To be held in Editorial Files

Mathematical details in section 4

"Low-Reynolds-number flow past an elliptic cylinder"

by K. Shintani, A. Umemura & A. Takano

Department of Mechanical Engineering
University of Electro-Communications
1-5-1 Chofugaoka
Chofu-City, Tokyo 182, Japan

1. Expression of $\psi_{1,n}^{(p)}$ in (4.4)

The particular solution $\psi_{1,n}^{(p)}$ is obtained as

$$
\begin{aligned}
\psi_{1,n}^{(p)} = {} & (A^2+B^2)\Big[-\tfrac{3}{64}\sinh 2(\xi-\xi_0)+\tfrac{1}{16}\{\xi-\xi_0\cosh 2(\xi-\xi_0)\}+\tfrac{1}{32}(\xi-\xi_0)\cosh 2(\xi-\xi_0)\Big]\sin 2\eta \\
& -(A^2-B^2)\Big[\Big\{-\tfrac{3}{64}(\xi-\xi_0)\cosh 2\xi+\tfrac{1}{32}(\xi-\xi_0)^2\sinh 2\xi+\tfrac{3}{128}\cosh 2\xi_0\sinh 2(\xi-\xi_0)\Big\}\sin 2\eta \\
& \qquad -\tfrac{1}{128}(\sinh 3\xi\sin\eta+\sinh\xi\sin 3\eta)\ln\frac{\cosh\xi-\cos\eta}{\cosh\xi+\cos\eta}+\tfrac{1}{64}\sinh 2\xi\sin 2\eta\ln(\cosh 2\xi-\cos 2\eta) \\
& \qquad +\Big\{a_2(e^{-2\xi}-e^{-2\xi_0})^2-k+\tfrac{1}{2}k'(e^{-2\xi+2\xi_0}-1)\Big\}\sin 2\eta+\sum_{m=2}^{\infty}\Big\{a_m e^{-2(m-1)\xi}+b_m e^{-2m\xi}+a_{m+1}e^{-2(m+1)\xi}\Big\}\sin 2m\eta\Big] \\
& +2AB\Big[-\tfrac{1}{16}(\xi-\xi_0)^2+\Big\{-\tfrac{1}{32}(\xi-\xi_0)^2\cosh 2\xi+\tfrac{3}{64}(\xi-\xi_0)\sinh 2\xi-\tfrac{3}{128}\sinh 2\xi_0\sinh 2(\xi-\xi_0)\Big\}\cos 2\eta \\
& \qquad +\tfrac{1}{128}(\cosh 3\xi\cos\eta+\cosh\xi\cos 3\eta)\ln\frac{\cosh\xi-\cos\eta}{\cosh\xi+\cos\eta}-\tfrac{1}{64}\cosh 2\xi\cos 2\eta\ln(\cosh 2\xi-\cos 2\eta) \\
& \qquad +(a_2' e^{-4\xi_0}-t-\tfrac{1}{2}t')\Big\{\cosh 2\xi-\cosh 2\xi_0-2(\xi-\xi_0)\sinh 2\xi_0\Big\}-\Big\{s+s'(\xi-\xi_0)\Big\} \\
& \qquad +\Big\{a_2'(e^{-2\xi}-e^{-2\xi_0})^2-t+\tfrac{1}{2}t'(e^{-2\xi+2\xi_0}-1)\Big\}\cos 2\eta+\sum_{m=2}^{\infty}\Big\{a_m' e^{-2(m-1)\xi}+b_m' e^{-2m\xi}+a_{m+1}' e^{-2(m+1)\xi}\Big\}\cos 2m\eta\Big]
\end{aligned}
$$

$$(n \geq 2) \quad , \qquad (1)$$

where we have defined

$$k=\frac{1}{128}\Big\{(4\xi_0-2\ln 2+2)\sinh 2\xi_0+\frac{2}{3}+\frac{1}{2}e^{-2\xi_0}-\frac{6}{5}e^{-4\xi_0}+\frac{1}{30}e^{-6\xi_0}\Big\} \quad ,$$

$$k'=\frac{1}{128}\Big\{(8\xi_0-4\ln 2+4)\cosh 2\xi_0+4\sinh 2\xi_0-e^{-2\xi_0}+\frac{24}{5}e^{-4\xi_0}-\frac{1}{5}e^{-6\xi_0}\Big\} \quad ,$$

$$s=-\frac{1}{128}\Big(2\sinh 2\xi_0-1+\frac{4}{3}e^{-2\xi_0}+\frac{1}{3}e^{-4\xi_0}\Big) \quad , \quad s'=-\frac{1}{128}\Big(4\cosh 2\xi_0-\frac{8}{3}e^{-2\xi_0}-\frac{4}{3}e^{-4\xi_0}\Big) \quad ,$$

$$t=-\frac{1}{128}\Big\{(4\xi_0-2\ln 2)\cosh 2\xi_0+2\sinh 2\xi_0+\frac{4}{3}+\frac{3}{2}e^{-2\xi_0}+\frac{6}{5}e^{-4\xi_0}+\frac{1}{30}e^{-6\xi_0}\Big\} \quad ,$$

$$t'=-\frac{1}{128}\Big\{(8\xi_0-4\ln 2)\sinh 2\xi_0+8\cosh 2\xi_0-3e^{-2\xi_0}-\frac{24}{5}e^{-4\xi_0}-\frac{1}{5}e^{-6\xi_0}\Big\} \quad ,$$

$$a_m=c_m e^{2\xi_0}+c_{m+1}e^{-2\xi_0}-d_m \quad , \quad b_m=-\frac{1}{2}c_m e^{4\xi_0}-2c_{m+1}-\frac{1}{2}c_{m+2}e^{-4\xi_0}+e_m \quad ,$$

$$a'_m = -c_m e^{2\xi_0} - c_{m+1} e^{-2\xi_0} - d_m \quad , \quad b'_m = \frac{1}{2} c_m e^{4\xi_0} + 2 c_{m+1} + \frac{1}{2} c_{m+2} e^{-4\xi_0} + e_m \quad ,$$

$$c_m = -\frac{1}{64(m-1)(2m-3)(2m-1)} \quad , \quad d_m = -\frac{2m-1}{64(2m-3)(2m+1)} \quad , \quad e_m = -\frac{m}{64(m-1)(m+1)}$$

$$(m \geq 2) \quad ,$$

$$A^2 = \sum_{i+j=n} A_i A_j \quad , \quad AB = \sum_{i+j=n} A_i B_j \quad , \quad B^2 = \sum_{i+j=n} B_i B_j \quad (n \geq 2) \quad . \tag{2}$$

Unfortunately we could not find biharmonic functions in their closed forms to make the logarithmic terms in (1) satisfy the no-slip condition on the cylinder surface. Hence, these terms were expanded in Fourier series and the biharmonic functions, which were constructed by considering the fact that the functions $x \cdot f$, $y \cdot f$ and $r^2 \cdot f$ are biharmonic if $f(\xi, \eta)$ is harmonic, were used in order to make each of the Fourier components satisfy the no-slip condition.

2. Evaluation of the constant C in (4.7)

The Jacobian of the right-hand side of (4.5) becomes

$$-\tilde{J}(\Psi_{0,1}\ , \tilde{\nabla}^2 \Psi_{0,1}) = e^{z\cos(\theta-\alpha)} \sum_{n=1}^{\infty} g_n(z) \sin n(\theta-\alpha) \quad , \tag{3}$$

where

$$g_1(z) = -\frac{1}{2z}\left\{2K_0K_1I_1 + 3K_1^2I_2 + K_0K_1I_3 + zK_0K_1I_2 + \frac{1}{2}zK_0^2(I_1+I_3)\right\} \quad ,$$

$$g_2(z) = \frac{1}{2z}\left\{K_0^2(I_1+3I_3) + K_0K_1(I_0+2I_2-I_4) + K_1^2(I_1-I_3) + \frac{1}{2}zK_1^2I_4 + \frac{1}{2}zK_0K_1I_3\right\} \quad ,$$

$$g_n(z) = \frac{1}{2z}\left\{K_1^2(I_{n-1}-I_{n+1}) + \left(\frac{1}{2}zK_0^2 + K_0K_1 - \frac{1}{2}zK_1^2\right)(I_{n-2}-I_{n+2})\right\} \quad (n \geq 3) \quad , \tag{4}$$

and $z = \tilde{r}/2$. If we make the integral representation of $\frac{1}{2}\frac{\partial^2 \Psi_{0,2}^{(p)}}{\partial X \partial Y}\Big|_{\tilde{r}=0}$ by use of (4.6) ($X = \tilde{r}\cos(\theta-\alpha)$, $Y = \tilde{r}\sin(\theta-\alpha)$), it does not have a finite value and its integrand is divergent in $O(\ln z_1/z_1)$ as $z_1=\tilde{r}_1/2\to 0$. We subtract a known function which has the same asymptotic form as that of $\Psi^{(p)}_{0,2}$ except the third term of (4.7). As a function which has the asymptotic form

$$\left\{ \frac{1}{16}(\ln\tilde{r})^2-\frac{1}{8}(\Gamma+\frac{1}{4})\ln\tilde{r}+C' \right\}\tilde{r}^2\sin 2(\theta-\alpha) \tag{5}$$

as $\tilde{r}\to 0$ and approaches zero in $O(1/\tilde{r})$ as $\tilde{r}\to\infty$, we find out

$$\hat{\Psi}=(K_0^2 I_1^2+3K_0 I_2)\sin 2(\theta-\alpha) \quad , \tag{6}$$

in which case

$$C'=\frac{1}{16}(2\ln 2-\gamma)^2+\frac{3}{32}(2\ln 2-\gamma) \quad . \tag{7}$$

Taking the Laplacian of (6), we have the equation

$$\tilde{\nabla}^2\hat{\Psi}=\hat{\Phi}(z)\sin 2(\theta-\alpha) \quad , \tag{8}$$

where

$$\begin{aligned}\hat{\Phi}(z)=&\frac{1}{4}K_0^2\left\{\frac{1}{2}I_0^2+\frac{7}{2}I_1^2+I_0I_2+\frac{1}{2}I_2^2+\frac{1}{2}I_1I_3+\frac{1}{z}I_1(I_0+I_2)-\frac{4}{z^2}I_1^2\right\}\\&-K_0K_1I_1(I_0+I_2)+\frac{1}{2}K_1^2I_1^2-\frac{3}{4}K_1(I_1+I_3)\\&+\frac{1}{4}K_0\left\{\frac{3}{4}I_0+\frac{9}{2}I_2+\frac{3}{4}I_4+\frac{3}{2z}(I_1+I_3)-\frac{1}{12z^2}I_2\right\} \quad .\end{aligned} \tag{9}$$

Therefore, the integral representation of $\hat{\Psi}$ becomes

$$\hat{\Psi}=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\tilde{r}_1\,d\tilde{r}_1\,d\theta_1\,\ln|\tilde{\mathbf{r}}-\tilde{\mathbf{r}}_1|\,\hat{\Phi}(\tfrac{1}{2}\tilde{r}_1)\sin 2(\theta_1-\alpha) \quad , \tag{10}$$

and the constant C in (4.7) can be calculated by

$$\begin{aligned}C&=\frac{1}{2}\frac{\partial^2\Psi^{(p)}_{0,2}}{\partial X\partial Y}\bigg|_{\tilde{r}=0}-\frac{1}{2}\frac{\partial^2\hat{\Psi}}{\partial X\partial Y}\bigg|_{\tilde{r}=0}+C'\\&=\frac{1}{2}\int_0^{\infty}\left\{(\frac{2}{z_1}I_1-z_1K_1)g_1+\sum_{n=2}^{\infty}g_n\frac{2n}{z_1}I_n+\frac{1}{2z_1}\hat{\Phi}\right\}dz_1+C' \quad .\end{aligned} \tag{11}$$

The integrand of (11) is nonsingular at $z_1=0$ and approaches zero in $O(1/z_1^2)$ as $z_1 \to \infty$. The numerical integration yields $C \doteqdot 0.0515$.

3. Results of matching to $O[R/(\ln R)^2]$

$M_{3,1}$ in $\psi_{1,3}$ also participates in the matching of $O[1/(\ln R)^2]$ in the description of the outer variables, and it should be determined by the matching about $\tilde{r}^2 \ln \tilde{r}$ of $O[1/(\ln R)^3]$. Although the particular solution of $\Psi_{0,3}$ would have to be evaluated in order to complete the matching of $O[1/(\ln R)^3]$, the close examination makes it clear that the particular solution of $\Psi_{0,3}$ has no function of $\tilde{r}$ alone and we need to consider only its homogeneous part in the determination of $M_{3,1}$. The matching about $\tilde{r}^2 \ln \tilde{r}$ of $O[1/(\ln R)^3]$ yields $M_{3,1} = -\frac{1}{16}(p-q)(2\Gamma-p-q)\sin 2\alpha$. After the whole procedure described in the text, we have

$$L_0 = -\frac{1}{32} e^{2\xi_0} \cos 2\alpha, \quad M_{0,1} = 0, \quad M_{0,2} = M_{0,3} = \frac{1}{32} e^{2\xi_0} \sin 2\alpha,$$

$$L_1 = -\frac{1}{64} e^{2\xi_0} \cos 2\alpha, \quad M_{1,1} = 0, \quad M_{1,2} = \left(\frac{1}{64} e^{2\xi_0} - \frac{1}{16} e^{-2\xi_0}\right) \sin 2\alpha, \quad M_{1,3} = \frac{1}{64} e^{2\xi_0} \sin 2\alpha,$$

$$L_2 = \frac{1}{16}\xi_0 + \frac{5}{128} - e^{2\xi_0}\left\{\frac{1}{32}\Gamma^2 + \frac{1}{32}\Gamma + \frac{1}{256}(p-q)^2 - \frac{1}{128}(p+q) + \frac{9}{256} + \frac{1}{64}(2\xi_0 - \ln 2) - \frac{1}{2}C\right\}\cos 2\alpha - \frac{1}{128}\cos 4\alpha,$$

$$M_{2,1} = -\frac{1}{16} e^{-2\xi_0} \sin 2\alpha, \quad M_{2,2} = M_{2,3} - \frac{1}{8}\left\{\frac{1}{2}(p-q)\Gamma - \frac{1}{4}(p^2-q^2) - \frac{1}{8} + 8\left(a_2' e^{-4\xi_0} - t - \frac{1}{2}t'\right)\right\}\sin 2\alpha,$$

$$M_{2,3} = -\frac{1}{8} e^{2\xi_0}\left[-\frac{1}{4}\Gamma^2 - \frac{1}{4}\Gamma + \frac{1}{32}\left\{(p-q)^2 + 2(p+q) - 9\right\} - \frac{1}{8}(2\xi_0 - \ln 2) - \frac{1}{8}(p-q)\cos 2\alpha + 4C\right]\sin 2\alpha. \quad (12)$$