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APPENDIX B

Nonlinear evolution equation at the Prandtl number $\eta = 1$

Equation (6.3) is a first approximation to the complete nonlinear evolution equation, which can be written as ($\Delta J = \frac{1}{4} - J$):

$$\frac{\partial \tilde{A}}{\partial t} = \Delta J \cdot \tilde{A} + \sum_{n=1}^{\infty} \alpha_n |\tilde{A}|^{2n} \tilde{A} \tag{B.1}$$

where α_1 evidently is the Landau constant and $\alpha_2 = a_4$ (see Paper 1). Let us show that in the general case $\alpha_n = O(\nu^{-n})$.

Let us consider the term in the expansion of Ψ which is proportional to A^{m+1}. Looking in (4.1) and (4.2) at the harmonics generation process (which is due to nonlinear terms on the right-hand sides) we see the main contribution to this term to be $O[(\varepsilon \varkappa^{1/6})^{m+1} (\varkappa^{-2/3})^m] = O(\varepsilon \varkappa^{1/6} \cdot \varepsilon^m \varkappa^{-m/2})$. The contribution to the fundamental (and hence to (B.1)) is made at even m ($m = 2n$) only. Therefore the main term in α_n is of order $O(\varkappa^{-n}) = O(\nu^{-n})$. In Particular, the Landau constant $a \equiv \alpha_1 = O(\nu^{-1})$, in agreement with our previous study, and the coefficient $a_4 \equiv \alpha_2 = O(\nu^{-2})$ (cf. Paper 1). But at the Prandtl number $\eta = 1$, as will be shown, these main contributions all vanish. That is why we have to calculate the next terms in α_n's expansions in powers of ν.

a) Landau constant evaluation at $\eta = 1$.

For this purpose, we first derive the more detailed expansion of the second harmonic of the outer solution as $y \to 0$ than in (3.12). It is convenient to write equation (3.10) in the form ($u = \tanh y$)

$$M\Phi \equiv \Phi'' - 2\left(2u + \frac{1}{u}\right)\Phi' + \left(4u^2 + \frac{9/4}{u^2} + 1\right)\Phi = G(y), \qquad G(y) = 2u^2 + \frac{3/8}{u^2} \tag{B.2}$$

where $\psi_2^{(1)} = \Phi(y) \cdot (\sinh y \cdot \cosh^2 y)^{-1} (A^{\pm})^2 \cdot S$

Let f_a and f_b be solutions of the homogeneous equation $Mf = 0$ such that

$$f_a \approx e^{q_a |y|}(1 + O(e^{-2|y|})), \quad f_b \approx e^{q_b |y|}(1 + O(e^{-2|y|})); \quad y \to \pm\infty$$

where $q_{a,b} = 3 \mp \sqrt{7}/2$. Without loss of generality one can define them as even functions of y. The solution of (B.2) obeying the boundary conditions (2.7) can be written as:

$$\Phi(y) = f(y) + C^{\pm} f_a(y) \tag{B.3}$$

where $f(y)$ is a particular solution of (B.2) such that it does not contain $e^{q_{a,b}|y|}$ as $y \to \pm\infty$. It is an even function of y with asymptotic representations

$$f \sim \frac{1}{6} - \frac{2}{3} y^2 + (\alpha + \beta l)|y|^{3/2} \qquad as \quad y \to \pm 0 \;,$$
$$f \sim \frac{19}{58} + O(e^{-2|y|}) \qquad as \quad y \to \pm\infty \;. \tag{B.4}$$

Here α and β are real and $l = \ln|y/2|$. As $y \to \pm 0$, we obtain for f_a and f_b:

$$f_{a,b}(y) \sim |y|^{3/2} (\alpha_{a,b} + \beta_{a,b} \cdot l) \tag{B.5}$$

with real $\alpha_{a,b}$ and $\beta_{a,b}$ which are fully determined by the properties of operator M. Using (B.3)-(B.5) we can write for Φ as $y \to \pm 0$

$$\Phi \sim \frac{1}{6} - \frac{2}{3} y^2 + (a_2^{\pm} + b_2^{\pm} l)\, y\, |y|^{1/2}$$

where

$$a_2^{+} = \alpha + \alpha_a C^{+}, \qquad a_2^{-} = -(\alpha + \alpha_a C^{-}),$$
$$b_2^{+} = \beta + \beta_a C^{+}, \qquad b_2^{-} = -(\beta + \beta_a C^{-}) \tag{B.6}$$

In terms of the inner variable Y we obtain instead of (3.12)

$$\psi_2 \sim \varepsilon^2 \varkappa^{-1/3} (6Y)^{-1} (A^{\pm})^2 S + \varepsilon^2 \varkappa^{1/6} (A^{\pm})^2 (a_2^{\pm} + b_2^{\pm} \ln|\frac{\varkappa^{1/3}}{2} Y|) \cdot |Y|^{1/2} \cdot S \tag{B.7}$$

Note that the coefficients $a_2^{\pm}$ and $b_2^{\pm}$ are fully determined by boundary conditions (2.7) and by rule of the singular point indentation.

Now we consider the inner solution. The nonlinear term in (4.19) is due to "interaction" of the fundamental with $\Psi_2^{(1)}$ and $\bar{\Psi}_0^{(1)}$ and, as shown previously, vanishes at $\eta = 1$. In order to obtain the Landau constant a (and the second term in (4.21)) we consider the next terms of expansion. From (3.13) and (B.7), for the inner solution we have

$$\Psi_2 = \varepsilon^2 \varkappa^{-1/3} \Psi_2^{(1)} + \varepsilon^2 \varkappa^{1/6} \Psi_2^{(2)} + \cdots ,$$
$$\Psi_0 = \varepsilon^2 \varkappa^{-1/3} \Psi_0^{(1)} + \varepsilon^2 \varkappa^{1/3} \Psi_0^{(2)} + \cdots$$

The first terms of these expansions are of the same order of magnitude and, therefore, the second and the zeroth harmonics make contribution to a that is also of the same order. The next terms of expansions are of different order and the Landau constant (a_1 in (4.22) and also the coefficient h_1 in (4.21)) is due to the second harmonic ($\Psi_2^{(2)}$) only.

For obtaining the nonlinear evolution equation at $\eta = 1$ it is necessary to reorder the variables so that the evolution time, the supercriticality and the nonlinearity should be of the same order, i.e.

$$\tau = \varepsilon^2 \varkappa^{-1/2} t , \quad \Delta\mathcal{J} = \frac{1}{4} - \mathcal{J} = - \varepsilon^2 \varkappa^{-1/2} \gamma^{(1)} \tag{B.8}$$

In the outer solution, this provides only formal changes: the expansion of fundamental becomes $\varphi_1 = \varepsilon \varphi_1^{(1)} + \varepsilon^3 \varkappa^{-1/2} \varphi_1^{(2)} + \cdots$, the $\varphi_0^{(1)}$ is evaluated in $\varepsilon^4 \varkappa^{-1/2}$ and so on but the $\varphi_n^{(m)}$ themselves do not change. The MSC (3.9) does not change either. In the inner solution we have to consider the additional terms of expansion: $\varepsilon^3 \varkappa^{-1/3} \Psi_1^{(4)}$ in fundamental and $\varepsilon^2 \varkappa^{1/6} \Psi_2^{(2)}$ in the second harmonic.

It is easy to see that for evaluating $b_1^{(2)+} + b_1^{(2)-}$, i.e. for obtaining the evolution equation, it is necessary to match the terms $\sim Y^{1/2} \ln\left(\frac{\varkappa^{1/3}}{2} Y\right)$ in the asymptotic expansion of the fundamental of the inner solution to (3.11).

It is convenient to write equations (4.1) and (4.2) in the form

$$\mathcal{N}\Psi \equiv \left[\mathcal{D}^3 - \left(Y\mathcal{D} - \frac{1}{2}\right)\frac{\partial}{\partial x}\right]\Psi = \frac{1}{2}\Pi_x + \varkappa^{-2/3}\{\Psi_Y, \Psi\}^* + \tag{B.9}$$
$$+ \varepsilon^3 \varkappa^{-5/6} \frac{\partial}{\partial \tau} \Psi_Y - \underline{2 \cdot \Delta\mathcal{J} \cdot (\Psi_x + \Pi_x)} ,$$

$$\mathcal{N}\Pi = \varkappa^{-2/3}\{\Pi_Y, \Psi\}^* + \varepsilon^2 \varkappa^{-5/6} \frac{\partial}{\partial \tau} \Pi_Y ; \qquad P = -2\mathcal{D}(\Psi + \Pi).$$

It will be recalled that the contribution $\sim Y^{1/2}\ln\left(\frac{\varkappa^{1/3}}{2}Y\right)$ to the asymptotic representation of the inner solution is due to W_b, which is the solution of the equation $\mathcal{N}_1 W_b = -ik_0 W_a$. It is easy to see that the presence of such a contribution may be due only to either $\Pi \neq 0$ or the underlined term in (B.9) (it is the term to which the linear contribution to the right-hand side of (2.3) (see also (4.19)) is due).

As was mentioned (see Section 4 after (4.20)) $\Pi_1^{(1)} = \Pi_2^{(1)} = \Pi_0^{(1)} = 0$. Generally $\Pi = 0$ is the solution of (B.9). But matching to (B.7) requires $\Pi_2^{(2)} \neq 0$. Equations (B.9) in this order ($\varepsilon^2 \varkappa^{1/6}$ of the second harmonic)

$$\mathcal{N}_2 \Psi_2^{(2)} = ik_0 \Pi_2^{(2)}, \quad \mathcal{N}_2 \Pi_2^{(2)} = 0$$

have a solution

$$\Psi_2^{(2)} = 2^{-1/6} A^2 \{m_2^{(2)} W_a(2^{1/3}Y) + n_2^{(2)} W_b(2^{1/3}Y)\} \equiv A^2 f_1 ,$$

$$\Pi_2^{(2)} = -2^{5/6} A^2 n_2^{(2)} W_a(2^{1/3}Y)$$

with an asymptotic representation as $Y \to \infty$, $Y \in \mathcal{M}$

$$\Psi_2^{(2)} \sim A^2 \{n_2^{(2)} \cdot \ln\left(\frac{\varkappa^{1/3}}{2}Y\right) + m_2^{(2)} + \tfrac{1}{3} n_2^{(2)} \ln 2\} Y^{1/2}.$$

Matching to the outer solution provides

$$n_2^{(2)} = b_2^+ = i b_2^- ; \qquad a_2^- = -i a_2^+ - \pi b_2^+$$

and using (B.6) we obtain

$$\pi\beta_a^2 C^\pm = (1 \mp i)(\alpha\beta_a - \alpha_a\beta) - \pi\beta\beta_a ,$$

$$(1-i)\, n_2^{(2)} = b_2^+ + b_2^- = \beta_a (C^+ - C^-) = \frac{2i}{\pi\beta_a}(\alpha_a\beta - \alpha\beta_a) \tag{B.10}$$

The contribution to the fundamental in the order $\varepsilon^3\varkappa^{-1/3}$ obeys the equation

$$\mathcal{N}_1 \Psi_1^{(4)} = \mathcal{R} ,$$

$$\mathcal{R} = ik_0|A|^2 A\{\mathcal{N}_1(f_1''\overline{W}_a + f_1'\overline{W}_a' - 2f_1\overline{W}_a'') - ik_0 n_2^{(2)}(U''\overline{W}_a + 2U'\overline{W}_a')\} + k_0^2 \mathcal{J}^{(1)} A W_a \tag{B.11}$$

where $U(Y) = 2^{-1/6} W_a(2^{1/3}Y)$ and a prime denotes the Y-derivative. The right-hand side of (B.11) is nonanalytic in the lower half-plane (see Appendix A); therefore in the asymptotic expansions of $\Psi_1^{(4)}$ as $Y \to \pm\infty$

$$\Psi_1^{(4)} = -\frac{1}{2}\mathcal{J}^{(1)} A Y^{1/2} \ln^2\left(\frac{\varkappa^{1/3}}{2}Y\right) + [m_1^{(4)\pm} + n_1^{(4)\pm} \cdot \ln\left(\frac{\varkappa^{1/3}}{2}Y\right)] A Y^{1/2} + O(Y^{-1}),$$

$n^+ \neq n^-$, $m^+ \neq m^-$ (cf. (4.15)). Matching to the outer solution provides (cf. (4.16))

$$b_1^{(2)+} + b_1^{(2)-} = -(n_1^{(4)+} - n_1^{(4)-}) + i\pi \mathcal{J}^{(1)} \tag{B.12}$$

For evaluating $n_1^{(4)+} - n_1^{(4)-}$ let us multiply (B.11) by the solution of conjugate problem v and taking into account that $\mathcal{N}_1^+ v = 0$ and $v = -4 W_a''$ we obtain

$$n_1^{(4)+} - n_1^{(4)-} = 4|A|^2 n_2^{(2)} \int_{-\infty}^{\infty} dY\, W_a'' (U''\overline{W_a} + 2U'\overline{W_a}').$$

Substitution of this into (B.12) and then in MSC (3.9) provides the nonlinear evolution equation for $\eta = 1$:

$$\frac{\alpha}{k_0}\frac{\partial \tilde{A}}{\partial \tau} = -\mathcal{J}^{(1)} A - \frac{4i}{\pi} n_2^{(2)} |A|^2 A \int_{-\infty}^{\infty} dY\, W_a'' (U''\overline{W_a} + 2U'\overline{W_a}') \tag{B.13}$$

where α is the same as in (6.3).

Thus, proceeding from asymptotic representation (B.7) of the second harmonic of outer solution (which was not obtained in Paper 1) we were to introduce into the inner solution the term $\varepsilon^2 \varkappa^{1/6} \psi_2^{(2)}$ which cannot be obtained from (B.9) through the harmonics generation process. And it is the term that provides the main contribution $O(\gamma^{-1/2})$ to the Landau constant a at $\eta = 1$ (see (4.22)).

Now, for completing the a calculation it is necessary to determine $n_2^{(2)}$ which can be expressed in terms of $b_2^{\pm}$ (see (B.10)). According to (B.3), one can write Φ in the form

$$\Phi(y) = f_a(y)\int_{\pm\infty}^{y} dz\, \frac{G(z)}{W(z)} f_b(z) - f_b(y)\int_{\pm\infty}^{y} dz\, \frac{G(z)}{W(z)} f_a(z) + C^{\pm} f_a(y)$$

where the Wronskian $W(y) = f_a' f_b - f_a f_b' = F\cdot \sinh^2 y \cdot \cosh^2 y \cdot s$, $F = \alpha_b \beta_a - \alpha_a \beta_b = -64\sqrt{7}$. To obtain the asymptotic representation as $y \to 0$ we pick out the singular terms. To do so, we introduce

$\tilde{f}(y) = f - \frac{1}{6}$ with asymptotic representation

$f \sim (\alpha + \beta \ell)|y|^{3/2} + O(y^2)$ as $y \to 0$. The $\tilde{f}$ obeys the equation $M\tilde{f} = G_1(y)$, where

$$\frac{G_1(y)}{W(y)} = \frac{1}{2F}\left(\frac{8/3}{\cosh^6 y} - \frac{1/3}{\cosh^4 y \cdot \sinh^2 y}\right) \tag{B.14}$$

Then

$$\tilde{f} = \int_{\pm\infty}^{y} dz\, \frac{G_1(z)}{W(z)} [f_a(y) f_b(z) - f_a(z) f_b(y)]$$

As $y \to 0$, we have

$$\alpha = \int_0^\infty dy \frac{G_1}{W}(\alpha_b f_a - \alpha_a f_b), \quad \beta = \int_0^\infty dy \frac{G_1}{W}(\beta_b f_a - \beta_a f_b) .$$

Substituting into (B.10) and using (B.14) we obtain

$$n_2^{(2)} = \frac{1-i}{2\pi\beta_a}\int_0^\infty dy \left(\frac{8/3}{\cosh^6 y} - \frac{1/3}{\cosh^4 y \cdot \sinh^2 y}\right) f_a .$$

Using equation (B.2) one can show that

$$n_2^{(2)} = \frac{1-i}{2\pi\beta_a}\int_0^\infty dy \left(\cosh^{-4} y - \frac{1}{2}\cosh^{-2} y\right) \tag{B.15}$$

The constant $n_2^{(2)}$ is completely determined by the outer solution and evidently depends on the initial flow model used. In particular, in the Holmboe model, equation (B.2) takes the form ($u = \tanh y$)

$$M\Phi \equiv \Phi'' - 2\left(2u + \frac{1}{u}\right)\Phi' + \left(4u^2 + \frac{9/4}{u^2} + \frac{7}{4}\right)\Phi = 2u^2 + \frac{3/8}{u^2} - \frac{1}{8}$$

and calculations similar to those performed above provide

$$n_2^{(2)} = \frac{1-i}{\pi\beta_a}\left[\frac{4}{3} - \int_0^\infty dy \left(\frac{6}{\cosh^6 y} - \frac{3/8}{\cosh^4 y}\right) f_a\right]$$

where f_a is the solution of the homogeneous equation $Mf = 0$ with asymptotic behaviour $f_a \sim e^{2|y|}(1 + O(e^{-2|y|}))$ as $y \to \pm\infty$ and β_a has the same meaning as in (B.5). Thus equation (B.13) may be written in the form

$$\frac{\alpha}{k_0}\frac{\partial A}{\partial \tau} = -\gamma^{(1)} A + d|A|^2 A \quad , \quad d = -B\cdot I$$

or in terms of the original variables (cf. (6.3))

$$\frac{\alpha}{k_0}\frac{\partial \tilde{A}}{\partial t} = \left(\frac{1}{4} - \gamma\right)\tilde{A} + \frac{d}{\varkappa^{1/2}}|\tilde{A}|^2\tilde{A} \tag{B.16}$$

Here

$$B = 2k_0^{-1/2}\int_0^\infty dY \cdot \mathrm{Re}\{W_a''(U''\overline{W}_a + 2U'\overline{W}_a')(1+i)\} = 0.44 ,$$

$$I = \begin{cases} \dfrac{2k_0^{1/2}}{\pi^2\beta_a}\displaystyle\int_0^\infty dy\left(\cosh^{-4} y - \frac{1}{2}\cosh^{-2} y\right) f_a = 1.014\cdot\frac{2k_0^2}{\pi^2} & \text{in the Drazin model} \\ \dfrac{4k_0^{1/2}}{\pi^2\beta_a}\left[\dfrac{4}{3} - \displaystyle\int_0^\infty dy\left(6\cosh^{-6} y - \frac{3}{8}\cosh^{-4} y\right) f_a\right] = 4.14\,\frac{2k_0^2}{\pi^2} & \text{in the Holmboe model.} \end{cases}$$

One can see that at $\eta = 1$ the Landau constant is the product of the coefficient B which does not depend on which model is chosen, and the integral I that is determined completely by the outer flow profile. As $I > 0$ the stabilization takes place. Finally, we obtain the numerical values of a_1 and h_1 in (4.21) and (4.22) (at $\eta = 1$ the $a_0(\eta)$ appearing in (6.2) and (6.3) is equal to 0.36):

in the Drazin model $a_1 = -5.4 \cdot 10^{-2}$, $h_1 = 0.47$,

in the Holmboe model $a_1 = -0.26$, $h_1 = 2.28$

b) General structure of the complete nonlinear evolution equation in the viscous CL regime

Now, we are able to consider the following nonlinear terms in (B.1). They are also obtained by matching (3.11) to the term $\sim Y^{1/2} \ln(\frac{\varkappa^{1/3}}{2} Y)$ in the asymptotic expansion of the inner solution (if such a term is absent, then all $\alpha_n = 0$). Such a term can be due only to either the underlined addendum in (B.9) (the "inner" reason) or $\Pi \neq 0$. Since $\Pi = 0$ obeys (B.9), $\Pi \neq 0$ can arise only if this is required by the matching to outer solution ("outer" reason). Accordingly, it is convenient to separate contributions to α_n connected with each of these reasons: $\alpha_n = \alpha_n^{in} + \alpha_n^{out}$. Let us show that $|\alpha_n| \ll \nu^{-n}$ at $\eta = 1$.

This is obvious for α_n^{in} since the underlined term in (B.9) contains a small factor ΔJ. The contribution due to the "inner" reason is therefore $\alpha_n^{in} = O(\Delta J \cdot \nu^{-n})$. The contribution due to the "outer" reason will be considered in more detail.

Let, as the outer solution is being constructed step-by-step, Π be zero up to some iteration and in the next iteration $\Pi \neq 0$ be required for matching to the term $O(\varepsilon^{p+1})$ of the ℓ-th harmonic of outer solution. The equation for Π at this step is evidently a homogeneous one: $\mathcal{N}_\ell \Pi = 0$. If $\ell \neq 0$, the asymptotic expansion of the outer solution ψ as $y \to 0$ must contain the term $\varepsilon^{p+1} \varkappa^{1/6} A^{p+1} |Y|^{1/2} \times$ $\times \ln \left|\frac{\varkappa^{1/3}}{2} Y\right|$ to which the term $\Psi \sim \varepsilon^{p+1} \varkappa^{1/6} A^{p+1} W_\ell (\ell^{1/3} Y)$ in the inner solution corresponds. As has already been shown, the main contribution to Ψ proportional to A^{p+1} is of order $O(\varepsilon^{p+1} \varkappa^{1/6} \varkappa^{-p/2})$ Hence, on incorporating the $\Pi \neq 0$ obtained in the process of harmonics generation, we may have nonzero contributions to α_n^{out} (as $2n \geq p+1$) but of order $O(\nu^{-n+p/2})$ rather than $O(\nu^{-n})$, as in the case $\eta \neq 1$). If, however, $\ell = 0$, then $\mathcal{N}_\ell = \mathcal{N}_0 = \mathcal{D}^3$ and Π' is the second-order polynomial in Y. In the general case, the matching to the outer solution provides $\Psi, \Pi \sim \varepsilon^{p+1} A^{p+1}$ (in special cases one may obtain $\Psi, \Pi \sim \varepsilon^{p+1} \varkappa^{1/3} A^{p+1}$ or $\Psi, \Pi \sim \varepsilon^{p+1} \varkappa^{2/3} A^{p+1}$). This contributes to α_n^{out} in the order $O(\nu^{-n+p/2-1/6})$ or less (in special cases).

Therefore, at any $p > 0$ and at any l the contribution to α_n is less than $O(\nu^{-n})$. This means that at $\eta = 1$, the leading terms in all α_n no longer appear (contrary to the statement in Paper 1 that $a_4 \equiv \alpha_2 = O(\nu^{-2})$ at $\eta = 1$). Based on the Landau constant evaluation performed above we can argue that the main contribution to Π due to the "outer" reason is $\varepsilon^2 \varkappa^{1/6} \Pi_2^{(2)}$. This provides $\alpha_n^{out} = O(\nu^{-n+1/2})$.

Hence equation (B.1) can be written in the form

$$\frac{\partial \tilde{A}}{\partial t} = \Delta \mathcal{J} \tilde{A} + N^{in} + N^{out} \tag{B.17}$$

where

$$N^{in} = \sum_{n=1}^{\infty} \alpha_n^{in} |\tilde{A}|^{2n} \tilde{A}, \quad \alpha_n^{in} = O\left(\frac{\Delta \mathcal{J}}{\nu^n}\right); \quad N^{out} = \sum_{n=1}^{\infty} \alpha_n^{out} |\tilde{A}|^{2n} A, \quad \alpha_n^{out} = O\left(\frac{\nu^{-1/2}}{\nu^{n-1}}\right)$$

It is easy to see that, as far as we consider the viscous CL ($\tilde{A}^2 \ll \nu$, $\Delta \mathcal{J} \ll \nu^{1/3}$), the leading term in each of these series is the first one ($n = 1$). Therefore, also at $\eta = 1$, we can keep only the cubic in $\tilde{A}$ term in the nonlinear evolution equation. This immediately leads to an equation of the form (2.3).

Note that if we represent $\Delta \mathcal{J} \cdot \tilde{A} + N^{in} = \left(-\frac{\varphi}{\pi}\right) \Delta \mathcal{J} \cdot \tilde{A}$ we obtain the phase shift across the CL φ (see Introduction) as a function of $\tilde{A}^2/\nu$ (for unstratified flow this phase shift was calculated by Haberman (1972, 1976)).

The question of the relative magnitudes of the terms on the right-hand side of (B.17) is also interesting. As $\alpha_1^{in} = O\left(\frac{\Delta \mathcal{J}}{\nu}\right)$ and $\alpha_1^{out} = O\left(\frac{\nu^{1/2}}{\nu}\right)$, so $N^{in} < N^{out}$ as $\Delta \mathcal{J} < \nu^{1/2}$ and $N^{in} > N^{out}$ as $\Delta \mathcal{J} > \nu^{1/2}$. Comparison with a linear term demonstrates that N^{in} is always less than it and N^{out} may be of the same order when the amplitude $\tilde{A} \sim (\Delta \mathcal{J} \cdot \nu^{1/2})^{1/2}$

Hence the nonlinearity in the viscous CL regime ($\tilde{A} \ll \nu^{1/2}$) is competitive only if $\Delta J < \nu^{1/2}$ and in this case the leading nonlinear term is N^{out}. In evaluating the Landau constant at $\eta = 1$ we have actually obtained α_1^{out}. Therefore the evolution equation (B.16) is valid at $\Delta J < \nu^{1/2}$ only, but not in the whole viscous CL region. In this area the nonlinearity is competitive and the instability is saturated at $\tilde{A} = \tilde{A}_0 \sim (\Delta J \cdot \nu^{1/2})^{1/2}$ (see Fig. 5).

Fig 5

As $\nu^{1/2} < \Delta J < \nu^{1/3}$ the leading term on the right-hand side of (B.17) is a linear one, but the leading nonlinear term ($N^{in} \sim \Delta J \cdot \nu^{-1} \tilde{A}^{3}$) is responsible only for the nonlinear correction to the growth rate.