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APPENDIX TO PAPER

PERTURBED BIFURCATION THEORY

FOR

POISEUILLE ANNULAR FLOW

by

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The first major numerical calculation involved the solution of the Orr-Sommerfeld system (and its adjoint) to determine the apexes of the neutral curves with their critical frequencies and Reynolds numbers for varying values of η. The technique employed to do this (Keller and Cebeci) consists of transforming the fourth order differential equation into a first order system of the form

$$\underline{Z}' = A(r;\alpha_o,c,R)\underline{Z} \tag{A.1}$$

where

$$c = \omega_o/\alpha_o$$

$$\underline{Z}(r) = \begin{bmatrix} \phi(r) \\ F(r) \\ \psi(r) \\ G(r) \end{bmatrix} \qquad \begin{aligned} \phi' &= F \\ F' &= \psi + \frac{1}{r}F \\ \psi' &= G \\ G &= \phi''' - \frac{1}{r}\phi'' + \frac{1}{r^2}\phi' \end{aligned}$$

and

$$A(r;\alpha_o,c,R) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{1}{r} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \{-\alpha_o^4 - i\alpha_o^3 R(U-c) & 0 & 2\alpha_o^2 + i\alpha_o R(U-c) & \frac{1}{r} \\ -i\alpha_o R(U'' - \frac{1}{r}U')\} & & & \end{bmatrix}$$

The boundary conditions can be written in the form

$$B_o\underline{Z}(a) = \underline{o} \qquad B_1\underline{Z}(b) = \underline{o} \tag{A.2}$$

where

$$B_o = B_1 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Bmatrix}$$

To solve this eigenvalue problem we drop one of the homogeneous boundary conditions and replace it by an inhomogeneous one independent of the two original constraints imposed. For example, we replace the condition

$$B_o\underline{Z}(a) = \underline{o}$$

by the condition

$$C_o\underline{Z}(a) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

where

$$C_o = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Bmatrix}$$

We then solve the problem

$$\underline{Z}' = A\underline{Z}$$

$$C_o\underline{Z}(a) = \begin{Bmatrix} 1 \\ o \end{Bmatrix} \qquad B_1\underline{Z}(b) = \underline{o} \tag{A.3}$$

We then use this solution in the dropped boundary condition, which can be written in the form:

$$D(\alpha_o, c, R) = o. \tag{A.4}$$

Since the eigenfunctions are complex, (A.4) is actually two real equations. By fixing a value of α_o, we search for the values of c and R that will make the above relation exact. We solve (A.4) by Newton's method; i.e. we determine the iterates (c^ν, R^ν) defined by

$$\begin{aligned} D_{r,c}^\nu \Delta c^\nu + D_{r,R}^\nu \Delta R^\nu &= -D_r^\nu \\ D_{i,c}^\nu \Delta c^\nu + D_{i,R}^\nu \Delta R^\nu &= -D_i^\nu \end{aligned} \tag{A.5}$$

where

$$D^\nu \equiv D_r^\nu + iD_i^\nu$$

$$c^{\nu+1} = c^\nu + \Delta c^\nu$$

$$R^{\nu+1} = R^\nu + \Delta R^\nu$$

We determine the partial derivatives by using the solutions of the variational problems:

a) $\underline{Z}_c' = A\underline{Z}_c + A_c\underline{Z}$ $\qquad C_o\underline{Z}_c(a) = -C_{o,c}\underline{Z}(a)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad B_1\underline{Z}_c(b) = -B_{1,c}\underline{Z}(b)$

b) $\underline{Z}_R' = A\underline{Z}_R + A_R\underline{Z}$ $\qquad C_o\underline{Z}_R(a) = -C_{o,R}\underline{Z}(a)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad B_1\underline{Z}_R(b) = -B_{1,R}\underline{Z}(b)$

in the forms

$$D_{r,c}+iD_{i,c} = Z_c(o)$$

$$D_{r,R} + iD_{i,R} = Z_R(o) \qquad \text{(A.6)}$$

Since Newton's method converges quadratically, very few iterations are required to solve these systems. Also, since the matrix A is unchanged, very little extra work is required to solve the variational problems once the original system is solved.

We can also obtain quite accurate initial guesses for new parameter values by solving other variational equations. If we call ζ the parameter to be varied, then we need only compute (c_ζ, R_ζ) from $dD/d\zeta = 0$ to get

$$D_{r,c}c_\zeta + D_{r,R}R_\zeta = -D_{r,\zeta}$$

$$D_{i,c}c_\zeta + D_{i,R}R_\zeta = -D_{i,\zeta} \qquad \text{(A.7)}$$

where the right hand side is evaluated using the solution of

$$\underline{Z}'_\zeta = A\underline{Z}_\zeta + A_\zeta\underline{Z} \qquad C_o\underline{Z}_\zeta(a) = -C_{o,\zeta}\underline{Z}(a)$$

$$B_1\underline{Z}_\zeta(b) = -B_{1,\zeta}\underline{Z}(b) \qquad \text{(A.8)}$$

Having solved this problem, we then use as our initial guess at

$$c(\zeta+\Delta\zeta) = c(\zeta) + \Delta\zeta c_\zeta$$
$$R(\zeta+\Delta\zeta) = R(\zeta) + \Delta\zeta R_\zeta \tag{A.9}$$

There are two parameters which we will vary, α and η. We first vary α to obtain the apex of the neutral curve for a fixed η value. The apex is characterized by the condition that $\partial R/\partial\alpha = 0$ there. Once we have found the apex for a particular value of η, we can determine accurate initial guesses for c and R as well as the location of the new apex for a slightly smaller value of η by solving additional variational equations. In particular, we compute $R_\eta, R_{\alpha\alpha}, R_{\eta\alpha}$. If at $\eta = \eta_o$ the apex is characterized by $\alpha = \alpha_o$ and $R = R_o$, then as our initial guesses for $\eta = \eta_o + \delta\eta$ we take

$$\alpha = \alpha_o - \frac{R_{\eta\alpha}(\alpha_o,\eta_o)}{R_{\alpha\alpha}(\alpha_o,\eta_o)}\delta\eta$$
$$R = R_o + R_\eta(\alpha_o,\eta_o)\delta\eta \tag{A.10}$$

This proved a very efficient method of determining the new apex locations for small η changes.

DIFFERENCE SCHEME

In order to approximate the solution $\underline{Z}$ of (A.3) we use the box scheme (centered-Euler). We first partition the interval (a,b) by setting

$$r_o = a \ , \ r_j = r_{j-1}+h_j \qquad 1\leq j\leq J \ , \ r_J = b \tag{A.11}$$

We then approximate $\underline{Z}(r_j;\alpha,c,R)$ by $\underline{Z}_j(\alpha,c,R)$ which satisfies

$$\frac{\underline{Z}_j - \underline{Z}_{j-1}}{h_j} = A_{j-\frac{1}{2}} \frac{\underline{Z}_j + \underline{Z}_{j-1}}{2} \qquad 1 \leq j \leq J \tag{A.12}$$

$$C_o \underline{Z}_o = \begin{Bmatrix} 1 \\ o \end{Bmatrix} \qquad B_1 \underline{Z}_j = \underline{o} \tag{A.13}$$

where

$$A_{j-\frac{1}{2}} = A(r_{j-\frac{1}{2}}; \alpha, c, R) \quad , \quad r_{j-\frac{1}{2}} = r_{j-1} + \frac{h_j}{2}$$

We can rewrite this as a linear system

$$A_h \underline{Z} = \underline{e}_1$$

where

$$A_h = \begin{bmatrix} C_o & 0 & & \\ L_1 & R_1 & & \\ & \ddots & \ddots & \\ 0 & & L_J & R_J \\ & & & B_1 \end{bmatrix}$$

$$L_j = -\frac{1}{h_j} I - \tfrac{1}{2} A_{j-\frac{1}{2}}$$

$$R_j = \frac{1}{h_j} I - \tfrac{1}{2} A_{j-\frac{1}{2}}$$

$$\underline{Z} = (\underline{Z}_o, \underline{Z}_1, \cdots, \underline{Z}_J)^T$$

$$\underline{e}_1 = (1,0,\cdots,0)^T$$

Equation (A.15) is solved by employing the LU decomposition of A . The variational problems are differenced in the same manner and solved using the decomposed form of A .

Throughout our η range, we used N=200 mesh points in the interval [a,b], where b-a=2. As a check, we ran the apex calculations with 400 points and found that the eigenvalues differed only in the fourth decimal place. As an additional check, the eigenvalues were computed from the adjoint equation; they agreed with those from the OSE also to the fourth decimal place.

CHEBYSHEV POLYNOMIAL APPROXIMATIONS

Although the finite difference scheme provides an efficient method of determining the critical apex locations, the eigenfunctions are given by means of a Chebyshev polynomial expansion (see Orszag, 1971). The n-th degree Chebyshev polynomial of the first kind, $T_n(y)$, is defined by

$$T_n(\cos\theta) = \cos n\theta \tag{A.17}$$

for all non-negative integers n.

An important difference between finite difference (finite element) approximations to the eigenfunctions and Chebyshev polynomial approximations lies in their order of accuracy. FDM (FEM) approximations give only a finite order of accuracy in the sense that errors behave asymptotically like $(\delta r)^p$ for some finite p as the mesh size δr approaches zero. However, if the laminar velocity profile $U(r)\epsilon C^\infty[-1,1]$, the Che-

byshev approximations are of infinite order accuracy in the sense that errors decrease more rapidly than any power of 1/N as N approaches infinity, where N is the number of Chebyshev polynomials used in the approximation. Chebyshev expansions were used not only for the O-S eigenfunctions, but also for the solutions of all subsequent differential equations. The integrals involved in all inner product evaluations were then solved using Gauss-Chebyshev quadrature. This technique has degree of precision 2n-1, whereas the standard trapezoidal or Simpson's rule has degree of precision n (n being the number of points used in the quadrature).