FM 4866

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From "Drag on a sphere moving slowly in rotating viscous fluid" by A. J. Weisenborn

Appendix B evaluation of $B^{(p+1,\ell+1)}$

In this appendix we shall evaluate the quantity $B^{(p+1,\ell+1)}$, given by

$$B^{(p+1,\ell+1)} = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, \frac{x^2 \, j_p(x) \, j_\ell(x)}{x^2 - 2iT\xi} \qquad (\xi \geqslant 0)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, \frac{x \, j_p(x) \, j_\ell(x)}{x-z}$$

with $z = (1+i)\,(T\xi)^{\frac{1}{2}}$, by means of complex integration. Using the notation of section 4.2 we can replace this integral by

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, \frac{x \, j_{\max(p,\ell)}(x) \, j_{\min(p,\ell)}(x[1+\varepsilon])}{x-z}$$

The outcome of the integration does not depend on the way in which the limit is taken (i.e. $\varepsilon \uparrow 0$ or $\varepsilon \downarrow 0$), in view of the principle of dominated convergence (see e.g. Feller 1971†); an integrable function which bounds the integrand is readily constructed. Using the relation (see Abramowitz and Stegun (1968))

$$j_n(x) = \tfrac{1}{2} \left(h_n^{(1)}(x) + h_n^{(2)}(x) \right)$$

we obtain for $B^{(p+1,\ell+1)}$:

$$B^{(p+1,\ell+1)} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \frac{x \, j_{\max(p,\ell)}(x) \, h^{(1)}_{\min(p,\ell)}(x[1+\varepsilon])}{x-z}$$

$$+ \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \frac{x \, j_{\max(p,\ell)}(x) \, h^{(2)}_{\min(p,\ell)}(x[1+\varepsilon])}{x-z}$$

† Feller, W. An Introduction to Probability Theory and its Applications, vol. 2 p.111. Wiley.

To evaluate the first integral we use as contour a large semicircle above the real axis with its centre at the origin, together with that part of the real axis which joins the ends of the semicircle; for the second integral we use the reflection of this contour with respect to the real axis. The integrals round the large semicircles both tend to zero as the radii tend to infinity. Applying now Cauchy's residue theorem we find that the second integral vanishes and that the first integral yields the expression for $B^{(p+1,\ell+1)}$ given in eq. (4.17).

Appendix C <u>proof of eq. (4.22)</u>

In this appendix we shall show that the tensors $\underline{a}_i^{(2\ell+1)}$, defined in eq. (4.21) for $\ell \geqslant 1$, satisfy the relation

$$\underline{a}_i^{(2\ell+1)} \circledcirc \overset{\curvearrowleft}{\underline{a}_j^{(2\ell+1)}} = (-1)^{i+1}\, \delta_{ij} \qquad\qquad i,j = 1,2,3$$

Many of the relations we shall use in this appendix can be found in chapter 2 of Hess and Koehler (1980). With the help of the relation

$$\overline{\hat{b}^{2\ell+1}} = \overline{\hat{b}^{2\ell}}\;\hat{b} - \frac{2\ell}{4\ell+1}\,\underline{\Delta}^{(2\ell)} \odot \overline{\hat{b}^{2\ell-1}}$$

and the identities

$$\hat{b}^{2\ell+1} \odot \underline{\Delta}^{(2\ell+1)} = \overline{\hat{b}^{2\ell+1}} \odot \underline{\Delta}^{(2\ell+1)} = \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \underline{\Delta}^{(2\ell+1)}$$

one easily verifies that

$$\overline{\hat{b}^{2\ell+1}} \circledcirc \overline{\hat{b}^{2\ell+1}} = \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \overline{\hat{b}^{2\ell+1}}$$

$$= \hat{b}\;\overline{\hat{b}^{2\ell}} \odot \left(\overline{\hat{b}^{2\ell}}\;\hat{b} - \frac{2\ell}{4\ell+1}\,\underline{\Delta}^{(2\ell)} \odot \overline{\hat{b}^{2\ell-1}} \right)$$

$$= \frac{2\ell+1}{4\ell+1}\ \overbrace{\hat{b}^{2\ell}}\ \odot\ \overbrace{\hat{b}^{2\ell}}$$

$$= \frac{(2\ell+1)!}{(4\ell+1)!!} \qquad \text{(C.1)}$$

Using eq. (C.1) and the fact that $(\hat{\Omega}\cdot\hat{U})^2 = 1$, one finds

$$\underline{a}_1^{(2\ell+1)} \odot \overbrace{\underline{a}_1^{(2\ell+1)}} = \frac{(4\ell+1)!!}{(2\ell+1)!}\ \hat{\Omega}^{2\ell}\ \hat{U}\ \odot \underline{\Delta}^{(2\ell+1)} \odot \underline{\Delta}^{(2\ell+1)} \odot\ \hat{U}\ \hat{\Omega}^{2\ell}$$

$$= \frac{(4\ell+1)!!}{(2\ell+1)!}\ \overbrace{\hat{\Omega}^{2\ell}\ \hat{U}}\ \odot\ \overbrace{\hat{U}\ \hat{\Omega}^{2\ell}} = 1$$

Combining the identity

$$\underline{\square}^{(2\ell)} \odot^{*} \underline{\square}^{(2\ell)} = -\frac{2\ell+1}{2\ell}\ \underline{\Delta}^{(2\ell)}$$

where $\odot^{*}$ denotes the $2\ell + 1$ - fold contraction, with eq. (C.1) it is clear that

$$\underline{a}_2^{(2\ell+1)} \odot \overbrace{\underline{a}_2^{(2\ell+1)}} = \frac{(4\ell+1)!!2\ell}{(2\ell+1)!}\ \hat{\Omega}^{2\ell-1}\ \hat{U} \odot \underline{\square}^{(2\ell)} \odot^{*} \underline{\square}^{(2\ell)} \odot \hat{U}\ \hat{\Omega}^{2\ell-1} = -1$$

Finally, using the relation

$$\left(\underline{\Delta}^{(2\ell)}\right)_{\mu_1,\ldots,\mu_{2\ell-1},\lambda,\mu_1',\ldots,\mu_{2\ell-1}',\lambda} = \frac{4\ell+1}{4\ell-1}\left(\underline{\Delta}^{(2\ell-1)}\right)_{\mu_1,\ldots,\mu_{2\ell-1},\mu_1',\ldots,\mu_{2\ell-1}'}$$

and eq. (C.1) one may check that

$$\underline{a}_3^{(2\ell+1)} \odot \overbrace{\underline{a}_3^{(2\ell+1)}} = \frac{(4\ell-1)!!}{(2\ell-1)!(4\ell+1)}\ \hat{\Omega}^{2\ell-2}\ \hat{U} \odot \underline{\Delta}^{(2\ell)} \odot^{*} \underline{\Delta}^{(2\ell)} \odot \hat{U}\ \hat{\Omega}^{2\ell-2} = 1$$

Appendix D

As preparation for appendices E and F we evaluate in this appendix the quantities $b_{i,j}^{(2p+1,2\ell+1)}$ for the cases $p,\ell = 0,1$ and $i,j = 1,2,3$.

Using eq. (4.17) and the relations (see e.g. Abramowitz and Stegun (1968))

$$j_{2n}(z) = z^{2n} \left(\frac{1}{z}\frac{d}{dz} \right)^{2n} \frac{\sin z}{z} \qquad ; \qquad h_{2n}^{(1)}(z) = z^{2n} \left(\frac{1}{z}\frac{d}{dz} \right)^{2n} \frac{e^{iz}}{iz}$$

one has, denoting $2(T\xi)^{\frac{1}{2}}$ by y, for $B^{(1,3)}$ and $B^{(3,3)}$:

$$B^{(1,3)} = \frac{1}{2y} \left(-1 + \frac{6}{y^2} - \left(1 + \frac{6}{y} + \frac{6}{y^2} \right) e^{-y} \sin y + \left(1 - \frac{6}{y^2} \right) e^{-y} \cos y \right)$$

$$+ \frac{i}{2y} \left(-1 + \frac{6}{y} - \frac{6}{y^2} + \left(1 - \frac{6}{y^2} \right) e^{-y} \sin y + \left(1 + \frac{6}{y} + \frac{6}{y^2} \right) e^{-y} \cos y \right)$$

$$= - \frac{1}{60} y^3 + \frac{1}{105} y^4 - \frac{1}{504} y^5 + i\left(\frac{1}{30} y^2 - \frac{1}{60} y^3 + \frac{1}{504} y^5 \right) + O\left(y^6 \right)$$

$$B^{(3,3)} = \frac{1}{2y} \left(1 + \frac{6}{y^2} - \frac{36}{y^4} + \left(1 + \frac{12}{y} + \frac{30}{y^2} - \frac{36}{y^4} \right) e^{-y} \sin y \right.$$

$$\left. + \left(-1 + \frac{30}{y^2} + \frac{72}{y^3} + \frac{36}{y^4} \right) e^{-y} \cos y \right)$$

$$+ \frac{i}{2y} \left(1 - \frac{6}{y^2} - \frac{36}{y^4} - \left(1 - \frac{30}{y^2} - \frac{72}{y^3} - \frac{36}{y^4} \right) e^{-y} \sin y \right.$$

$$\left. - \left(1 + \frac{12}{y} + \frac{30}{y^2} + \frac{36}{y^4} \right) e^{-y} \cos y \right)$$

$$= \frac{1}{5} + \frac{i}{105} y^2 + 0 (y^4)$$

In view of the symmetry relation (4.24) we have to evaluate $b_{i,j}^{(2p+1,2\ell+1)}$ only for $p \leqslant \ell$, while in the case $p = \ell$ only for $i \geqslant j$. For the cases $p,\ell = 0,1$ the contractions in eq. (4.23) can be carried out very easily. Below we have listed expressions for $b_{i,j}^{(2p+1,2\ell+1)}$ for $p,\ell = 0,1$ and $i,j = 1,2,3$ for small and large values of T.

$$b_{1,1}^{(1,1)} = \int_0^1 d\xi \, (1-\xi^2) \, \mathrm{Re} \, B^{(1,1)}$$

$$= \frac{2}{3} (1 - \frac{4}{7} T^{\frac{1}{2}} + \frac{8}{45} T^{3/2} - \frac{8}{75} T^2 + \frac{32}{1155} T^{5/2} - \frac{32}{12285} T^{7/2}) + 0 (T^4)$$

$$b_{1,1}^{(1,3)} = \frac{3}{2} \sqrt{10} \int_0^1 d\xi \, (1 - \xi^2)(1 - 5\xi^2) \, \mathrm{Re} \, B^{(1,3)}$$

$$= \frac{32}{975} \sqrt{10} \, T^{3/2} (1 - \frac{52}{49} T^{\frac{1}{2}}) + 0 (T^{5/2}) \qquad (T \ll 1)$$

$$= - \frac{4}{15} \sqrt{10} \, T^{-\frac{1}{2}} (1 - \frac{135}{64} \pi T^{-\frac{1}{2}}) + 0 (T^{-3/2}) \qquad (T \to \infty)$$

$$b_{1,2}^{(1,3)} = 15 \int_0^1 d\xi \, (1 - \xi^2) \, \xi \, \mathrm{Im} \, B^{(1,3)}$$

$$= \frac{4}{15} T (1 - \frac{60}{77} T^{\frac{1}{2}} + \frac{200}{819} T^{3/2}) + 0 (T^3)$$

$$= - \frac{10}{7} T^{-\frac{1}{2}} (1 - \frac{21}{4} T^{-\frac{1}{2}}) + 0 (T^{-3/2})$$

$$b_{1,3}^{(1,3)} = \sqrt{15} \int_0^1 d\xi \, (1 - \xi^2) \, \mathrm{Re} \, B^{(1,3)}$$

$$= -\frac{16}{675}\sqrt{15}\,T^{3/2}\left(1 - \frac{6}{7}T^{\frac{1}{2}}\right) + 0\left(T^{5/2}\right)$$

$$= -\frac{2}{5}\sqrt{15}\,T^{-\frac{1}{2}}\left(1 - \frac{15}{16}\pi\,T^{-\frac{1}{2}}\right) + 0\left(T^{-3/2}\right)$$

$$b_{1,1}^{(3,3)} = \frac{45}{2}\int_0^1 d\xi\,\left(1 - \xi^2\right)\left(1 - 5\xi^2\right)^2 \operatorname{Re} B^{(3,3)}$$

$$= \frac{24}{7} + 0\left(T^2\right)$$

$$= \frac{112}{13}T^{-\frac{1}{2}} + 0\left(T^{-3/2}\right)$$

$$b_{1,2}^{(3,3)} = \frac{45}{2}\sqrt{10}\int_0^1 d\xi\,\left(1 - \xi^2\right)\left(1 - 5\xi^2\right)\xi \operatorname{Im} B^{(3,3)}$$

$$= -\frac{32}{245}\sqrt{10}\,T + 0\left(T^2\right)$$

$$= \frac{60}{77}\sqrt{10}\,T^{-\frac{1}{2}} + 0\left(T^{-3/2}\right)$$

$$b_{1,3}^{(3,3)} = \frac{15}{2}\sqrt{6}\int_0^1 d\xi\,\left(1 - \xi^2\right)\left(1 - 5\xi^2\right) \operatorname{Re} B^{(3,3)}$$

$$= 0\left(T^2\right)$$

$$= \frac{4}{3}\sqrt{6}\,T^{-\frac{1}{2}} + 0\left(T^{-3/2}\right)$$

$$b_{2,2}^{(3,3)} = -225\int_0^1 d\xi\,\left(1 - \xi^2\right)\xi^2 \operatorname{Re} B^{(3,3)}$$

$$= -6 + 0\left(T^2\right)$$

$$= -10\,T^{-\frac{1}{2}} + 0\left(T^{-3/2}\right)$$

$$b_{2,3}^{(3,3)} = 15\sqrt{15}\int_0^1 d\xi\,\left(1 - \xi^2\right)\xi \operatorname{Im} B^{(3,3)}$$

$$= \frac{8}{105}\sqrt{15}\ T + 0\ (\ T^2\ )$$

$$= \frac{10}{7}\sqrt{15}\ T^{-\frac{1}{2}} + 0\ (\ T^{-3/2}\ )$$

$$b_{3,3}^{(3,3)} = 15 \int_0^1 d\xi\ (\ 1 - \xi^2\ )\ \mathrm{Re}\ B^{(3,3)}$$

$$= 2 + 0\ (\ T^2\ )$$

$$= 6\ T^{-\frac{1}{2}} + 0\ (\ T^{-3/2}\ )$$

Appendix E asymptotic behaviour of $\mu^T(1)$

In this appendix we show that the asymptotic behaviour of $\mu^T(1)$ for large values of of the Taylor number is given by

$$\mu^T(1) = \frac{3}{16}\left(\eta a T\right)^{-1}\ \{\ 1 + 0\ (\ T^{-\frac{1}{2}}\ )\ \} \tag{E.1}$$

After substituting in eq. (4.26) the asymptotic expressions for $b_{i,j}^{(2p+1,2\ell+1)}$ for $p,\ell\ =\ 0,1$; $i,j\ =\ 1,2,3$, evaluated in appendix D, and carrying out some standard manipulations one obtains for $\mu^T(1)$

$$\mu^T(1) = (\ 4\pi\eta a\ )^{-1} \begin{vmatrix} \frac{112}{13}\ T^{-\frac{1}{2}} & \frac{60}{77}\sqrt{10}\ T^{-\frac{1}{2}} & \frac{4}{3}\sqrt{6}\ T^{-\frac{1}{2}} \\ \frac{60}{77}\sqrt{10}\ T^{-\frac{1}{2}} & -\ 10\ T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\ T^{-\frac{1}{2}} \\ \frac{4}{3}\sqrt{6}\ T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\ T^{-\frac{1}{2}} & 6\ T^{-\frac{1}{2}} \end{vmatrix}^{-1} \times$$

$$\begin{vmatrix} \frac{6\pi}{8} T^{-1} & \frac{9\pi}{16}\sqrt{10}\, T^{-1} & \frac{15}{2} T^{-1} & \frac{3\pi}{8}\sqrt{15}\, T^{-1} \\ \frac{9\pi}{16}\sqrt{10}\, T^{-1} & \frac{112}{13} T^{-\frac{1}{2}} & \frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} \\ \frac{15}{2} T^{-1} & \frac{60}{77}\sqrt{10}\, T^{-\frac{1}{2}} & -10\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} \\ \frac{3\pi}{8}\sqrt{15}\, T^{-1} & \frac{4}{3}\sqrt{6}\, T^{-\frac{1}{2}} & \frac{10}{7}\sqrt{15}\, T^{-\frac{1}{2}} & 6\, T^{-\frac{1}{2}} \end{vmatrix}$$

With this expression it is easily shown that the asymptotic behaviour of $\mu^T(1)$ is given by eq. (E.1).

Appendix F <u>proof of eq. (4.34)</u>

In this appendix we shall derive the power series expansion for μ^T given in eq. (4.34) . The expansion of the first term at the right hand side of eq. (4.33) can easily be obtained from eq. (4.28) :

$$\hat{U} \cdot \underline{\underline{B}}^{(1,1)} \cdot \hat{U} = -\frac{2}{3}\left(1 - \frac{4}{7} T^{\frac{1}{2}} + \frac{8}{45} T^{3/2} - \frac{8}{75} T^2 + \frac{32}{1155} T^{5/2} - \frac{32}{12285} T^{7/2} \right) + 0\left(T^4 \right) \qquad (F.1)$$

We now turn to the other term in eq. (4.33) ,

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U}$$

Since the leading term in the expansion of $\underline{\underline{B}}^{(1,3)}$ in powers of $T^{\frac{1}{2}}$ is proportional to T (see eq. (4.18)), $\underline{\underline{B}}^{(3,3)^{-1}}$ must be expanded up to order $T^{3/2}$ and $\underline{\underline{B}}^{(1,3)}$ and $\underline{\underline{B}}^{(3,1)}$ up to order $T^{5/2}$. The tensor $\underline{\underline{B}}^{(3,1)} \cdot \hat{U}$ of rank three is traceless symmetric in its first two indices and contains only one unit vector $\hat{U}$. This tensor may therefore be decomposed in a similar way as $\underline{F}^{(3)}$ (see section 4.2) . Using eqs. (4.22) - (4.24) it may be verified that

$$\underline{\underline{B}}^{(3,1)} \cdot \hat{U} = \sum_{i=1}^{3} (-1)^{i+1} \, b_{1,i}^{(1,3)} \; \underline{a}_i^{(3)} \tag{F.2}$$

and also that

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} = \sum_{i=1}^{3} (-1)^{i+1} \, b_{1,i}^{(1,3)} \; \underline{a}_i^{(3)} \tag{F.3}$$

Consider now the tensor $\underline{\underline{B}}^{(3,3)^{-1}}$. In view of eqs. (F.2) and (F.3) it is sufficient to determine the elements of the matrix $\underline{\beta}$ defined by the relation

$$\underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{a}_i^{(3)} \equiv \sum_{j=1}^{3} \left(\underline{\beta}\right)_{i,j} \underline{a}_j^{(3)} \qquad i = 1,2,3 \tag{F.4}$$

up to order $T^{3/2}$. Eq. (F.4) is equivalent with

$$\underline{\underline{B}}^{(3,3)} \odot \underline{a}_i^{(3)} = \sum_{j=1}^{3} \left(\underline{\beta}^{-1}\right)_{i,j} \underline{a}_j^{(3)} \qquad i = 1,2,3$$

Using eq. (4.22) and the expressions for $b_{i,j}^{(3,3)}$ for small values of T, given in appendix D, one finds that

$$\underline{\beta}^{-1} = \begin{pmatrix} \frac{24}{7} & \frac{32}{245}\sqrt{10}\,T & 0 \\ -\frac{32}{245}\sqrt{10}\,T & 6 & \frac{8}{105}\sqrt{15}\,T \\ 0 & -\frac{8}{105}\sqrt{15}\,T & 2 \end{pmatrix} + 0\left(T^2\right)$$

Upon inversion one obtains for the matrix $\underline{\beta}$:

$$\underline{\beta} = \begin{pmatrix} \frac{7}{24} & -\frac{2}{315}\sqrt{10}\,T & 0 \\ \frac{2}{315}\sqrt{10}\,T & \frac{1}{6} & -\frac{2}{315}\sqrt{15}\,T \\ 0 & \frac{2}{315}\sqrt{15}\,T & \frac{1}{2} \end{pmatrix} + 0\left(T^2\right) \tag{F.5}$$

The evaluation of the term $\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U}$ is now straightforward : using eqs. (F.2) – (F.4) one obtains for this term :

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U} = \sum_{i=1}^{3} \sum_{j=1}^{3} (-1)^{i+1} \, b_{1,i}^{(1,3)} \, b_{1,j}^{(1,3)} \, (\underline{\beta})_{i,j}$$

Substituting eq. (F.5) and the expressions for $b_{1,i}^{(1,3)}$ for small values of T, listed in appendix D, in this equation, one gets:

$$\hat{U} \cdot \underline{\underline{B}}^{(1,3)} \odot \underline{\underline{B}}^{(3,3)^{-1}} \odot \underline{\underline{B}}^{(3,1)} \cdot \hat{U} =$$

$$\frac{2}{3} \left(\frac{4}{225} T^2 - \frac{32}{1155} T^{5/2} - \frac{485504}{2029052025} T^3 + \frac{1216}{36855} T^{7/2} \right) + o \left(T^4 \right) \qquad (F.6)$$

Substitution of eq. (F.1) and (F.6) into eq. (4.33) yields eq. (4.34).