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Appendix A. Unbounded parabolic profile.

Consider the basic velocity

$$U(y) = \tfrac{1}{2}U''_m y^2 \qquad \text{for} \quad -\infty < y < \infty, \tag{A1}$$

where $U''_m$ is some constant. Thus we seek to solve

$$(\tfrac{1}{2}U''_m y^2 - c)(\phi'' - \alpha^2\phi) - (U''_m - \beta)\phi = 0 \tag{A2}$$

and

$$\phi(y) \to 0 \qquad \text{as} \qquad y \to \pm\infty. \tag{A3}$$

(Equation (A2) and condition (3) imply that $\phi$ vanishes at infinity.)

It may be seen that this problem is symmetric in $y$ and that no eigenvalue $c$ belongs to more than one independent eigenfunction $\phi$; it follows that each eigenfunction is either even or odd; therefore the boundary conditions (A3) may conveniently be replaced by

$$\phi(y) \to 0 \qquad \text{as} \qquad y \to \infty \tag{A4}$$

and either

$$\phi'(0) = 0 \quad \text{or} \quad \phi(0) = 0 \tag{A5}$$

respectively.

Note that $U'' - \beta = U''_m - \beta$ is constant and therefore does not change sign in the flow; therefore the flow is stable. It follows, by elimination of the possible classes of modes, that if $U''_m < 0$ then all bound states belong to the continuum of singular stable modes. If, however, $U''_m > 0$ then there may exist modified Rossby waves with $c < 0$. To study these modes it is helpful to define

$$Y = \sqrt{(-U''_m/2c)}y, \qquad k = \sqrt{(-2\alpha^2 c/U''_m)} > 0, \tag{A6}$$

and

$$\mu = \sqrt{2\beta/U''_m - 9/4} \quad \text{or} \quad \nu = \sqrt{9/4 - 2\beta/U''_m}. \tag{A7}$$

Then the problem may be rewritten without approximation as

$$\phi_{YY} + \left(-k^2 + \frac{\frac{1}{4} + \mu^2}{Y^2 + 1}\right) \phi = 0, \tag{A8}$$

where

$$\phi_Y = 0 \quad \text{or} \quad \phi = 0 \quad \text{at} \quad Y = 0 \tag{A9}$$

and

$$\phi(Y) \to 0 \qquad \text{as} \quad Y \to \infty. \tag{A10}$$

(We use subscripts now to denote derivatives.)

Further, it can be shown that if $\psi$ is defined by $\phi(Y) = \sqrt{Y^2 + 1}\,\psi(Y)$ then $\psi$ satisfies an oblate spheroidal wave equation. Unfortunately little seems to be known about its eigenvalues with the present boundary conditions, in spite of its having a name, so we must examine its spectrum ourselves.

Regarding (A8)-(A10) as a Sturm-Liouville problem to determine $\mu^2$ for any given value of $k^2 > 0$, one may see that $\mu^2$, and therefore $\beta$, decreases monotonely from infinity as $k$ decreases from infinity. For use in §4, we now seek the behaviour of $\beta$ as $k^2 \downarrow 0$. One can see at once that if $0 > \frac{1}{4} + \mu^2 = \frac{1}{4} - \nu^2$ then all solutions of equation (A8) are exponential in character and therefore none can satisfy boundary condition (A9) as well as (A10). So a lower bound of $\mu^2$ for the existence of eigensolutions is $-\frac{1}{4}$. To establish that the greatest lower bound is in fact zero, we make the ansatz that $k \downarrow 0$ more rapidly than any power of $\mu$ as $\mu \downarrow 0$ for each value of $n$, and will justify the ansatz plausibly below by constructing the asymptotic solution by the method of matched asymptotic expansions.

To try to solve the problem by regular-perturbation theory, one would expand the solution of equation (A8) in powers of $\mu^2$ for small $\mu$. However, one can see at once that this expansion is not valid uniformly at infinity, because equation (A8) and boundary condition (A10) give

$$\phi(Y) \sim \text{constant} \times e^{-kY} \quad \text{as} \quad Y \uparrow \infty. \tag{A11}$$

This suggests the use of a power series in $\mu^2$ to satisfy equation (A8) and boundary condition (A9) at $Y = 0$ as an inner approximation, together with an outer approximation in terms of the outer variable

$$\eta \equiv kY = \sqrt{(-2\alpha^2 c/U_m'')}\,Y = \alpha y. \tag{A12}$$

In terms of this transformed independent variable, equation

(A8) becomes

$$\phi_{\eta\eta} + \left(-1 + \frac{\frac{1}{4} + \mu^2}{\eta^2 + k^2}\right)\phi_o = 0 \tag{A13}$$

without approximation. Thus the outer equation in the limit as $k \downarrow 0$ for fixed $\eta$ is

$$\phi_{o\eta\eta} + \left(-1 + \frac{\frac{1}{4} + \mu^2}{\eta^2}\right)\phi_o = 0, \tag{A14}$$

and the outer solution $\phi_o(y) \to 0$ as $\eta \uparrow \infty$. In fact the complete expansion in powers of $\mu^2$ may be found, because it can be shown that

$$\phi_o(\eta) = A\eta^{\frac{1}{2}}K_{i\mu}(\eta) \tag{A15}$$

for some constant $A$ of normalization, where $K_{i\mu}$ is a modified Bessel function of the second kind. This gives

$$\phi_o(\eta) \sim \sqrt{\tfrac{1}{2}\pi}\, Ae^{-\eta} = \sqrt{\tfrac{1}{2}\pi}\, Ae^{-kY} \quad \text{as} \quad \eta \uparrow \infty,$$

of the form (A11), and

$$\phi_o(\eta) = \frac{1}{\sqrt{2}} A\left\{\Gamma(i\mu)(\tfrac{1}{2}\eta)^{\frac{1}{2}-i\mu} + \Gamma(-i\mu)(\tfrac{1}{2}\eta)^{\frac{1}{2}+i\mu} + O(\eta^{\frac{5}{2}\pm i\mu})\right\} \quad \text{as} \quad \eta \downarrow 0, \tag{A16}$$

where $\Gamma$ is the gamma function (cf. Abramowitz & Stegun 1964, §§9.7.2, 9.6.2, 6.1.17, 9.6.7, 9.6.10).

The inner problem is found by putting $k = 0$ for fixed $Y$ formally to get

$$\phi_{iYY} + \frac{\frac{1}{4} + \mu^2}{Y^2 + 1}\phi_i = 0, \tag{A17}$$

where the inner solution $\phi_i$ satisfies boundary condition (A9) at $Y = 0$ but not (A10). Again, the problem is simple enough to be solved explicitly for all $\mu$, it being easily shown that

$$\phi_i(Y) = (Y^2 + 1)^{\frac{1}{2}} \left\{ BP^1_{-\frac{1}{2}+i\mu}(iY) + B^*P^1_{-\frac{1}{2}-i\mu}(-iY) \right\} \tag{A18}$$

is the general real solution of equation (A17) for an arbitrary complex constant $B$, where $P^1_{-\frac{1}{2}\pm i\mu}$ are the associated Legendre functions and $B^*$ is the complex conjugate of $B$. Then boundary condition (A9) gives

$$B^* = \pm B$$

according to whether $\phi$ is an even or an odd function respectively. Therefore

$$\phi_i(Y) \sim \frac{Bi^{\frac{1}{2}}}{(-2\pi)^{\frac{1}{2}}} \left[ \frac{2^{-i\mu}\Gamma(-i\mu)}{\Gamma(-i\mu - \frac{1}{2})} \left\{ i^{-i\mu} \pm (-i)^{-i\mu} \right\} Y^{\frac{1}{2}-i\mu} \right.$$

$$\left. + \frac{2^{i\mu}\Gamma(i\mu)}{\Gamma(i\mu - \frac{1}{2})} \left\{ i^{i\mu} \pm (-i)^{i\mu} \right\} Y^{\frac{1}{2}+i\mu} \right] \quad \text{as} \quad Y \uparrow \infty \tag{A19}$$

(cf. Abramowitz & Stegun 1964, §8.1.5 after correction).

We next match the inner limit (A16) of the outer solution $\phi_o$ with the outer limit (A19) of the inner solution $\phi_i$ in a region where $Y$ is large and $\eta$ is small when $k$ is small. One may identify terms in $Y^{\frac{1}{2}\pm i\mu}$ in each limit, so that the ratio of the coefficients of $Y^{\frac{1}{2}\pm i\mu}$ in (A16) may be made equal to the ratio in (A19). It follows at length that one must take the limit as $\mu \downarrow 0$ in order to make the matching possible.

Then relation (A16) gives

$$\phi_o(\eta) \sim -2\mu^{-1}A\eta^{\frac{1}{2}}\sin(\mu\ell n\tfrac{1}{2}\eta) \quad \text{as} \quad \eta \downarrow 0 \quad \text{and} \quad \mu \downarrow 0$$

and (A19) gives

$$\phi_i(Y) \sim (-2i)^{\frac{1}{2}}BY^{\frac{1}{2}} \times \begin{cases} i(\pi\mu)^{-1}\cos(\mu\ell nY) \\ \tfrac{1}{2}\sin(\mu\ell nY) \end{cases} \quad \text{as} \quad Y \uparrow \infty \quad \text{and} \quad \mu \downarrow 0$$

respectively according as $\phi$ is even or odd. Noting that $\eta = kY$ and matching, we deduce that

$$\mu\ell n\tfrac{1}{2}k \to -\tfrac{1}{2}(n+1)\pi \quad \text{as} \quad \mu \downarrow 0$$

for $n = 1, 2, \ldots$, where $n$ is odd if $\phi$ is an even function and $n$ is even if $\phi$ is an odd function, so that the solution has $n - 1$ zeros in the overlap region where $-\infty < Y < \infty$. Therefore

$$c_n \sim -2U_m''\alpha^{-2}b_n e^{-(n+1)\pi/\mu} \quad \text{as} \quad \beta \downarrow 9U_m''/8 \tag{A20}$$

for some constants $b_n > 0$ and for fixed $\alpha^2$ and $n = 1, 2, \ldots$ .

Formula (A20) gives

$$c_n - U_m \sim -2U_m''\alpha^{-2}b_n e^{-(n+1)\pi/\mu} \tag{A21}$$

more generally when $U(y) = U_m - \tfrac{1}{2}U_m''y^2$, and therefore

$$c_g = \frac{\partial(\alpha c_n)}{\partial\alpha} = U_m + 2U_m''\alpha^{-2}b_n e^{-(n+1)\pi/\mu} + \ldots$$

so that

$$c_g - c_n \sim 2(U_m - c_n) \qquad \text{as} \quad \mu \downarrow 0 \qquad \text{(A20)}$$

for $n = 1, 2, \ldots$ . Thus $c_g$ is a little greater than the phase velocity and lies within the range of $U$.

Appendix B. Semi-bounded linear profile.

Consider next the basic velocity

$$U(y) = U'_m y \qquad \text{for} \quad 0 \leq y < \infty, \tag{B1}$$

where $U'_m$ is some constant. Then the eigenvalue problem is that

$$\phi'' + \{-\alpha^2 + \beta/(U'_m y - c)\}\phi = 0, \tag{B2}$$

$$\phi(0) = 0 \tag{B3}$$

and

$$\phi(y) \to 0 \qquad \text{as} \quad y \to \infty. \tag{B4}$$

Again, this flow can be shown to be stable. Also modified Rossby waves occur only when $U'_m > 0$ and $c < 0$. Further, if $\alpha = 0$ then there is no modified Rossby wave, because no solution of equation (B2) is compatible with the boundary condition (B4) at infinity.

So we may suppose without loss of generality that $U'_m > 0$, $\alpha > 0$ and $c < 0$ in order to find the modified Rossby waves. Then it is convenient to define

$$Y = 2\alpha(y - c/U'_m), \quad b = \beta/2\alpha U'_m \quad \text{and} \quad \lambda = -2\alpha c/U'_m \tag{B5}$$

so that the problem may be rewritten as

$$\phi_{YY} + (-\tfrac{1}{4} + \frac{b}{Y})\phi = 0, \tag{B6}$$

$$\phi(Y) \to 0 \qquad \text{as} \quad Y \to \infty \tag{B7}$$

and

$$\phi(\lambda) = 0 . \tag{B8}$$

Now equation (B6) is Whittaker's equation, whose solution satisfying condition (B7) at infinity is

$$\phi(Y) = e^{-\frac{1}{2}Y}\, U(-b,0,Y), \tag{B9}$$

where U is the second solution of Kummer's equation (cf. Abramowitz & Stegun 1964, §§13.1.31, 13.1.33). Then the boundary condition at the wall gives the eigenvalue relation,

$$c = -U_m'\lambda_n/2\alpha \quad \text{for} \quad n = 1,2, \ldots, \tag{B10}$$

where $\lambda_n$ is the nth positive zero of $U(-b,0,\lambda)$ for each given value of b.

Again, we shall find $\lambda_n$ and thence c asymptotically by making the ansatz that $\lambda_n \downarrow 0$ as $b \downarrow b_n > 0$. Then

$$0 = U(-b,0,\lambda) = \frac{1}{\Gamma(1-b)} + O(-b\lambda \ln\lambda) \quad \text{as} \quad \lambda \downarrow 0$$

(cf. Abramowitz & Stegun 1964, §13.5.11), so that

$$\lambda_n \downarrow 0 \quad \text{as} \quad b \downarrow b_n = n, \tag{B11}$$

because the $b_n$ are the poles of the gamma function.