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APPENDIX B

The kernel $G(x,y)$

Each formula (3.24) - (3.26) contains the product of the amplitude functions in the form $B(\tau-z_1)B(\tau-z_2)B(\tau-z_3)B(\tau-z_4)$ and it is easy to see that $z_4=z_1+z_2+z_3$. Let us use for the integrals of (3.24) - (3.26) the new variables:

$x_1=t+t_1,\ x_2=t+t_2,\ x_3=t_4-t-t_1-t_2,\ t$ in (3.24) and (3.25),

$x_1=t+t_3+t_1,\ x_2=t+t_3+t_2,\ x_3=t_4-t-2t_3-t_1-t_2,\ t,\ t_3$ in (3.26)

then, we integrate over t and over t and t_3, respectively. The integrals of (3.24) - (3.26) will take the form (S = 11, 02, 31):

$$(M^+-M^-)_S=\int_0^\infty dx_3\int_0^\infty dx_1\int_0^\infty dx_2\, g_S(x_1,x_2,x_3)B(\tau-x_1)B(\tau-x_2)B(\tau-x_3)\bar B(\tau-x_1-x_2-x_3)=$$

$$=\int_0^\infty dx_1\int_0^{x_1}dx_2\int_0^{x_2}dx_3\,[g_S(x_1,x_2,x_3)+g_S(x_1,x_3,x_2)+g_S(x_2,x_3,x_1)]\times$$

$$\times B(\tau-x_1)B(\tau-x_2)B(\tau-x_3)\bar B(\tau-x_1-x_2-x_3).$$

Using new variables $x_1=t$, $x_2=xt$, and $x_3=xyt$, after due manipulations, we obtain

$$(M^+-M^-)_S=\left(\frac{8k^9}{\pi^3}\right)^{1/2}e^{i\pi/4}\int_0^\infty dt\cdot t^{7/2}\int_0^1 dx\cdot x\int_0^1 dy\,G_S(x,y)B(\tau-t)\times \tag{B1}$$

$$\times B(\tau-xt)B(\tau-xyt)\bar B(\tau-(1+x+xy)t)$$

where

$$G_{11}=\frac{1}{3}x^{1/2}y\int_0^1 d\sigma\,\sigma^{3/2}(1+\sigma)^{-1/2}(1-\sigma xy)^{-1/2}(1+x+xy-\sigma xy)^{-3/2}\Big\{\Big[(1+xy)^2-$$

$$-\frac{(1-x)(1+xy)}{1-\sigma xy}(1+x+2xy)\Big](1+x+xy-\sigma xy)+x^3y(1+y)^2(1+\sigma)\Big\}f_1(\sigma)-$$

$$-x^{1/2}y\int_0^1 d\sigma\,\sigma^{1/2}(1-\sigma)^{-1/2}(1-\sigma y)^{-1/2}(1-\sigma xy)^{-1/2}(1+x+xy-\sigma xy)^{-1/2}\Big\{\frac{x^2(1+y)^2}{1+\sigma y}+\frac{2+\sigma xy}{1+\sigma xy}(1+xy)^2\Big\}+$$

$$+\frac{2}{3}x^{3/2}y^2\int_0^1 d\sigma\cdot\sigma^{3/2}(1-\sigma)^{-1/2}(1-\sigma y)^{-1/2}(1-\sigma xy)^{-1/2}(1+x+xy-\sigma xy)^{-3/2}\Big\{\Big[(1+x+2xy)(1+x+xy-\sigma xy)-$$

$$-2(1+x+2xy)\left(\frac{1-x}{1+x-2\sigma xy}\right)^{1/2}\frac{1-\sigma xy}{1+x-2\sigma xy}\left(1+x+\frac{1}{2}xy-\frac{3}{2}\sigma xy\right)f_2\left(\frac{x(1-\sigma y)}{1-\sigma xy}\right)\Big]f_3(\sigma)+$$

$$+\Big[(1+x+xy-\sigma xy)\frac{1+x}{1+\sigma y}\Big((2+y+\sigma y)\frac{1-xy}{1-\sigma xy}-(1+y)\Big)+(1+2x+xy)(1+x+xy-\sigma xy)-$$

$$-x^2(1+y)^2-2(1+2x+xy)\left(\frac{1-xy}{1+xy-2\sigma xy}\right)^{1/2}\frac{1-\sigma xy}{1+xy-2\sigma xy}\left(1+xy+\frac{x}{2}-\frac{3}{2}\sigma xy\right)f_2\left(\frac{xy(1-\sigma)}{1-\sigma xy}\right)\Big]f_3(\sigma y)+$$

$$+\Big[(1+x+xy-\sigma xy)\Big((1+x)\frac{1+xy}{1+\sigma xy}-(1+x)y\frac{2+x+xy}{1+\sigma xy}\cdot\frac{1-\sigma}{1-\sigma y}+2+x+xy\Big)-(1+xy)^2-$$

$$-(2+x+xy)\Big(\frac{1-y}{1+y-2\sigma y}\Big)^{1/2}\frac{1-\sigma y}{1+y-2\sigma y}(1+2x+2xy-3\sigma xy)f_2\Big(\frac{y(1-\sigma)}{1-\sigma y}\Big)\Big]f_3(\sigma xy)\Big\}-$$

$$-\pi^{1/2}\frac{\Gamma(3/4)}{\Gamma(1/4)}x^{3/2}(1-y)^{1/2}\int_0^1 d\sigma(1-\sigma)^{-1/2}\Big[\frac{y}{2y+(1-y)\sigma}\Big]^{1/2}[1+x-x(1-y)\sigma]^{-3/2}[1-xy-x(1-y)\sigma]^{-1/2}\times$$

$$\times\Big\{[1+x-x(1-y)\sigma]\Big[\frac{(1+xy)(1+3y)(1-x)}{1-x+x(1-y)(1-\sigma)}-\frac{(1-\sigma)(1-xy)(1-y)(1+xy)}{(1-\sigma)(1-xy)+\sigma(1-x)}+2xy(1+y)+(1-y)(1+x+2xy)\sigma\Big]$$

$$-x^2(1+y)^2[2y+(1-y)\sigma]-2(1+x+2xy)\Big[\frac{1-x}{1-x+2x(1-y)(1-\sigma)}\Big]^{1/2}\frac{1-xy-x(1-y)\sigma}{1-xy-x(1-y)\sigma+x(1-y)(1-\sigma)}\times$$

$$\times[2y+(1-y)\sigma]\Big[1+x-xy-\frac{3}{2}x(1-y)\sigma\Big]\cdot f_2\Big(\frac{x(1-y)(1-\sigma)}{1-xy-x(1-y)\sigma}\Big)\Big\}\quad, \qquad \text{(B2)}$$

$$G_{02}=2x^{1/2}y\int_0^1 d\sigma\,\sigma^{1/2}(1-\sigma)^{-1/2}(1-\sigma y)^{-1/2}(1-\sigma xy)^{-1/2}(1+x+xy-\sigma xy)^{-1/2}\times$$

$$\times\Big\{(1+x)(1+x-\sigma xy)\Big(\frac{1-x}{1+x-2\sigma xy}\Big)^{1/2}\frac{1-\sigma xy}{1+x-2\sigma xy}f_2\Big(\frac{x(1-\sigma y)}{1-\sigma xy}\Big)+$$

$$+(1+xy)(1+xy-\sigma xy)\Big(\frac{1-x}{1+x-2\sigma xy}\Big)^{1/2}\frac{1-\sigma xy}{1+x-2\sigma xy}f_2\Big(\frac{x(1-\sigma y)}{1-\sigma xy}\Big)+$$

$$+x^2(1+y)(1+y-\sigma y)\Big(\frac{1-y}{1+y-2\sigma y}\Big)^{1/2}\frac{1-\sigma y}{1+y-2\sigma y}\cdot f_2\Big(\frac{y(1-\sigma)}{1-\sigma y}\Big)\Big\} \qquad \text{(B3)}$$

$$G_{31}=\frac{1}{3}\frac{x^{3/2}y^2}{1+x+xy}\int_0^1 d\sigma\cdot\sigma^{3/2}(1-\sigma)^{-1/2}(1-\sigma y)^{-1/2}\Big(\frac{1+x+xy-3\sigma xy}{1-\sigma xy}\Big)^{1/2}f_0\times$$

$$\times\Big\{\Big(\frac{1-x}{1+x-2\sigma xy}\Big)^{1/2}\frac{1-\sigma xy}{1+x-2\sigma xy}(1+x-2xy)\Big(3\frac{1+x-xy-\sigma xy}{1+xy-2\sigma xy}-4\frac{1+x-\frac{3}{2}xy-\frac{1}{2}\sigma xy}{1+x-2\sigma xy}\Big)f_2\Big(\frac{x(1-\sigma y)}{1-\sigma xy}\Big)$$

$$+\frac{2}{3}xy^2\frac{(1-\sigma)^2}{1-\sigma y}\cdot\frac{1-2x+xy}{1-\sigma xy}\Big(\frac{1-xy}{1+xy-2\sigma xy}\Big)^{1/2}\frac{1+x-xy-\sigma xy}{1+xy-2\sigma xy}\cdot f_1\Big(\frac{xy(1-\sigma)}{1-\sigma xy}\Big)+$$

$$+\Big(\frac{1-y}{1+y-2\sigma y}\Big)^{1/2}\frac{1-\sigma y}{1+y-2\sigma y}\cdot\frac{1-x-xy+\sigma xy}{1-\sigma xy}(2-x-xy)f_2\Big(\frac{y(1-\sigma)}{1-\sigma y}\Big)+$$

$$+y\frac{1-\sigma}{1-\sigma y}\cdot\frac{(1-xy)(1-2x+xy)}{(1+x-2\sigma xy)(1+xy-2\sigma xy)}(3-x-2\sigma xy)-\frac{(1-x)(1+x-2xy)}{(1+x-2\sigma xy)(1+xy-2\sigma xy)}\times$$

$$\times\Big[x(1-y)\frac{1-xy}{1-\sigma xy}+2(1-x)\Big]\Big\}-2x^{1/2}y(1-x)^{1/2}(1+x+xy)\int_0^1 d\sigma\cdot\sigma^{1/2}(1-\sigma)^{-1/2}(1-\sigma y)^{-1/2}\times$$

$$\times\Big(\frac{1-\sigma xy}{1+x-2\sigma xy}\Big)^{1/2}\cdot\frac{1+x-2xy}{1+x-2xy\sigma}\cdot\frac{1+x-xy-\sigma xy}{1+x+xy-3\sigma xy}(1+x+xy-2\sigma xy)\times$$

$$\times(1+x+xy-\sigma xy)^{-3/2}f_2\Big(\frac{x(1-\sigma y)}{1-\sigma xy}\Big)\,. \qquad \text{(B4)}$$

Here we have used the notations

$$f_0 = F_1\left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \frac{\sigma x y}{1+x+xy}, \frac{3\sigma x y}{1+x+xy}\right) + F_1\left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}; \frac{\sigma x y}{1+x+xy}, \frac{3\sigma x y}{1+x+xy}\right)$$

$$f_1(z) = F\left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; z^2\right), \quad f_2(z) = F\left(-\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; z^2\right), \quad f_3(z) = F\left(\frac{1}{4}, 1; \frac{7}{4}; z^2\right);$$

On adding the contributions according to (3.13), we obtain (3.30). Calculation by formulae (B2) - (B4) shows that G_{02} is everywhere positive, while G_{11} and G_{31} are everywhere negative; however, their sum $G(x,y) = G_{11} + G_{02} + G_{31} \geq 0$ is everywhere on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$. In view of this, we also have $H(x,y,z) \geq 0$ (see (3.31)).