Appendix B. Kernels of the evolution equations (3.25) – (3.27)

When choosing the normalization of the kernels and the form of their representation, we are guided by the following considerations. Firstly, when $\alpha = 1/2$ the kernels $G_\nu(\sigma)$ and $G(x, y)$ must to a maximum extent coincide with those reported in Paper I. Secondly, the kernels must remain of order unity in the limit of weak stratification ($\alpha \to 0$). Thirdly, in cubic nonlinear terms a dependence on σ when $\sigma \to 0$ is singled out, so that the kernels here have finite and nonzero limits. And fourthlt, for the purposes of greater illustration and to ease the reproduction of results, the kernels are represented as the sum of contributions forming the appropriate nonlinear term.

B.1. *Dissipative cubic nonlinearity.*

The kernel $G_\nu(\sigma)$ involved in NEE (3.25) is defined by

$$\sigma G_\nu(\sigma) = G_0{}^{(\nu)}(\sigma) + G_2{}^{(\nu)}(\sigma), \tag{B 1}$$

where

$$\begin{aligned}
G_0{}^{(\nu)} = &-\frac{2}{\Gamma(4-2\alpha)}\left\{\frac{3}{4}(2-\alpha)\sigma\int_0^1 \mathrm{d}u u^{1-\alpha}\right.\\
&\times\left[8(3-2\alpha)(\sigma+u)^{1-\alpha}F_1\left(2-\alpha,\alpha-2,\alpha-1,2-2\alpha;\sigma,\frac{\sigma u}{\sigma+u}\right)\right.\\
&\left.-2\alpha\sigma(1+\sigma u)^{1-\alpha}F_1\left(2-\alpha,\alpha+1,\alpha-1,5-2\alpha;\sigma,\frac{\sigma u}{1+\sigma u}\right)\right]\\
&+(1+\sigma)^{-1-\alpha}\left[3\alpha\sigma(1+\sigma)^2(1+\sigma^2)F_1\left(2-\alpha,\alpha,\alpha-1,4-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right)\right.\\
&+3\alpha\sigma^3(1-\sigma^2)F_1\left(2-\alpha,\alpha,\alpha,4-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right)+\sigma\{\alpha(1-3\sigma+9\sigma^2-6\sigma^3+\sigma^4)\\
&-2(1-2\alpha)\sigma(2-3\sigma+2\sigma^2)\}F_1\left(2-\alpha,\alpha,\alpha+1,4-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right)\\
&+\frac{3-2\alpha}{1-\alpha}\{\alpha(1-\sigma)(1-4\sigma+6\sigma^2+\sigma^3)-2(1-2\alpha)\sigma(2-3\sigma+2\sigma^2)\}\\
&\left.\left.\times F_1\left(2-\alpha,\alpha,\alpha+1,3-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right)\right]\right\},
\end{aligned}$$

$$\begin{aligned}
G_2{}^{(\nu)} = &\frac{\alpha(1+\sigma)(1-\sigma)^{1-2\alpha}}{2[\Gamma(2-\alpha)]^2}\int_0^1 \mathrm{d}t\frac{t^{-\alpha}(1-t)^{1-\alpha}(1-\sigma t)^{3\alpha-1}}{(1+\sigma t)^{\alpha+1}(1-\sigma^2 t)^{\alpha+1}}\left\{\left[2\left(\frac{1-\sigma^2 t}{1-\sigma t}\right)^4\right.\right.\\
&\left.+(1-\sigma)^4\left(\frac{1+\sigma t}{1-\sigma t}\right)^4+2\sigma^3\left(\frac{1-t}{1-\sigma t}\right)^3\frac{2-\sigma+\sigma t-2\sigma^2 t}{1-\sigma t}\right]F\left(-\alpha,-\frac{\alpha}{2};1-\frac{\alpha}{2};\sigma^2 t^2\right)\\
&\left.+\frac{(1-\sigma)^4}{(1-\sigma t)^{2\alpha}}(1+\sigma t)^{3\alpha}F_1\left(-\alpha,\alpha+1,2-3\alpha,1-\alpha;\frac{\sigma t}{1+\sigma t},\frac{2\sigma t}{1+\sigma t}\right)\right\}
\end{aligned}$$

are the contributions from the interaction of the fundamental with the zeroth and second harmonics, respectively. Here

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{z^n}{n!},$$

$$F_1(a,b,c,d;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\Gamma(a+m+n)\Gamma(b+m)\Gamma(c+n)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d+m+n)} \frac{x^m y^n}{m!n!}$$

are hypergeometric functions (Erdelelyi 1953).

When $\alpha \to 0$

$$G_0{}^{(\nu)} = 2\sigma^3(2-\sigma) + O(\alpha), \qquad G_2{}^{(\nu)} = O(\alpha),$$

so that

$$G_\nu = 2\sigma^2(2-\sigma) + O(\alpha) \tag{B 2}$$

and (3.25) becomes NEE obtained by Shukhman & Churilov (1997, equation (3.8)) for the case of a weak stratification.

If α is not small, then when $\sigma \to 0$ the components

$$G_0{}^{(\nu)} = -\frac{\alpha}{(1-\alpha)^2\Gamma(2-2\alpha)} - \frac{4\alpha\sigma}{\Gamma(4-2\alpha)} - \frac{2\alpha\sigma^2}{(1-\alpha)\Gamma(4-2\alpha)} + O(\sigma^3),$$

$$G_2{}^{(\nu)} = \frac{\alpha}{(1-\alpha)^2\Gamma(2-2\alpha)} - \frac{4\alpha\sigma}{\Gamma(4-2\alpha)} + \frac{2\alpha(4-3\alpha)\sigma^2}{(1-\alpha)\Gamma(4-2\alpha)} + O(\alpha\sigma^3)$$

have nonzero limits, but in order for the limit of the complete kernel $G_\nu(\sigma)$ to be nonzero, in (B1) from the sum of the components, unlike Paper I, the factor σ is singled out, so that

$$G_\nu(\sigma) = \frac{-8\alpha + 6\alpha\sigma}{\Gamma(4-2\alpha)} + O(\sigma^2).$$

Otherwise there is a complete correspondence between the results: when $\alpha = 1/2$ the components $G_0{}^{(\nu)}(\sigma)$ and $G_2{}^{(\nu)}(\sigma)$ are precisely the same as the kernel components G_0 and G_2 obtained in Paper I (formulas (3.16)).

In the limit $\sigma \to 1$, the components

$$G_0{}^{(\nu)} = -\frac{2\Gamma(2\alpha+1)(1-\sigma)^{1-2\alpha}}{(1-\alpha)\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{2(2-5\alpha)}{(1-\alpha)\Gamma(3-2\alpha)}$$
$$+ \frac{12\alpha\Gamma(2-2\alpha)}{\Gamma(2-\alpha)\Gamma(4-3\alpha)}\, {}_3F_2(2-\alpha, 2-2\alpha, \alpha-1; 3-\alpha, 4-3\alpha; -1) + O\left[(1-\sigma)^{2-2\alpha}\right],$$

$$G_2{}^{(\nu)} = \frac{2\Gamma(2\alpha+1)(1-\sigma)^{1-2\alpha}}{(1-\alpha)\Gamma(\alpha+1)\Gamma(2-\alpha)} + O\left[(1-\sigma)^{1-\alpha}\right]$$

are finite when $\alpha \le 1/2$ and are singular when $\alpha > 1/2$, but

$$G_\nu = \frac{2(2-5\alpha)}{(1-\alpha)\Gamma(3-2\alpha)} + \frac{12\alpha\Gamma(2-2\alpha)}{\Gamma(2-\alpha)\Gamma(4-3\alpha)}\, {}_3F_2(2-\alpha, 2-2\alpha, \alpha-1; 3-\alpha, 4-3\alpha; -1)$$
$$+ O\left[(1-\sigma)^{1-\alpha}\right]$$

remains finite at any $\alpha \in [0,1)$. Here

$$ {}_3F_2(a,b,b';c,c';x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(b'+n)\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c+n)\Gamma(c'+n)} \frac{x^n}{n!} $$

is a generalized hypergeometric function (Erdelyi 1953).

B.2. *Nondissipative cubic nonlinearity.*

Results presented in this paragraph are valid when $0 \le \alpha < 1/2$ (recall that the nonlinearity itself plays an important role in the development of an instability when $\alpha \le 1/4$). The kernel of the nonlinear term of NEE (3.26) is defined in a manner like (B1):

$$ \sigma G_Q(\sigma) = G_0{}^{(Q)}(\sigma) + G_2{}^{(Q)}(\sigma), \tag{B 3} $$

where

$$ \begin{aligned} G_0{}^{(Q)} = & \frac{(1+\sigma)^{1-\alpha}}{(1-\alpha)^2} \left[2(1-2\alpha-2\alpha^2)F_1\left(1-\alpha,\alpha,\alpha,2-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right) \right. \\ & - 2(1+\alpha)(1-2\alpha)F_1\left(1-\alpha,\alpha,\alpha-1,2-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right) \\ & \left. + \alpha(1+\alpha)F_1\left(1-\alpha,\alpha,\alpha+1,2-2\alpha;\sigma,\frac{\sigma}{1+\sigma}\right)\right], \\ G_2{}^{(Q)} = & \frac{\alpha\Gamma(2-2\alpha)(1-\sigma)^{1-2\alpha}}{2[\Gamma(2-\alpha)]^2} \int_0^1 \frac{\mathrm{d}t\, t^{-\alpha}(1-t)^{-\alpha}}{(1-\sigma^2 t)^{\alpha+1}(1-\sigma^2t^2)^{1-\alpha}} \left\{ -\sigma(1+\sigma)(1-t) \right. \\ & \times [(1-\alpha)F_1(-\alpha,-3\alpha,\alpha,1-\alpha;\sigma t,-\sigma t) + 2\alpha F_1(-\alpha,2-3\alpha,\alpha-2,1-\alpha;\sigma t,-\sigma t)] \\ & + \frac{(1-\sigma t)^{2\alpha+1}}{(1+\sigma t)^{2\alpha}} \left[2(1-\alpha)(1+\sigma)^2 - 3(1-3\alpha)\sigma(1+\sigma)\frac{1-t}{1-\sigma t} \right. \\ & \left.\left. - 8\alpha\sigma^2\left(\frac{1-t}{1-\sigma t}\right)^2 \right] F\left(-\alpha,-\frac{\alpha}{2};1-\frac{\alpha}{2};\sigma^2t^2\right) \right\}. \end{aligned} $$

The kernel $G_Q(\sigma)$ is positive at any $\alpha \in [0,\frac{1}{2})$. In the limit of a weak stratification

$$ G_0{}^{(Q)} = \sigma + O(\alpha), \qquad G_2{}^{(Q)} = O(\alpha), $$

so that

$$ G_Q(\sigma) = 1 + O(\alpha). $$

If α is not small, then when $\sigma \to 0$ we get

$$ \begin{aligned} G_0{}^{(Q)} &= -\frac{\alpha}{1-\alpha}\left[1 - \frac{2-3\alpha-\alpha^2}{2\alpha(1-\alpha)}\sigma - \frac{2+\alpha}{3-2\alpha}\sigma^2\right] + O(\sigma^3), \\ G_2{}^{(Q)} &= \frac{\alpha}{1-\alpha}\left[1 - \frac{1-3\alpha}{2(1-\alpha)}\sigma - \frac{2+\alpha}{3-2\alpha}\sigma^2\right] + O(\alpha\sigma^3), \\ G_Q(\sigma) &= 1 + O(\sigma^2), \end{aligned} $$

and in the limit $\sigma \to 1$ we have

$$G_0{}^{(Q)} = \frac{(1+\alpha)(1-2\alpha)}{(1-\alpha)^2} + \frac{4\alpha^2\Gamma(\frac{3}{2}-\alpha)\Gamma(1-2\alpha)}{(1-\alpha)\Gamma(2-\alpha)\Gamma(\frac{3}{2}-2\alpha)} - \frac{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})\Gamma(1-2\alpha)}{\pi^{1/2}\Gamma(2-\alpha)}(1-\sigma)^{1-2\alpha} + O(1-\sigma),$$

$$G_2{}^{(Q)} = \frac{4\alpha^2(1-2\alpha)}{(1-\alpha)^2(1-3\alpha)} - \frac{4\alpha^2\Gamma(\frac{3}{2}-\alpha)\Gamma(1-2\alpha)}{(1-\alpha)\Gamma(2-\alpha)\Gamma(\frac{3}{2}-2\alpha)} - \frac{\Gamma(3-2\alpha)\Gamma(\frac{3\alpha+1}{2})}{2^{3-3\alpha}(1-3\alpha)\Gamma(\frac{3-\alpha}{2})}(1-\sigma)^{1-3\alpha} + \frac{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})\Gamma(1-2\alpha)}{\pi^{1/2}\Gamma(2-\alpha)}(1-\sigma)^{1-2\alpha} + O\left[1-\sigma+(1-\sigma)^{2-3\alpha}\right],$$

$$G_Q = \frac{1-2\alpha}{1-3\alpha} - \frac{\Gamma(3-2\alpha)\Gamma(\frac{3\alpha+1}{2})}{2^{3-3\alpha}(1-3\alpha)\Gamma(\frac{3-\alpha}{2})}(1-\sigma)^{1-3\alpha} + O\left[1-\sigma+(1-\sigma)^{2-3\alpha}\right].$$

B.3. *Quintic nonlinearity.*

In accordance with (3.12), we represent the kernel $G(x,y)$ of equation (3.13) as the sum of the contributions

$$G(x,y) = G_{02}(x,y) + G_{11}(x,y) + G_{31}(x,y) \tag{B 4}$$

from the interaction of the zeroth and the second harmonic and the fundamental with the fundamental and the third harmonic. Here

$$G_{31} = \frac{2\alpha}{\alpha+1}\frac{x^{2-\alpha}y^2}{(1+x+xy)^{2\alpha}}\int_0^1 \frac{\mathrm{d}\sigma\sigma^{\alpha+1}(1-\sigma)^{-\alpha}}{(1-\sigma y)^\alpha}\left(\frac{1+x+xy-3\sigma xy}{1-\sigma xy}\right)^\alpha f_2\left\{\frac{1+x-2xy}{1+x-2\sigma xy}\right.$$
$$\times\left(\frac{1-x}{1+x-2\sigma xy}\right)^{1-\alpha}\left(\frac{1-\sigma xy}{1+x-2\sigma xy}\right)^{2\alpha}\left[3\alpha\frac{(1+x-2\sigma xy)(1+x-xy-\sigma xy)}{1+x+xy-3\sigma xy}\right.$$
$$\left.-(1+2\alpha)(1+x)+2(1+\alpha)xy+2\alpha\sigma xy\right] f_0\left(\frac{x(1-\sigma y)}{1-\sigma xy}\right) + \frac{2\alpha^2(1-\sigma)^2}{(2-\alpha)(1-\sigma y)}xy^2$$
$$\times\left(\frac{1-xy}{1-\sigma xy}\right)^{1-\alpha}\left(\frac{1-\sigma xy}{1+xy-2\sigma xy}\right)^\alpha\frac{(1+xy-2x)(1+xy-x-\sigma xy)}{(1-\sigma xy)(1+xy-2\sigma xy)}f_3\left(\frac{xy(1-\sigma)}{1-\sigma xy}\right)$$
$$+\alpha\left(\frac{1-y}{1+y-2\sigma y}\right)^{1-\alpha}\left(\frac{1-\sigma y}{1+y-2\sigma y}\right)^{2\alpha}\frac{1-x-xy+\sigma xy}{1-\sigma xy}(2-x-xy)f_0\left(\frac{y(1-\sigma)}{1-\sigma y}\right)$$
$$+\alpha y\frac{(1-\sigma)(1-xy)(1-2x+xy)(3-x-2\sigma xy)}{(1-\sigma y)(1+x-2\sigma xy)(1+xy-2\sigma xy)} - \frac{\alpha(1-x)(1+x-2xy)}{(1+x-2\sigma xy)(1+xy-2\sigma xy)}$$
$$\left.\times\left[x(1-y)\frac{1-xy}{1-\sigma xy}+2(1-x)\right]\right\} - 4\alpha[x(1-x)]^{1-\alpha}y(1+x+xy)\int_0^1\frac{\mathrm{d}\sigma\sigma^\alpha(1-\sigma)^{-\alpha}}{(1-\sigma y)^\alpha}$$
$$\times\left(\frac{1-\sigma xy}{1+x-2\sigma xy}\right)^\alpha\frac{(1+x-2xy)(1+x-xy-\sigma xy)(1+x+xy-2\sigma xy)}{(1+x-2\sigma xy)(1+x+xy-3\sigma xy)(1+x+xy-\sigma xy)^{\alpha+1}}$$
$$\times f_0\left(\frac{x(1-\sigma y)}{1-\sigma xy}\right),$$

$$
\begin{aligned}
G_{11} = {} & \frac{2\alpha(1-\alpha)}{\alpha+1} x^{1-\alpha} y \int_0^1 \frac{\mathrm{d}\sigma \sigma^{\alpha+1}(1+\sigma)^{\alpha-1}(1-\sigma y)^{-\alpha}}{(1-\sigma xy)^\alpha (1+x+xy-\sigma xy)^\alpha} \Big[(1+xy)^2 \\
& + \frac{x^3 y(1+y)^2(1+\sigma)}{1+x+xy-\sigma xy} - \frac{1-x}{1-\sigma xy}(1+xy)(1+x+2xy) \Big] F\left(\alpha, \frac{\alpha+1}{2}; \frac{\alpha+3}{2}; \sigma^2\right) \\
& - 2x^{1-\alpha} y \int_0^1 \frac{\mathrm{d}\sigma \sigma^\alpha (1-\sigma)^{-\alpha}(1-\sigma y)^{-\alpha}}{(1-\sigma xy)^\alpha (1+x+xy-\sigma xy)^\alpha} \Big[x^2(1+y)^2 \left(2\alpha - 1 + \frac{\alpha}{1+\sigma y}\right) \\
& + \alpha \frac{2+\sigma xy}{1+\sigma xy}(1+xy)^2 \Big] + \frac{4\alpha^2}{\alpha+1} x^{2-\alpha} y^2 \int_0^1 \frac{\mathrm{d}\sigma \sigma^{\alpha+1}(1-\sigma)^{-\alpha}(1-\sigma y)^{-\alpha}}{(1-\sigma xy)^\alpha (1+x+xy-\sigma xy)^\alpha} \\
& \times \Bigg\{ (1+x+2xy) \Bigg[1 - 2\left(\frac{1-x}{1+x-2\sigma xy}\right)^{1-\alpha} \left(\frac{1-\sigma xy}{1+x-2\sigma xy}\right)^{2\alpha} \frac{1+x+\frac{xy}{2}-\frac{3}{2}\sigma xy}{1+x+xy-\sigma xy} \\
& \times f_0\left(\frac{x(1-\sigma y)}{1-\sigma xy}\right) \Bigg] f_1(\sigma) + \Bigg[\frac{(1-xy)(1+x)}{(1-\sigma xy)(1+\sigma y)}(2+y+\sigma y) - \frac{(1+x)(1+y)}{1+\sigma y} \\
& - \frac{x^2(1+y)^2}{1+x+xy-\sigma xy} + 1 + 2x + xy - 2\left(\frac{1-xy}{1+xy-2\sigma xy}\right)^{1-\alpha} \left(\frac{1-\sigma xy}{1+xy-2\sigma xy}\right)^{2\alpha} \\
& \times \frac{1+\frac{x}{2}+xy-\frac{3}{2}\sigma xy}{1+x+xy-\sigma xy}(1+2x+xy) f_0\left(\frac{xy(1-\sigma)}{1-\sigma xy}\right) \Bigg] f_1(\sigma y) + \Bigg[\frac{(1+x)(1+xy)}{1+\sigma xy} \\
& - \frac{y(1-\sigma)(2+x+xy)}{(1-\sigma y)(1+\sigma xy)}(1+x) - \frac{(1+xy)^2}{1+x+xy-\sigma xy} + 2 + x + xy - \left(\frac{1-\sigma y}{1+y-2\sigma y}\right)^{2\alpha} \\
& \times \left(\frac{1-y}{1+y-2\sigma y}\right)^{1-\alpha} \frac{1+2x+2xy-3\sigma xy}{1+x+xy-\sigma xy}(2+x+xy) f_0\left(\frac{y(1-\sigma)}{1-\sigma y}\right) \Bigg] f_1(\sigma xy) \Bigg\} \\
& - \frac{2\alpha\Gamma(1-\alpha)\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{1-\alpha}{2})} x^{2-\alpha}(1-y)^{1-\alpha} \int_0^1 \mathrm{d}\sigma (1-\sigma)^{-\alpha} \left[\frac{y}{2y+(1-y)\sigma}\right]^{1-\alpha} \\
& \times [1+x-x(1-y)\sigma]^{-\alpha} [1-xy-x(1-y)\sigma]^{-\alpha} \Bigg\{ \frac{(1-x)(1+xy)(1+3y)}{1-x+x(1-y)(1-\sigma)} \\
& - \frac{(1-\sigma)(1-x^2y^2)(1-y)}{(1-\sigma)(1-xy)+\sigma(1-x)} - \frac{x^2(1+y)^2[2y+(1-y)\sigma]}{1+x-x(1-y)\sigma} + 2xy(1+y) \\
& + (1-y)(1+x+2xy)\sigma - 2(1+x+2xy) \left[\frac{1-x}{1-x+2x(1-y)(1-\sigma)}\right]^{1-\alpha} \\
& \times \left[\frac{1-xy-x(1-y)\sigma}{1-xy-x(1-y)\sigma+x(1-y)(1-\sigma)}\right]^{2\alpha} \frac{2y+(1-y)\sigma}{1+x-x(1-y)\sigma} \\
& \times \left[1+x-xy-\frac{3}{2}x(1-y)\sigma\right] f_0\left(\frac{x(1-y)(1-\sigma)}{1-x+x(1-y)(1-\sigma)}\right) \Bigg\},
\end{aligned}
$$

$$G_{02} = 4\alpha x^{1-\alpha} y \int_0^1 \frac{\mathrm{d}\sigma \sigma^\alpha (1-\sigma)^{-\alpha}(1-\sigma y)^{-\alpha}}{(1-\sigma xy)^\alpha (1+x+xy-\sigma xy)^\alpha}$$

$$\times \left\{ (1+x)(1+x-\sigma xy) \left(\frac{1-x}{1+x-2\sigma xy} \right)^{1-\alpha} \left(\frac{1-\sigma xy}{1+x-2\sigma xy} \right)^{2\alpha} f_0\left(\frac{x(1-\sigma y)}{1-\sigma xy} \right) \right.$$

$$+ (1+xy)(1+xy-\sigma xy) \left(\frac{1-xy}{1+xy-2\sigma xy} \right)^{1-\alpha} \left(\frac{1-\sigma xy}{1+xy-2\sigma xy} \right)^{2\alpha} f_0\left(\frac{xy(1-\sigma)}{1-\sigma xy} \right)$$

$$\left. + x^2(1+y)(1+y-\sigma y) \left(\frac{1-y}{1+y-2\sigma y} \right)^{1-\alpha} \left(\frac{1-\sigma y}{1+y-2\sigma y} \right)^{2\alpha} f_0\left(\frac{y(1-\sigma)}{1-\sigma y} \right) \right\};$$

$$f_0(t) = F\left(-\alpha, -\frac{\alpha}{2}; 1-\frac{\alpha}{2}; t^2\right);$$

$$f_1(t) = F\left(1, \frac{1-\alpha}{2}; \frac{\alpha+3}{2}; t^2\right);$$

$$f_2 \quad = F_1(\alpha+1, \alpha, \alpha+1, \alpha+2; z, 3z) + F_1(\alpha+1, \alpha+1, \alpha, \alpha+2; z, 3z), \quad z = \frac{\sigma xy}{1+x+xy};$$

$$f_3(t) = F\left(1-\alpha, 1-\frac{\alpha}{2}; 2-\frac{\alpha}{2}; t^2\right).$$

When $\alpha = 1/2$ these formulas coincide with those written in Appendix B of Paper I.

In the limit of a weak stratification

$$G_{11} = 2x^3 y(1+y)^2 + O(\alpha), \quad G_{02} = O(\alpha), \quad G_{31} = O(\alpha);$$

$$G(x,y) = 2x^3 y(1+y)^2 + O(\alpha).$$