots & \ddot F_{1112} & -\cos\phi_1 & 0 & 0 & 0 & -a_2 & 0 \\
\dddot C_{1201} & \dddot A_{1202} & \dddot C_{1202} & \dddot F_{1202} & \dddot C_{1211} & \dddot D_{1211} & \dddot A_{1212} & \cdots & \dddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dddot C_{2201} & \dddot A_{2202} & \dddot C_{2202} & \dddot F_{2202} & \dddot C_{2211} & \dddot D_{2211} & \dddot A_{2212} & \cdots & \dddot F_{2212} & 0 & -1 & 0 & a_2\cos\phi_1 & 0 & 0 \\
\dot C_{1201} & \dot A_{1202} & \dot C_{1202} & \dot F_{1202} & \dot C_{1211} & \dot D_{1211} & \dot A_{1212} & \cdots & \dot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dot C_{2201} & \dot A_{2202} & \dot C_{2202} & \dot F_{2202} & \dot C_{2211} & \dot D_{2211} & \dot A_{2212} & \cdots & \dot F_{2212} & 0 & -\sin\phi_1 & 0 & 0 & 0 & 0 \\
\ddot C_{1201} & \ddot A_{1202} & \ddot C_{1202} & \ddot F_{1202} & \ddot C_{1211} & \ddot D_{1211} & \ddot A_{1212} & \cdots & \ddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \ddot C_{2201} & \ddot A_{2202} & \ddot C_{2202} & \ddot F_{2202} & \ddot C_{2211} & \ddot D_{2211} & \ddot A_{2212} & \cdots & \ddot F_{1112} & 0 & -\cos\phi_1 & 0 & a_2 & 0 & 0 \\
\dddot C_{1201} & \dddot A_{1202} & \dddot C_{1202} & \dddot F_{1202} & \dddot C_{1211} & \dddot D_{1211} & \dddot A_{1212} & \cdots & \dddot F_{1212} & 0 & 0 & 0 & 0 & 0 & 0 & \dddot C_{2201} & \dddot A_{2202} & \dddot C_{2202} & \dddot F_{2202} & \dddot C_{2211} & \dddot D_{2211} & \dddot A_{2212} & \cdots & \dddot F_{2212} & 1 & 0 & 0 & 0 & a_2\cos\phi_1 & 0 \\
\multicolumn{30}{c}{\vdots} \\
\multicolumn{30}{c}{\text{nine similar equations for sphere 2 and boundary ring at } \phi_k = \phi_2} \\
\multicolumn{30}{c}{\vdots}
\end{array}\right]
\bullet
\left[\begin{array}{c}
C_{101} \\ A_{102} \\ C_{102} \\ F_{102} \\ C_{111} \\ D_{111} \\ A_{112} \\ \cdot \\ \cdot \\ F_{112} \\ U_1 \\ V_1 \\ W_1 \\ (\Omega_x)_1 \\ (\Omega_y)_1 \\ (\Omega_z)_1 \\ C_{201} \\ A_{202} \\ C_{202} \\ \cdot \\ \cdot \\ \cdot \\ U_2 \\ V_2 \\ W_2 \\ (\Omega_x)_2 \\ (\Omega_y)_2 \\ (\Omega_z)_2
\end{array}\right]
=
\left[\begin{array}{c}
a_1^3(\rho_s-\rho)/3\cdot(\dot E_{1101}+\dot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\ddot E_{1101}+\ddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dddot E_{1101}+\dddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dot E_{1101}+\dot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\ddot E_{1101}+\ddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dddot E_{1101}+\dddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dot E_{1101}+\dot E_{2101}) \\
\cdot \\ \cdot \\
a_1^3(\rho_s-\rho)/3\cdot(\dddot E_{1101}+\dddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dot E_{1101}+\dot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\ddot E_{1101}+\ddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dddot E_{1101}+\dddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dot E_{1101}+\dot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\ddot E_{1101}+\ddot E_{2101}) \\
a_1^3(\rho_s-\rho)/3\cdot(\dddot E_{1101}+\dddot E_{1201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\dot E_{1201}+\dot E_{2201}) \\
a_1^3(\rho_s-\rho)/3\cdot(\ddot E_{1201}+\ddot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\dddot E_{1201}+\dddot E_{2201}) \\
\cdot \\ \cdot \\ \cdot \\
a_2^3(\rho_s-\rho)/3\cdot(\dot E_{1201}+\dot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\ddot E_{1201}+\ddot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\dddot E_{1201}+\dddot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\dot E_{1201}+\dot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\ddot E_{1201}+\ddot E_{2201}) \\
a_2^3(\rho_s-\rho)/3\cdot(\dddot E_{1201}+\dddot E_{2201})
\end{array}\right]
$$

Appendix E Symmetry classification of the unknown constants introduced in (2.4).

The unknown constants in the collocation series for two symmetric spheres are equal in magnitude and equal or opposite in sign depending on the type of symmetry between them. They are listed as follows:

Symmetry about the Y Plane:

Equal: B_{jkmn}, C_{jkmn}, E_{jkmn}, U_j, W_j & $(\Omega_y)_j$.

Opp.: A_{jkmn}, D_{jkmn}, F_{jkmn}, V_j, $(\Omega_x)_j$ & $(\Omega_z)_j$.

Symmetry about the X Plane:

Equal: B_{jkmn}, C_{jkmn} & E_{jkmn} for m=even;

A_{jkmn}, D_{jkmn} & F_{jkmn} for m=odd; V_j, W_j & $(\Omega_x)_j$.

Opp.: B_{jkmn}, C_{jkmn} & E_{jkmn} for m=odd;

A_{jkmn}, D_{jkmn} & F_{jkmn} for m=even; U_j, $(\Omega_y)_j$ & $(\Omega_z)_j$.

Symmetry about the Z Plane:

Equal: A_{jkmn} & B_{jkmn} for (m+n)=odd; U_j, V_j & $(\Omega_z)_j$.

C_{jkmn}, D_{jkmn}, E_{jkmn} & F_{jkmn} for (m+n)=even.

Opp.: A_{jkmn} & B_{jkmn} for (m+n)=even; W_j, $(\Omega_x)_j$ & $(\Omega_y)_j$.

C_{jkmn}, D_{jkmn}, E_{jkmn} & F_{jkmn} for (m+n)=odd.

Anti-symmetry: In case of anti-symmetry the existing relations are reversed so the equal constants become opposite and vice-versa. For instance when two spheres are settling under gravity we have $F_{z1} = F_{z2}$ (here F_z= gravitational force) and we reverse the symmetry conditions about the Z plane i.e. for symmetry about the Z plane A_{jkmn} & B_{jkmn} for (m+n)=odd are opposite; U_j, V_j & $(\Omega_z)_j$ are opposite; W_j, $(\Omega_x)_j$ & $(\Omega_y)_j$ are equal and so on.

Spheres in plane: When the spheres lie in any of the symmetry planes then the opposite constants for that plane of symmetry become zero.